ON RIEMANN-LIOUVILLE INTEGRAL OF ULTRA-HYPERBOLIC TYPE

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1. Introduction.

Riesz has persuited the many types, elliptic and hyperbolic types, of Riemann-Liouville integral since 1933. Now let r_{PQ} be the distance between two points P and Q, then we call the following integral the Riemann-Liouville integral

$$I^{\alpha}f(\mathrm{P})=\frac{1}{H_m(\alpha)}\int f(\mathrm{Q})r_{\mathrm{PQ}}^{\alpha-m}\,d\mathrm{Q}.$$

Here the range of integration is taken suitably according to the above mentioned types. Further $H_m(\alpha)$ corresponds to the gamma function in the old theory of Riemann-Liouville integral and it depends on the dimension m of the space and on the constant α. About this there hold the fundamental formulas

$$I^{\alpha}(I^{\beta}f(\mathrm{P}))=I^{\alpha+\beta}f(\mathrm{P}),\ \Delta I^{\alpha+2}f(\mathrm{P})=\pm I^{\alpha}f(\mathrm{P})\quad\text{and}\quad I^{0}(\mathrm{P})=f(\mathrm{P}).$$

Using the Riemann-Liouville integral, Riesz [6] established the general potential theory in m-dimensional Euclidean space. Further Frostman [3] proved elegantly the fundamental theorem on the equilibrium potential in his α-dimensional potential theory. In his proof a lemma played its essential part. This lemma can be obtained from the theory of the Riemann-Liouville integral and it is

$$\int_{\Omega_3}\frac{1}{r_{\mathrm{PM}}^{k}}\frac{1}{r_{\mathrm{MQ}}^{l}}\,d\mathrm{M}=H_m(k,l)\frac{1}{r_{\mathrm{PQ}}^{k+l-3}}.$$

In addition, this equality has many applications to other branches of analysis; cf. Nozaki [5].

Next let the distance r_{PQ} of two points P(x) and Q(ξ) be

$$r_{\mathrm{PQ}}{}^2=(x_1-\xi_1)^2-(x_2-\xi_2)^2-\cdots-(x_m-\xi_m)^2.$$

Then Riesz called the space with this distance r_{PQ} (hyperbolic distance) the Lorentzian space. In this space he constructed the theory of Riemann-Liouville integral. Using this integral he solved Cauchy problem which is one of the branches of the theory of the hyperbolic partial differential equations. Riesz's theories were given in his splended paper [7].

Now in the present paper, the author will intend to extend Riesz's results more

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generally. Let the distance r_{PQ} of two points $P(x)$ and $Q(\xi)$ be

$$r_{PQ}^2=r_{QP}^2=(x_1-\xi_1)^2+(x_2-\xi_2)^2+\cdots+(x_\mu-\xi_\mu)^2-(x_{\mu+1}-\xi_{\mu+1})^2-\cdots-(x_m-\xi_m)^2.$$

We call the space with this distance (ultra-hyperbolic distance) "an ultra-hyperbolic space" or "a generalized Lorentzian space". In this space we shall introduce the Riemann-Liouville integral and shall derive its fundamental properties.

Our theory is much indebted to the Riesz's one. Introducing the vector suitably in our space, the geometrical properties of our space—surface area and volume of the solid body etc.—can be obtained by the modified methods of Riesz. Therefore we can obtain the analytic continuation of $J^\alpha f(P)$ as similar as that of Riesz, but then the complicated calculations must be necessary.

Also taking $H_m(\alpha)$ suitably in every space, we may obtain the distribution $r_+^{\alpha-m}/H_m(\alpha)=\Phi_\alpha$. Then the composition theorem

$$\Phi_\alpha * \Phi_\beta = \Phi_{\alpha+\beta}$$

holds. This is one of the characterizations of Riemann-Liouville integral.

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2. Generalized Lorentzian space and $J^\alpha f(P)$.

Let the distance r_{PQ} between two points $P(x)$ and $Q(\xi)$ be

$$(1)\qquad \begin{aligned} r_{PQ}^2=r_{QP}^2=&(x_1-\xi_1)^2+(x_2-\xi_2)^2+\cdots+(x_\mu-\xi_\mu)^2\\ &-(x_{\mu+1}-\xi_{\mu+1})^2-\cdots-(x_{\mu+\nu}-\xi_{\mu+\nu})^2 \qquad (\mu+\nu=m).\end{aligned}$$

Also let the Laplacian operator be

$$\Delta\equiv\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\cdots+\frac{\partial^2}{\partial x_\mu^2}-\frac{\partial^2}{\partial x_{\mu+1}^2}-\cdots-\frac{\partial^2}{\partial x_{\mu+\nu}^2}.$$

In the sequel we use the same terminologies and the same notations as those in the Lorentzian space; cf. Riesz [7] Chap. III.

Let the point P be fixed and the point Q be variable. We denote by D^P the inverse cone with the vertex at P which is defined by the inequalities

$$r_{PQ}^2>0, \qquad x_1-\xi_1>0.$$

Also we denote by D^{PM} the common region which is bounded by the inverse and direct cones, that is

$$r_{PQ}^2>0, \qquad x_1-\xi_1>0, \qquad \text{and} \qquad r_{QM}^2>0, \qquad y_1-\xi_1<0$$

where the point $M(y)$ is an inner point of D^P.

Next we define the scalar product of two vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ by

$$(2)\qquad (\boldsymbol{X}, \boldsymbol{Y})=X_1Y_1+X_2Y_2+\cdots+X_\mu Y_\mu-X_{\mu+1}Y_{\mu+1}-\cdots-X_{\mu+\nu}Y_{\mu+\nu} \quad (\mu+\nu=m),$$

where X's and Y's are components of the vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ respectively. If we take ε's satisfying

$$\varepsilon_1=\varepsilon_2=\cdots=\varepsilon_\mu=1, \qquad \varepsilon_{\mu+1}=\cdots=\varepsilon_{\mu+\nu}=-1, \tag{3}$$

then we can write the scalar product (2) in the form

$$(\boldsymbol{X}, \boldsymbol{Y})=\sum_{k=1}^{m} \varepsilon_k X_k Y_k. \tag{2'}$$

Using those notations, we can represent the distance between two points briefly by

$$r_{\mathrm{PQ}}{}^2=(\boldsymbol{X}-\boldsymbol{\Xi}, \boldsymbol{X}-\boldsymbol{\Xi}), \qquad r_{\mathrm{OP}}{}^2=(\boldsymbol{X}, \boldsymbol{X}).$$

We say that two vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ are orthogonal when their product vanishes. We can also take ε's more generally in the place of those in (3) such that

$$\varepsilon_i>0 \ (i=1, 2, \cdots, \mu), \qquad \varepsilon_j<0 \ (j=\mu+1, \cdots, m) \ \text{ and } \ |\varepsilon_1\varepsilon_2\cdots\varepsilon_m|=1.$$

Using those ε's, the formula (2′) remains in the same form.

Now we consider the linear homogeneous transformation which remains $(\boldsymbol{X}, \boldsymbol{X})$ invariant. Since $(\boldsymbol{X}, \boldsymbol{Y})=(1/4)\{(\boldsymbol{X}+\boldsymbol{Y}, \boldsymbol{X}+\boldsymbol{Y})-(\boldsymbol{X}-\boldsymbol{Y}, \boldsymbol{X}-\boldsymbol{Y})\}$ holds, by such transformations $(\boldsymbol{X}, \boldsymbol{Y})$ also remains invariant. Moreover a set of these transformations constitutes a transformation group. We call it a transformation of Lorentz in a wider sense.

As regard to the derivations of a function $\varphi(\boldsymbol{x})$, there holds evidently the relations

$$(\operatorname{grad}\varphi, \operatorname{grad}\varphi)=\sum_{k=1}^{m} \varepsilon_k{}^{-1}\left(\frac{\partial\varphi}{\partial x_k}\right)^2, \tag{4}$$

and

$$\Delta\varphi=\left(\sum_{k=1}^{m} \varepsilon_k{}^{-1}\frac{\partial^2}{\partial x_k^2}\right)\varphi=\sum_{k=1}^{m} \varepsilon_k{}^{-1}\frac{\partial^2\varphi}{\partial x_k^2}. \tag{5}$$

Here the formulas (4) and (5) are also invariant by generalized Lorentzian transformation. By the above method we can proceed our vector analysis completely analogous to that of Lorentzian space.

We call m dimensional vector space with the distance (1), "a generalized Lorentzian space" or "an ultra-hyperbolic space". Let D be any given region which is enclosed by some surface T ($m-1$ dimensional varieties). We suppose that the the surface T is sufficiently smooth. Then the volume element $d\mathrm{Q}$ at any point Q of D can be represented, like that of Euclidean space, as

$$d\mathrm{Q}=d\xi_1\xi_2\cdots d\xi_m.$$

Also when the surface T is represented by means of parameters $\lambda_1, \lambda_2, \cdots, \lambda_{m-1}$ by

$$\xi_i=\xi_i(\lambda_1, \lambda_2, \cdots, \lambda_{m-1}) \qquad (i=1, 2, \cdots, m),$$

then we put

$$J_k=(-1)^{k-1}\frac{\partial(\xi_1, \xi_2, \cdots, \xi_{k-1}, \xi_{k+1}, \cdots, \xi_m)}{\partial(\lambda_1, \lambda_2, \cdots, \lambda_{m-1})} \quad \text{and} \quad J=\left(\sum_{k=1}^{m} \varepsilon_k{}^{-1}J_k^2\right)^{1/2}$$

Then the surface element dS at points Q of T becomes

$$dS=Jd\lambda_1 d\lambda_2 \cdots d\lambda_{m-1}.$$

Similarily we can express an element $d\sigma$ of the curves σ (p-dimensional varieties) by

$$d\sigma=|(d\xi, d\xi)|.$$

When $\boldsymbol{n}$ denotes the unit normal vector at point Q of the surface, as to the directional derivatives of the function in generalized Lorentzian space, we have as similar as the derivatives in the usual Euclidean space

$$\frac{dF(\xi)}{d\boldsymbol{n}}=\sum_{k=1}^{m}\frac{\partial F}{\partial n}n_k=(\text{grad } F, \boldsymbol{n}).$$

Also let $\mathbf{1}$ denotes a unit vector at any point of the surface, then we have

$$\frac{dF(\xi)}{d\mathbf{1}}=\sum_{k=1}^{m}\frac{\partial F}{\partial l}l_k=(\text{grad } F, \mathbf{1}).$$

Now let $r^2=r_{\mathrm{PQ}}{}^2$ and consider the integral

$$J^\alpha f(\mathrm{P})=\frac{1}{K_m(\alpha)}\int_{D^{\mathrm{P}}}f(\mathrm{Q})r_{\mathrm{PQ}}{}^{\alpha-m}\,d\mathrm{Q}. \tag{6}$$

If we use the distribution

$$\Phi_\alpha(\mathrm{P};\mathrm{Q})=\frac{K_m(\alpha)}{r_+{}^{\alpha-m}},$$

where r_+ is equal to r when $r>0$ and to 0 when $r<0$. Then (6) becomes

$$J^\alpha f(\mathrm{P})=(f*\Phi_\alpha)(\mathrm{P}).$$

We call $J^\alpha f(\mathrm{P})$ the Riemann-Liouville integral of the ultra-hyperbolic type. Hereafter we investigate the properties of this integral.

3. $J^\alpha f(\mathrm{P})$.

In the following sections, we are now to prove the fundamental theorems on our Riemann-Liouville integral.

THEOREM 1. *Let $f(\mathrm{P})$ be continuous and vanish rapidly at infinity in D^{P}. Then we have*

$$J^\alpha f(\mathrm{P})=\frac{1}{K_m(\alpha)}\int_{D^{\mathrm{P}}}f(\mathrm{Q})r_{\mathrm{PQ}}{}^{\alpha-m}\,d\mathrm{Q}=(f*\Phi_\alpha)(\mathrm{P}), \tag{1}$$

where

$$K_m(\alpha)=\pi^{(m-1)/2}\frac{\Gamma((2+\alpha-m)/2)\Gamma((1-\alpha)/2)\Gamma(\alpha)}{\Gamma((2+\alpha-\mu)/2)\Gamma((\mu-\alpha)/2)}$$

and

$$\Phi_\alpha(\mathrm{P};\mathrm{Q})=\frac{r_{\mathrm{PQ}}{}^{\alpha-m}}{K_m(\alpha)}=\frac{r_+{}^{\alpha-m}}{K_m(\alpha)}.$$

LEMMA (Extension of Boole's theorem). *We have*

$$\tag{2}\begin{aligned}&\iint_{x_1^2+x_2^2+\cdots+x_m^2\leq c^2}\cdots\int(c^2-x_1^2-x_2^2-\cdots x_m^2)^{-\lambda/2}\,dx_1\,dx_2\cdots dx_m\\&=c^{m-1}\frac{\Gamma(1-\lambda/2)}{\Gamma((m-\lambda)/2)}\,\pi^{m/2}.\end{aligned}$$

We can easily verify the validity of (2); Edwards [2] pp. 162–163.

Proof of the theorem. Putting $f(\mathrm{P})=e^{x_1}$ we shall determine $K_m(\alpha)$ so that the equality $J^\alpha e^{x_1}=e^{x_1}$ holds. By a linear transformation which leads P to the origin O, let Q be translated to another point which we write $\mathrm{Q}(\xi)$ again. Then we can write (1) in the form

$$\tag{3} J^\alpha e^{x_1}=\frac{e^{x_1}}{K_m(\alpha)}\int_{D^0}e^{\xi_1}\{(\xi_1^2+\xi_2^2+\cdots+\xi_\mu^2)-(\xi_{\mu+1}^2+\cdots+\xi_m^2)\}^{(\alpha-m)/2}d\xi_1d\xi_2\cdots d\xi_m.$$

If we put $\xi_1^2+\xi_2^2+\cdots+\xi_m^2=\rho^2$ and integrate with respect to $\xi_{\mu+1},\cdots,\xi_m$ in the region $\rho^2-(\xi_{\mu+1}^2+\cdots+\xi_m^2)>0$, then (3) becomes

$$J^\alpha e^{x_1}=\frac{e^{x_1}}{K_m(\alpha)}\iint\cdots\int e^{\xi_1}d\xi_1d\xi_2\cdots d\xi_\mu\iint\cdots\int\{\rho^2-(\xi_{\mu+1}^2+\cdots+\xi_m^2)\}^{(\alpha-m)/2}d\xi_{\mu+1}d\xi_{\mu+2}\cdots d\xi_m.$$

Using the above lemma, we have

$$\tag{4} J^\alpha e^{x_1}=\frac{e^{x_1}}{K_m(\alpha)}\,\frac{\Gamma((2+\alpha-m)/2)}{\Gamma((2+\alpha-\mu)/2)}\,\pi^{(m-\mu)/2}\iint\cdots\int e^{\xi_1}\rho^{\alpha-\mu}\,d\xi_1\,d\xi_2\cdots d\xi_\mu.$$

In the last integral, since Q is a variable point of D^0 with $\xi_1<0$ and since ρ varies from 0 to ∞, we may put $\xi_1=-\sin\theta$, $\xi_2^2+\cdots+\xi_\mu^2=\rho^2\cos^2\theta$. Then the integral of (4) become

$$\frac{2\pi^{(\mu-1)/2}}{\Gamma((\mu-1)/2)}\int_0^{\pi/2}\int_0^\infty e^{-\rho\sin\theta}\rho^{\alpha-1}\cos^{\mu-2}\theta\,d\theta\,d\rho$$

Putting $\rho\sin\theta=t$ again, we have

$$\tag{5}\begin{aligned}&\iint\cdots\int e^{\xi_1}\rho^{\alpha-\mu}d\xi_1\,d\xi_2\cdots d\xi_\mu=\frac{2\pi^{(\mu-1)/2}}{\Gamma((\mu-1)/2)}\int_0^{\pi/2}\int_0^\infty e^{-t}t^{\alpha-1}\sin^{-\alpha}\theta\cos^{\mu-2}\theta\,dt\,d\theta\\&=\frac{2\pi^{(\mu-1)/2}}{\Gamma((\mu-1)/2)}\int_0^{\pi/2}\sin^{(1-\alpha)-1}\theta\cos^{(\mu-1)-1}\theta\,d\theta\int_0^\infty e^{-t}t^{\alpha-1}\,dt=\pi^{(\mu-1)/2}\frac{\Gamma(\alpha)\Gamma((1-\alpha)/2)}{\Gamma((\mu-\alpha)/2)}.\end{aligned}$$

From (4) and (5) we have

$$J^{\alpha}e^{x_1}=\frac{e^{x_1}}{K_m(\alpha)}\pi^{(m-1)/2}\frac{\Gamma((2+\alpha-m)/2)\Gamma((1-\alpha)/2)\Gamma(\alpha)}{\Gamma((2+\alpha-\mu)/2)\Gamma((\mu-\alpha)/2)}. \tag{6}$$

Since $J^{\alpha}e^{x_1}=e^{x_1}$ must hold, we obtain finally from (6)

$$K_m(\alpha)=\pi^{(m-1)/2}\frac{\Gamma((2+\alpha-m)/2)\Gamma((1-\alpha)/2)\Gamma(\alpha)}{\Gamma((2+\alpha-\mu)/2)\Gamma((\mu-\alpha)/2)}. \tag{7}$$

REMARK 1. If we put $r^2=r_{PQ}^2$ and denote the convolution of $f(Q)$ with the distribution $\Phi_{\alpha}(P;\ Q)=r_+^{\alpha-m}/K_m(\alpha)$ by $(f*\Phi_{\alpha})(P)$, we may write the Riemann-Liouville integral in the form

$$J^{\alpha}f(P)=(f*\Phi_{\alpha})(P).$$

REMARK 2. If we put $\mu=1$ in (7) we can show that $K_m(\alpha)$ can be reduced to that of hyperbolic type by using the relation, $\Gamma(\alpha)/\Gamma((\alpha+1)/2)=2^{\alpha-1}\pi^{-1/2}\Gamma(\alpha/2)$,

$$K_m(\alpha)=\pi^{(m-1)/2}\frac{\Gamma((2+\alpha-m)/2)\Gamma((1-\alpha)/2)\Gamma(\alpha)}{\Gamma((1+\alpha)/2)\Gamma((1-\alpha)/2)}=\pi^{(m-2)/2}2^{\alpha-1}\Gamma\left(\frac{2+\alpha-m}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)=H_m(\alpha).$$

Concerning the compositions of the distributions Φ_{α} and Φ_{β}, we have the following theorem.

THEOREM 2. *Under the same conditions on* $f(P)$ *and the same notations as in Theorem* 1, *we have*

$$(f*\Phi_{\alpha})*\Phi_{\beta}=(f*\Phi_{\beta})*\Phi_{\alpha}=f*\Phi_{\alpha+\beta}. \tag{8}$$

Proof. The first term of (8) may be written explicitly as

$$\begin{aligned}(f*\Phi_{\alpha})*\Phi_{\beta}&=\frac{1}{K_m(\alpha)}\int_{DP}r_{PQ}^{\alpha-m}\,dQ\left\{\frac{1}{K_m(\beta)}\int_{DQ}f(M)r_{QM}^{\beta-m}\,dM\right\}\\&=\frac{1}{K_m(\alpha)K_m(\beta)}\int_{DP}f(M)\,dM\int_{DPM}r_{PQ}^{\alpha-m}r_{QM}^{\beta-m}\,dQ.\end{aligned} \tag{9}$$

Now consider the integral

$$(\Phi_{\alpha}*\Phi_{\beta})=\int_{DPM}\frac{r_{PQ}^{\alpha-m}}{K_m(\alpha)}\frac{r_{QM}^{\beta-m}}{K_m(\beta)}dQ.$$

By a linear transformation which leads P to the origin O, M to $(1, 0, \cdots, 0)$ and Q to a point T with $r_{OT}=1$ respectively, we obtain from (9)

$$\int_{DPM}r_{PQ}^{\alpha-m}r_{QM}^{\beta-m}\,dQ=r_{PM}^{\alpha+\beta-m}\int_{DO1}f_{OT}^{\alpha-m}r_{T1}^{\beta-m}\,dT=r_{PM}^{\alpha+\beta-m}B_m(\alpha,\ \beta), \tag{10}$$

where $B_m(\alpha,\ \beta)$ indicates the integral of the second member. Then (9) becomes

$$(f*\Phi_{\alpha})*\Phi_{\beta}=\frac{B_m(\alpha,\ \beta)}{K_m(\alpha)K_m(\beta)}\int_{DP}f(M)r_{PM}^{\alpha+\beta-m}\,dM=\frac{B_m(\alpha,\ \beta)K_m(\alpha+\beta)}{K_m(\alpha)K_m(\beta)}f*\Phi_{\alpha+\beta}. \tag{11}$$

If we put $f(\mathrm{P})=e^{x_1}$, then by Theorem 1, we have $(e^{x_1}*\Phi_\alpha)*\Phi_\beta=e^{x_1}$ and $e^{x_1}*\Phi_{\alpha+\beta}=e^{x_1}$. Therefore from (10) and (11), we obtain

$$B_m(\alpha,\ \beta)=\frac{K_m(\alpha)K_m(\beta)}{K_m(\alpha+\beta)}\int_{D^{\mathrm{O1}}} r_{\mathrm{OT}}{}^{\alpha-m}r_{\mathrm{T1}}{}^{\beta-m}\,d\mathrm{T}. \tag{12}$$

Hence from (10) and (11), we obtain finally $(f*\Phi_\alpha)*\Phi_\beta=f*\Phi_{\alpha+\beta}$. Similarly we can show that the relation $(f*\Phi_\beta)*\Phi_\alpha=f*\Phi_{\alpha+\beta}$ holds, so that the theorem is proved.

From (10) and (12) we obtain the following corollary.

Corollary.

$$r_{\mathrm{PM}}{}^{\alpha+\beta-m}=B_m(\alpha,\ \beta)\int_{D^{\mathrm{PM}}} r_{\mathrm{PQ}}{}^{\alpha-m}r_{\mathrm{QM}}{}^{\beta-m}\,d\mathrm{Q}$$

and

$$B_m(\alpha,\ \beta)=\frac{K_m(\alpha)K_m(\beta)}{K_m(\alpha+\beta)}.$$

This corollary is an extension of Frostman's Lemma stated in the introduction.

4. $\boldsymbol{H_m(\alpha)}$ and $\boldsymbol{H_m(\beta)}$.

There are many kinds of extensions of beta and gamma functions and we can refer to them Whittaker-Watson's book [8], Chap. XII. The results of Riesz and the present work are the extensions of their functions in another way, namely $B_m(\alpha,\ \beta)$ and $H_m(\alpha)$ or $K_m(\alpha)$ are extensions of them to the Lorentzian space or to the generalized Lorentzian space respectively. And our present results are the extensions of them in m-dimensional generalized Lorentzian space, and they are as follows:

$$B_m(\alpha,\ \beta)=\int_{\mathrm{O1}} r_{\mathrm{OT}}{}^{\alpha-m}r_{\mathrm{T1}}{}^{\beta-m}\,d\mathrm{T},\ K_m(\alpha)=\int_{D^{\mathrm{O}}} e^{\xi_1}r_{\mathrm{OQ}}{}^{\alpha-m}\,d\mathrm{Q},$$

$$\int_{D^{\mathrm{PQ}}} r_{\mathrm{PQ}}{}^{\alpha-m}r_{\mathrm{QM}}{}^{\beta-m}\,d\mathrm{Q}=B_m(\alpha,\ \beta)r_{\mathrm{PM}}{}^{\alpha+\beta-m},\qquad \text{and}\qquad B_m(\alpha,\ \beta)=\frac{K_m(\alpha)K_m(\beta)}{K_m(\alpha+\beta)}.$$

From this point of view the corollary of Theorem 2 is interesting and useful in analysis. In this section we use the notations for a while, H_m, $\bar{H}_m$ and K_m in the place of the functions corresponding to the gamma functions, in m-dimensional Euclidean, Lorentzian or generalized Lorentzian space respectively. Now let us investigate the relations among H_m, $\bar{H}_m$ and K_m. Since

$$H_m(\alpha)=\pi^{m/2}\,2^\alpha\frac{\Gamma(\alpha/2)}{\Gamma((m-\alpha)/2)}. \tag{1}$$

and

$$\bar{H}_m(\alpha)=\pi^{(m-2)/2}\,2^\alpha\frac{\Gamma(\alpha/2)}{\Gamma((\alpha+2-m)/2)}. \tag{2}$$

between (1) and (2) there exists the relation (Riesz)

$$\frac{H_m(\alpha)}{\overline{H}_m(\alpha)}=2e^{\iota\alpha\pi/2}\sin\frac{(m-\alpha)\pi}{2}.$$

We have seen that $\overline{H}_m(\alpha)$ is a special case of $K_m(\alpha)$.

To indicate the behaviour of $K_m(\alpha)$ precisely we use the notation in this section $K_m^{(\mu)}(\alpha)$ for $K_m(\alpha)$. Then we have

$$(3)\qquad K_m^{(\mu)}(\alpha)=\pi^{(m-1)/2}\frac{\Gamma((2+\alpha-m)/2)\Gamma(\alpha)}{\Gamma((1+\alpha)/2)}\frac{\sin\{(\mu-\alpha)\pi/2\}}{\cos(\alpha\pi/2)}.$$

Let p be a positive integer, then (3) becomes

$$K_m^{(\mu)}(\alpha)=\begin{cases}(-1)^{p-1}\pi^{(m-1)/2}\dfrac{\Gamma((2+\alpha-m)/2)\Gamma(\alpha)}{\Gamma((1+\alpha)/2)} & \text{for}\quad \mu=2p-1,\\[2ex] (-1)^{p-1}\pi^{(m-1)/2}\dfrac{\Gamma((2+\alpha-m)/2)}{\Gamma((1+\alpha)/2)}\tan\dfrac{\alpha\pi}{2} & \text{for}\quad \mu=2p.\end{cases}$$

Let $\Re(2+\alpha-m)>0$, then $\Gamma((2+\alpha-m)/2)\Gamma(\alpha)/\Gamma((1+\alpha)/2)$ is an analytic function of α. In addition if α is real and positive, this factor takes the positive value. But by the cause of the factor $\tan(\alpha\pi/2)$, $K_m^{(2p)}(\alpha)$ has other singularities other than those of the previous factor. For example

$$K_4^{(2)}(\alpha)=\pi^{3/2}\frac{2}{\alpha-2}\tan\frac{\alpha\pi}{2}\frac{\Gamma(\alpha/2)\Gamma(\alpha)}{\Gamma((1+\alpha)/2)},$$

has poles of order 1 at points $\alpha=1, 3, 5, \cdots$ other than the singularities of $\Gamma(\alpha)\Gamma(\alpha/2)/\Gamma((1+\alpha)/2)$.

In the following sections we consider that the functions $K_m(\alpha)$ and $B_m(\alpha, \beta)$ are analytic functions of α or of α and β.

5. Properties of $J^\alpha f(\mathrm{P})$.

We shall prove the following theorem.

THEOREM 3. *Under the same assumptions on $f(\mathrm{P})$ as in Theorem* 1, *it holds*

$$(1)\qquad \Delta(f*\Phi_{\alpha+2})=f*\Phi_\alpha,$$

where

$$\Delta\equiv\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\cdots+\frac{\partial^2}{\partial x_\mu^2}-\frac{\partial^2}{\partial x_{\mu+1}^2}-\cdots-\frac{\partial^2}{\partial x_{\mu+\nu}^2}\qquad(\mu+\nu=m).$$

Proof. Since there hold

$$\Delta r_{\mathrm{PQ}}{}^{\alpha+2-m}=\alpha(\alpha+2-m)r_{\mathrm{PQ}}{}^{\alpha-m}\quad\text{and}\quad K_m(\alpha+2)=\alpha(\alpha+2-m)K_m(\alpha),$$

if we put

$$\Phi_\alpha = \frac{r_{PQ}{}^{\alpha-m}}{K_m(\alpha)} = \frac{r_+{}^{\alpha-m}}{K_m(\alpha)} \qquad (Q \in D_P),$$

we have

$$\Delta\Phi_{\alpha+2} = \Phi_\alpha. \tag{2}$$

By differentiating under the integral sign and by using (2), we have

$$\Delta(f*\Phi_{\alpha+2}) = \Delta\left\{1/K_m(\alpha+2)\int_{D^P} f(Q) r_{PQ}{}^{\alpha+2-m}\, dQ\right\} = \int_{D^P} f(Q)\Delta(\Phi_{\alpha+2})\, dQ = \int_{D^P} f(Q)\Phi_\alpha\, dQ$$
$$= f*\Phi_\alpha.$$

Therefore we have (1).

Iterating (1) we obtain the more general form of (1):

$$\Delta^k(f*\Phi_{\alpha+2k}) = f*\Phi_\alpha \qquad (k=1, 2, \cdots).$$

REMARK. We shall investigate in the following sections the conditions on α in order to differentiate under the integral sign.

6. Further properties of $J^\alpha f(P)$.

Here we write as such

$$r_{PQ}{}^2 = (x_1-\xi_1)^2 + (x_2-\xi_2)^2 + \cdots + (x_\mu-\xi_\mu)^2 - (y_1-\eta_1)^2 - \cdots - (y_\nu-\eta_\nu)^2 \quad (\mu+\nu=m).$$

Now let us consider

$$K_m(\alpha)J^\alpha f(P) = \int_{D^P} f(\bar\xi, \bar\eta)\{(\bar\xi_1-x_1)^2+\cdots+(\bar\xi_\mu-x_\mu)^2-(\bar\eta_1-y_1)^2 - \cdots - (\bar\eta_\nu-y_\nu)^2\}^{(\alpha-m)/2}\, d\bar\xi_1\cdots d\bar\xi_\mu\, d\bar\eta_1\cdots d\bar\eta_\nu. \tag{1}$$

Let $\bar D^P$ be the direct cone with vertex at P. To consider the integral over $\bar D^P$ also we make an even extension of f with respect to $\bar\xi_1-x_1$ and denote the resulting function by u. Then from (1) we have

$$K_m(\alpha)J^\alpha f(P) = \frac{1}{2}\int_{D^P+\bar D^P} u(\bar\xi, \bar\eta) r_{PQ}{}^{\alpha-m}\, d\bar\xi_1\cdots d\bar\xi_\mu\, d\bar\eta_1\cdots d\bar\eta_\nu. \tag{2}$$

Then using the polar coordinates

$$\bar\xi_i - x_i = r\xi_i, \qquad \bar\eta_j - y_j = s\eta_j, \qquad \sum_{i=1}^{\mu}(\bar\xi_i-x_i)^2 = r^2 \quad \text{and} \quad \sum_{j=1}^{\nu}(\bar\eta_j-y_j)^2 = s^2,$$

(2) becomes

$$(3) \qquad 2K_m(\alpha)J^\alpha f(\mathrm{P})=\int_D\int_{\Omega_\mu}\int_{\Omega_\nu} u(x+r\xi,\ y+s\eta)(r^2-s^2)^{(\alpha-m)/2}r^{\mu-1}s^{\nu-1}\,drdsd\Omega_\mu d\Omega_\nu,$$

where $d\Omega_\mu$ and $d\Omega_\nu$ denote the surface elements of the μ- and ν-dimensional Euclidean unit spheres respectively and D denotes the region which is constituted by $0\leqq s\leqq r$ and $r\geqq 0$. Putting

$$(4) \qquad w(x,\ y;\ r,\ s)=\frac{1}{\omega_\mu\omega_\nu}\int_{\Omega_\mu}\int_{\Omega_\nu} u(x+r\xi,\ y+s\eta)\,d\Omega_\mu d\Omega_\nu,$$

then (3) becsmes

$$(5) \qquad J^\alpha f(\mathrm{P})=\frac{\omega_\mu\omega_\nu}{2K_m(\alpha)}\int_D w(x,\ y;\ r,\ s)r^{\mu-1}s^{\nu-1}(r^2-s^2)^{(\alpha-m)/2}\,drds,$$

where ω_μ and ω_ν denote the surface areas of μ- and ν-dimensional Euclidean unit spheres respectively.

Now let Q tend to P or Q tend to any surface point of D^{P} from within D^{P}, i.e., let $s\to t$, then $r_{\mathrm{PQ}}{}^2=r^2-s^2\to 0$. Hence when $m\geqq 2$ and when u, that is f, is continuous in D^{P} and vanishes rapidly at infinity in D^{P}, it can be easily verified that the integral (5) converges under the condition $\alpha>m-2$. We have thus the following theorem.

THEOREM 4. *If $f(\mathrm{P})$ be continuous and vanish rapidly at infinity within D^{P}, then $J^\alpha f(\mathrm{P})$ converges when $\alpha>m-2$ for $m\geqq 2$.*

Next we are now to prove the relation $J^0 f(\mathrm{P})=f(\mathrm{P})$. For that purpose we can take O for P without loss of generality in (5). Then it becomes

$$(6) \qquad J^\alpha f(0)=\frac{\omega_\mu\omega_\nu}{2K_m(\alpha)}\int_0^\infty\int_0^r w(0,\ 0;\ r,\ s)(r^2-s^2)^{(\alpha-m)/2}r^{\mu-1}s^{\nu-1}\,dsdr$$

and

$$w(0,\ 0;\ r,\ s)=\frac{1}{\omega_\mu\omega_\nu}\int_{\Omega_\mu}\int_{\Omega_\nu} u(r\xi,\ s\eta)\,d\Omega_\mu d\Omega_\nu.$$

Under the condition that u is differentiable $2n+1$ times, we can easily verify that $w(0,\ 0;\ r,\ s)$ can be expressed by

$$w(0,\ 0;\ r,\ s)=\sum_{p,q=0}^{n} C_{p,\,q}r^{2p}s^{2q}+O(r^p s^{2n+1-p}),$$

where $C_{p,\,q}$ is a constant independent of r, s and p, q are both non-negative integers. Moreover we have

$$w(0,\ 0;\ 0,\ 0)=f(0).$$

Putting $s=tr$, then $t^2=T$ in (6), we have

$$J^{\alpha}f(0)=\frac{\Gamma((2+\alpha-\mu)/2)\Gamma((\mu-\alpha)/2)}{2\pi^{(m-1)/2}\Gamma((2+\alpha-m)/2)\Gamma((1-\alpha)/2)}\frac{1}{\Gamma(\alpha)}$$

$$\cdot\int_0^{\infty}r^{\alpha-1}\,dr\int_0^1\left\{\sum_{p,q=0}^{n}C_{p,q}T^{p+q}+O(r^{2n+1}T^{(2n+1-p)/2})\right\}T^{\nu/2-1}(1-T)^{(\alpha-m)}\,dT.$$

Now we can suppose without loss of generality that $f(\mathrm{P})$ vanishes when $r_{\mathrm{OQ}}\geqq R$ (R being a sufficiently large number). Hence

$$J^{0}f(0)=\frac{\Gamma((2-\mu)/2)\Gamma(\mu/2)}{4\pi^{m/2}\Gamma((2-m)/2)\Gamma(1/2)}\lim_{\alpha\to0}\left\{\frac{1}{\Gamma(\alpha)}\int_0^{R}r^{\alpha-1}\,dr\int_0^1\left[\sum_{p,q=0}^{n}C_{p,q}T^{p+q}\right.\right.$$

$$\left.+O(r^{2n+1}T^{(2n+1-p)/2})\right]\times T^{\nu/2-1}(1-T)^{(\alpha-m)/2}\,dT$$

$$=\frac{\Gamma((2-\mu)/2)\Gamma(\mu/2)}{4\pi^{m/2}\Gamma((2-m)/2)}\frac{\Gamma(\nu/2)\Gamma((2-m)/2)}{\Gamma(\nu/2+(2-m)/2)}\left\{\lim_{\alpha\to0}\frac{1}{\Gamma(\alpha)}\int_0^{R}r^{\alpha-1}\,dr\right\}f(0)=f(0).$$

Hence we have $J^0f(0)=f(0)$.

Using the relation $\Delta J^{\alpha+2k}f(0)=J^{\alpha}f(0)$, and letting $\alpha\to2k$, we have $J^{-2k}f(0)=\Delta^{k}f(0)$. As to the analytic continuation of $J^{\alpha}f(\mathrm{P})$, we use the following lemma.

LEMMA. *Let $f(x)$ be continuous and let*

$$I^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x f(t)(x-t)^{\alpha-1}\,dt,$$

If $\alpha>0$, then $I^{\alpha}f(x)$ represents an analytic function of α. And if $f(x)$ is differentiable n times, then $I^{\alpha}f(x)$ can be continued as for as $\alpha>-n$ (n being a non-negative integer) and we have

$$I^0f(x)=f(x)\quad\textit{and}\quad I^{-k}f(x)=f^{(k)}(x)\qquad(k=0,1,2,\cdots,n-1).$$

This is the principle of analytic continuation by Hadamard (cf. Riesz [7] Chap. I).

Considering (6), by the lemma, $J^{\alpha}f(\mathrm{P})$ can be continued as far as $\alpha>-2k-1$, $f(\mathrm{P})$ must have continuous derivatives of order $[(m+1)/2]+2k$ ($k=0,1,2,\cdots$). Thus we have the following theorem.

THEOREM 5. *Let $f(\mathrm{P})$ be continuous and vanish rapidly at infinity within D^{P}, and $f(\mathrm{P})$ be continuously differentiable $[(m+1)/2]+2k$ times ($k=0,1,2,\cdots$). Then we can continue $J^{\alpha}f(\mathrm{P})$ as far as $\alpha>-2k-1$ and it represents the analytic function of α. And we have*

$$(7)\qquad J^{-2k}f(\mathrm{P})=\Delta^{k}f(\mathrm{P})\qquad\textit{and}\qquad J^{0}f(\mathrm{P})=f(\mathrm{P})=f*\Phi_0.$$

REMARK 1. The relation (7) shows that Φ_0 is the Dirac's distribution $\delta(\mathrm{P})$, i.e., $\Phi_0=\delta(\mathrm{P})$.

REMARK 2. If we put $\mu=\nu=m$ in $2m$-dimensional space, then (4) becomes

$$w(x, y;\ r, s)=\frac{1}{\omega_m^2}\int_{\Omega_m}\int_{\Omega_m} u(x+r\xi,\ y+s\eta)\, d\Omega_\mu d\Omega_\nu.$$

The last integral is the mean value over the unit spheres with respect to ξ and η respectively. If we put

$$\sigma(x, y;\ r)=w(x, y;\ r, 0) \qquad \text{and} \qquad \tau(x, y;\ r)=w(x, y;\ 0, r),$$

then both functions σ and τ satisfy the partial differential equation $\Delta_x\sigma=\Delta_y\tau$ with initial conditions $\sigma(x, y;\ 0)=u(x, y)$ and $\tau(x, y;\ 0)=0$. Further σ and τ may be found to be the continuous solution of Darboux's equation

$$\Delta_x\sigma-\frac{m-1}{r}\sigma_r-\sigma_{rr}=0;$$

Cf. [1] pp. 411–412.

7. $J^\alpha_{D^{\rm P}}$, $J^\alpha_{D^{\rm P}_S}$ and Green's formula.

Let the surface S be $S(\xi_1, \xi_2, \cdots, \xi_m)=0$, and suppose that it can be differentiable any times for a while. Let $D^{\rm P}_S$ denote the region bounded by the surface of $D^{\rm P}$ which contains the vertex P and the surface S, and let $S^{\rm P}$ be the part of S which is cut by S and which lies within $D^{\rm P}$. Moreover let $C^{\rm P}$ denote the part of the surface $D^{\rm P}$ which is cut by S and which contains P. Then if u and v are continuously differentiable twice, we can apply the Green's theorem upon $D^{\rm P}_S$ and it follows

$$\int_{D^{\rm P}_S}(u\Delta v-v\Delta u)\, d{\rm Q}=-\int_{S^{\rm P}+C^{\rm P}}\left(u\frac{\partial v}{\partial n}-v\frac{\partial u}{\partial n}\right)dS, \tag{1}$$

where n denote the inner normal direction.

Now let $v=r_{\rm PQ}{}^{\alpha+2-m}$. Since $r_{\rm PQ}=0$ on $C^{\rm P}$ and $\Delta\Phi_{\alpha+2}=\Phi_\alpha$, (1) becomes

$$\begin{aligned}\frac{1}{K_m(\alpha)}\int_{D^{\rm P}_S} u({\rm Q})r_{\rm PQ}{}^{\alpha-m}\, d{\rm Q}&=\frac{1}{K_m(\alpha)}\int_{D^{\rm P}_S}\Delta u({\rm Q})r_{\rm PQ}{}^{\alpha+2-m}\, d{\rm Q}\\ &+\frac{1}{K_m(\alpha)}\int_{S^{\rm P}}\left\{\frac{\partial u({\rm Q})}{\partial n}r_{\rm PQ}{}^{\alpha+2-m}-u({\rm Q})\frac{\partial r_{\rm PQ}{}^{\alpha+2-m}}{\partial n}\right\}ds \quad (\alpha>m).\end{aligned} \tag{2}$$

We put

$$J^\alpha_{D^{\rm P}}f({\rm P})=\frac{1}{K_m(\alpha)}\int_{D^{\rm P}_S} f({\rm Q})r_{\rm PQ}{}^{\alpha-m}\, d{\rm Q}=(f*\Phi_\alpha)({\rm P})$$

and

$$J^\alpha_{D^{\rm P}_S}f({\rm P})=\frac{1}{K_m(\alpha)}\int_{D^{\rm P}_S} f({\rm Q})r_{\rm PQ}{}^{\alpha-m}\, d{\rm Q}=(f*\bar{\Phi}_\alpha)({\rm P}).$$

Suppose that $f(\mathrm{Q})$ vanishes outside D_S^{P}, then we easily see that the following relations hold:

$$(3) \qquad (f*\bar{\Phi}_\alpha)*\bar{\Phi}_\beta=(f*\Phi_\alpha)*\Phi_\beta=f*\Phi_{\alpha+\beta}=f*\bar{\Phi}_{\alpha+\beta},$$

and

$$(4) \qquad \Delta(f*\bar{\Phi}_{\alpha+2})=\Delta(f*\Phi_{\alpha+2})=(f*\Phi_\alpha)=f*\bar{\Phi}_\alpha.$$

Also we use the expressions by Riesz in order to write (2) simply,

$$J_*^\alpha\overline{f,g,h}(\mathrm{P})=\frac{1}{K_m(\alpha)}\int_{D_S^{\mathrm{P}}}f(\mathrm{Q})r_{\mathrm{PQ}}{}^{\alpha-m}\,d\mathrm{Q}+\frac{1}{K_m(\alpha)}\int_{S^{\mathrm{P}}}\left(g(\mathrm{Q})r_{\mathrm{PQ}}{}^{\alpha-m}-h(\mathrm{Q})\frac{\partial r_{\mathrm{PQ}}{}^{\alpha-m}}{\partial n}\right)dS.$$

Then (2) can be written

$$J_{D_S^{\mathrm{P}}}^\alpha f(\mathrm{P})=J_*^{\alpha+2}\overline{\Delta u,\ \frac{\partial u}{\partial n},\ u(\mathrm{P})}.$$

From (3) and (4) we have at once

$$J_*^\alpha(J_*^\beta)=J_*^{\alpha+\beta} \qquad \text{and} \qquad \Delta J_*^{\alpha+2}=J_*^\alpha.$$

8. J_*^α.

In this section we shall use the same notations as in §7, and we shall now investigate the behaviour of J_*^α. Now let us consider the "hyperboloid"

$$H\colon (\boldsymbol{Z},\boldsymbol{Z})=1 \qquad (z_1<0).$$

Putting

$$0\leq t\leq 1, \qquad \sum_{i=1}^{\mu}\varphi_j^2=1, \qquad \sum_{j=\mu+1}^{m}\varphi_j^2=1$$

and

$$z_1=-\frac{t^{-1}+t}{2}\varphi_1\ (\varphi_1>0), \qquad z_i=-\frac{t^{-1}+t}{2}\varphi_i\ (i=2,3,\cdots,\mu)$$

and $$z_j=-\frac{t^{-1}-t}{2}\varphi_j \qquad (j=\mu+1,\cdots,m),$$

the hyperboloid H may be represented by the parameter t. Let the surface S be

$$S(\boldsymbol{a})=S(a_1,a_2,\cdots,a_m)=0.$$

We project the point $\boldsymbol{a}$ of S upon the surface $(\boldsymbol{z}-\boldsymbol{x})^2=1$, by the segment which connects P and Q. Also we put $\boldsymbol{a}-\boldsymbol{x}=r_a\boldsymbol{z}$ i.e. $\boldsymbol{a}-\boldsymbol{x}=r\boldsymbol{z}$ for simplicity. Now we shall represent $J_{D_S^{\mathrm{P}}}^\alpha f(\mathrm{P})$by the integral upon $(\boldsymbol{z}-\boldsymbol{x})^2=1$. In fact, since

$$(1) \qquad d\mathrm{Q}=r_{\mathrm{Q}}^{m-1}\,dH dr_{\mathrm{Q}} \qquad (r_{\mathrm{Q}}=r_{\mathrm{PQ}})$$

where dH denote the surface element of the surface H, and further since $\xi-\boldsymbol{x}=\sigma(\boldsymbol{a}-\boldsymbol{x})$ $(0\leqq\sigma\leqq 1)$ i.e, $\xi=(1-\sigma)\boldsymbol{x}+\sigma\boldsymbol{a}$, (1) becomes $dQ=r^m dH\sigma^{m-1}d\sigma$. Hence we have

$$J_{D_S^P}^{\alpha} f(\mathrm{P})=\frac{1}{K_m(\alpha)}\int_H F(\boldsymbol{a},\alpha)r^\alpha\, dH \tag{2}$$

where

$$F(\boldsymbol{a},\alpha)=\int_0^1 f(\boldsymbol{a},\sigma)\sigma^{\alpha-1}\,d\sigma.$$

Next putting $z_1^2+z_2^2+\cdots+z_\mu^2=\cosh^2\theta$, $z_{\mu+1}^2+\cdots+z_m^2=\sinh^2\theta$, it becomes

$$dH=\sin^{m-\mu-1}\theta\,\cos^{\mu-1}\theta\, d\Omega_{m-\mu}d\Omega_\mu d\theta. \tag{3}$$

Now we can proceed by the way of Riesz as follows. Putting $T=e^{-2\theta}$, we have from (3)

$$r^\alpha dH=2^{1-m}v^\alpha T^{(\alpha-m)/2}(1-T)^{m-\mu-1}(1+T)^{\mu-1}d\Omega_{m-\mu}d\Omega_\mu dT=k(T)v^\alpha T^{(\alpha-m)/2}d\Omega_{m-\mu}d\Omega_\mu dT,$$

where

$$k(T)=2^{1-m}(1-T)^{m-\mu-1}(1+T)^{\mu-1}\qquad\text{and}\qquad v=\frac{r_a}{t}=\frac{r}{t}.$$

Then (2) becomes

$$J_{D_S^P}^{\alpha} f(\mathrm{P})=\frac{1}{K_m(\alpha)}\int_0^1 K(T,\alpha)T^{(\alpha-m)/2}\,dT, \tag{4}$$

where

$$K(T,\alpha)=k(T)\int_{\Omega_\mu}\int_{\Omega_\nu} v^\alpha F(\boldsymbol{a},\alpha)\,d\Omega_\mu d\Omega_\nu \qquad (m=\mu+\nu).$$

Next since

$$dS=\left|\frac{dr_a}{dn}\right| r_a^{m-1}\,dH=|M|^{-1}Nr_a^m\,dH$$

where

$$M=(\operatorname{grad} S,\ \boldsymbol{a}-\boldsymbol{x}),\quad N=(\operatorname{grad} S,\ \operatorname{grad} S)^{1/2}\ \text{ and }\ r_a^{m-1}\left|\frac{dr_a}{dn}\right| dS=dH,$$

we have similar formulas for $J_*^\alpha\overline{0,g,0}(\mathrm{P})$ and $J_*^\alpha\overline{0,0,h}(\mathrm{P})$ as that of $J_*^\alpha\overline{f,0,0}(\mathrm{P})$ respectively. Thus we have

$$J_*^\alpha\overline{0,g,0}(\mathrm{P})=\frac{1}{K_m(\alpha)}\int_0^1 T^{(\alpha-m)/2}K(T,\alpha)\,dT, \tag{5}$$

with

$$K(T,\alpha)=k(T)\int_{\Omega_\mu}\int_{\Omega_\nu}|M|^{-1}Nv^\alpha g(\boldsymbol{a})\,d\Omega_\mu d\Omega_\nu,$$

and

$$J_*^\alpha\overline{0,\,0,\,h}(\mathrm{P})=\frac{1}{K_m(\alpha)}\int_0^1 K(T,\,\alpha)T^{(\alpha-m-2)/2}\,dT, \tag{6}$$

with

$$K(T,\,\alpha)=(\alpha-m)k(T)\int_{\Omega_\mu}\int_{\Omega_\nu} v^{\alpha-2}h(\boldsymbol{a})\,d\Omega_\mu d\Omega_\nu.$$

Therefore we may write (4), (5) and (6) in the form

$$J^\alpha=\frac{1}{K_m(\alpha)}\int_0^1 T^{\beta-1}K(T,\,\alpha)\,dT, \tag{7}$$

where β is equal respectively to $(\alpha+2-m)/2$, $(\alpha+2-m)/2$ and $(\alpha-m)/2$. These formulas (4), (5) and (6) are the same as those of Riesz [7], p. 58.

9. Analytic continuation of J^α.

Now we can easily show that the existence of the continuous derivatives with respect to T and x_j of a_k and v of the last section, depends on the continuous differentiability of the surface S. Also the existence of the continuous derivatives of $K(T,\,\alpha)$ depends on those of the functions f, g, h and the surface S.

As already stated, the formula (7) in § 8 is the same form as that of Riesz, so that (4) in § 8 can be written

$$J^\alpha=\frac{\Gamma^2(\alpha+2+\beta)}{\pi^{(m-3)/2}\sin\{(\mu-\alpha)/2\}\Gamma((1-\alpha)/2)\Gamma(\alpha)\Gamma(\beta)}\int_0^1 T^{\beta-1}K(\mathrm{P},\,\alpha)\,d\mathrm{P}$$

where $\beta=\alpha+2-m$.

Using the principle of continuation, we can conclude that $f(\mathrm{P})$ and $S(\boldsymbol{a})$ have both continuous derivatives of order $[(m+1)/2]$, and J^α can be continued as far as $\alpha>-1$. Further if f and S have both the continuous derivatives of order $[(m+1)]+3p$, then J^α can be continued as fai as $\alpha>-2p-1$ $(p=1,\,2,\,\cdots)$. As to (5) and (6) in § 8, we can conclude in the same way as above and we have the following theorem.

THEOREM 6. 1°. *If the function f and S be both continuously differentiable $[(m+1)/2]+3p$ times, then J^α can be continued as far as $\alpha>-2p-1$ and we have*

$$J_*^0\overline{f,\,0,\,0}(\mathrm{P})=2^{2-m}\frac{\Gamma((2-\mu)/2)\Gamma(m-\mu)}{\Gamma((m-\mu)/2)\Gamma((2-m)/2)}\left\{\sum_{j=0}^{\mu-1}\binom{\mu-1}{j}\frac{\Gamma((2-m+2j)/2)}{\Gamma((2+m-2\mu+2j)/2)}\right\}f(\mathrm{P}). \tag{1}$$

2°. *If g and S be continuously differentiable $[(m+1)/2]+p$ and $[(m+3)/2]+p$ times respectively, then J^α can be continued as far as $\alpha>-2p-1$ and we have*

$$J^{-2p}=J_*^{-2p}\overline{0,\,g,\,0}(\mathrm{P})=0 \qquad (p=0,\,1,\,2,\,\cdots).$$

3°. *If the function h and S be continuously differentiable $[(m+3)/2]+p$ and*

$[(m+1)/2]+p$ times respectively, then J^{α} can be continued as far as $\alpha>-2p-1$ and we have

$$J^{-2p}=J_{*}^{-2p}\overline{0,\ 0.\ h}(\mathrm{P})=0 \qquad (p=0,1,2,\cdots).$$

REMARK 1. If we put $\mu=1$, then (1) becomes $J_{*}^{0}\overline{f,\ 0,\ 0}(\mathrm{P})=f(\mathrm{P})$, but otherwise we cannot show the relation $J_{*}^{0}\overline{f,\ 0,\ 0}(\mathrm{P})=f(\mathrm{P})$.

REMARK 2. In our present case, we take the range of integration D^{P} or D_{S}^{P} and further we treat the values of function on the surface S, so that our theory of Riemann-Liouville integral is somewhat different from the work of Gelfand and Shilov [4].

I0. Applications

Abelian integral equations. In this section, using the Riemann-Liouville integral, we shall investigate the integral equation of Abelian type.

Now we consider the Abelian integral equation

$$\text{(1)} \qquad \int_{\Omega} f(\mathrm{Q})r_{\mathrm{PQ}}{}^{\alpha-m}\,d\mathrm{Q}=\psi(\mathrm{P}) \qquad m>\alpha>m-2,$$

where Ω denote the m-dimensional Euclidean space and $\psi(\mathrm{P})$ is a given continuous function. We are now to obtain the continuous solution $f(\mathrm{P})$, Next S^{*} denote the closed unit sphere of Ω. We put

$$\text{(II)} \qquad \int_{S^{*}} f(\mathrm{Q})r_{\mathrm{PQ}}{}^{\alpha-m}d\mathrm{Q}=\psi(\mathrm{P}).$$

First, to reduce (II) to (I), we use the Poisson kernel with respect to S^{*} (cf. Riesz [6]):

$$\lambda_{\mathrm{P}}(\mathrm{Q})=\pi^{-(m/2+1)}\Gamma(m/2)\sin\frac{\alpha\pi}{2}\,|1-r_{\mathrm{P}}{}^{2}|^{\alpha/2}|1-r_{\mathrm{Q}}{}^{2}|^{-\alpha/2}r_{\mathrm{PQ}}{}^{\alpha-m}$$

where we mean $r_{\mathrm{P}}=r_{\mathrm{OP}}$ etc. We can extend the function $\psi(\mathrm{P})$ continuously in the whole space as follows. We put

$$\text{(1)} \qquad \bar{\psi}(\mathrm{P})=\int_{S^{*}}\psi(\mathrm{Q})\lambda_{\mathrm{P}}(\mathrm{Q})\,d\mathrm{Q}.$$

Then if $\mathrm{P}\in S^{*}$, we know that $\bar{\psi}(\mathrm{P})=\psi(\mathrm{P})$ by the property of $\lambda_{\mathrm{P}}(\mathrm{Q})$. Therefore

$$\Psi(\mathrm{P})=\begin{cases}\psi(\mathrm{P}) & \mathrm{P}\in S^{*},\\ \bar{\psi}(\mathrm{P}) & \mathrm{P}\notin S^{*}.\end{cases}$$

Then $\Psi(\mathrm{P})$ becomes the continuous extension of $\psi(\mathrm{P})$ in the whole space. But in order to verify the convergence of the integral (1), let $v(\mathrm{P})$ be potential of the positive mass distribution on S^* and let c be any positive constant. It is sufficient to suppose that $|\psi(\mathrm{P})| \leqq v(\mathrm{P})+c$. Thus we can reduce the equation (II) to (I), that is, we have only to solve the equation of the type

$$\int_{\Omega} f(\mathrm{Q}) r_{\mathrm{PQ}}{}^{\alpha-m}\, d\mathrm{Q}=\Psi(\mathrm{P}).$$

The following method of our argument may apply not only in Ω but also in the generalized Lorentzian space. In the last case it is sufficient to replace D^{P} for Ω and

$$\Delta J^{\alpha+2} f(\mathrm{P})=J^{\alpha} f(\mathrm{P}) \quad \text{for} \quad \Delta I^{\alpha+2} f(\mathrm{P})=-I^{\alpha} f(\mathrm{P}).$$

Now let us consider the equation (I) which can be written

$$(I') \qquad I^{\alpha} f(\mathrm{P})=\psi(\mathrm{P}) \frac{1}{H_m(\alpha)} \quad \text{or} \quad (f * \Phi_{\alpha})(\mathrm{P})=\frac{1}{H_m(\alpha)} \psi(\mathrm{P}),$$

where $\Phi_{\alpha}(\mathrm{P}, \mathrm{Q})=r_{\mathrm{PQ}}{}^{\alpha-m}/H_m(\alpha)$. Hence it becomes

$$(f * \Phi_{\alpha+\beta})(\mathrm{M})=\frac{1}{H_m(\alpha)}(\psi * \Phi_{\beta})(\mathrm{M}).$$

By operating the Laplacian operator, we have

$$(2) \qquad (f * \Phi_{\alpha+\beta-2})(\mathrm{M})=-\frac{1}{H_m(\alpha)} \Delta_{\mathrm{M}}(\psi * \Phi_{\beta})(\mathrm{M})=\frac{1}{H_m(\alpha)}(\psi * \Phi_{\beta-2})(\mathrm{M}).$$

Since $f * \Phi_0=f$, by letting $\beta \rightarrow 2-\alpha$, we have

$$(3) \qquad f(\mathrm{M})=\frac{1}{H_m(\alpha)}(\psi * \Phi_{\alpha})(\mathrm{M}),$$

where $H_m(\alpha)=\pi^{m/2} 2^{\alpha} \Gamma(\alpha/2)/\Gamma((m-\alpha)/2)$.

In general, when $\psi(\mathrm{P})$ can be differentiable $2p$ ($p=0, 1, 2, \cdots$) times I^{α} can be continued to I^{-2p}. Then the integral of the right hand member of (2) or (3) converges absolutely, and therefore the function $f(\mathrm{M})$ may represent the solution of (I′), and then the integral of (3) converges.

11. Potential in ultra-hyperbolic space.

We put

$$U^{(\alpha)}(\mathrm{P})=\frac{1}{K_m(\alpha)} \int_{D^{\mathrm{P}}} r_{\mathrm{PQ}}{}^{\alpha-m} d\mu(\mathrm{Q})=(f * \Phi_{\alpha})(\mathrm{P}), \qquad d\mu(\mathrm{Q})=f(\mathrm{Q}) d\mathrm{Q}.$$

We call the function $U^{(\alpha)}(\mathrm{P})$, α-dimensional potential in the ultra-hyperbolic space. Concerning this potential, we have as a usual potential,

$$(U^{(\alpha)}*\Phi_\beta)(\mathrm{P})=U^{(\alpha+\beta)}(\mathrm{P}), \qquad \Delta U^{(\alpha+2)}(\mathrm{P})=U^{(\alpha)}(\mathrm{P}),$$

that is, in this case the composition theorem and Poisson's relation are valid.

Further if we put

$$V^{(\beta)}(\mathrm{P})=\frac{1}{K_m(\beta)}\int_{D^{\mathrm{P}}} r_{\mathrm{PQ}}{}^{\beta-m}d\nu(\mathrm{Q}), \qquad d\nu(\mathrm{Q})=g(\mathrm{Q})\,d\mathrm{Q},$$

then we have

$$I(\mu,\nu)=\int_{D^{\mathrm{R}}} U^{(\alpha)}(\mathrm{T})V^{(\beta)}(\mathrm{T})\,d\mathrm{T}=\int_{D^{\mathrm{R}}} d\mathrm{T}\left\{\frac{1}{K_m(\alpha)}\int_{D^{\mathrm{P}}} r_{\mathrm{PT}}{}^{\alpha-m}\,d\mu(\mathrm{P})\right\}$$

$$(1) \qquad \left\{\frac{1}{K_m(\beta)}\int_{D^{\mathrm{Q}}} r_{\mathrm{QT}}{}^{\beta-m}\,d\nu(\mathrm{Q})\right\}$$

$$=\frac{1}{K_m(\alpha+\beta)}\int_{D^{\mathrm{R}}} r_{\mathrm{PQ}}{}^{\alpha+\beta-m}\,d\mu(\mathrm{P})d\nu(\mathrm{Q}).$$

Therefore we have

$$(2) \qquad \int_{D^{\mathrm{R}}} r_{\mathrm{PQ}}{}^{\alpha+\beta-m}\,d\nu(\mathrm{P})d\nu(\mathrm{Q})=K_m(\alpha+\beta)\int_{D^{\mathrm{R}}} U^{(\alpha)}(\mathrm{T})V^{(\beta)}(\mathrm{T})\,d\mathrm{T}.$$

From (2) we obtain

$$(3) \qquad \int_{D^{\mathrm{R}}} U^{(\alpha)}\,d\nu(\mathrm{P})=\int_{D^{\mathrm{R}}} V^{(\beta)}\,d\mu(\mathrm{P}),$$

that is the reciprocal formula is valid here also.

The formulas (1) and (3) are useful for a variation method. Further if we put $\alpha=\beta=\alpha/2$, $\mu=\nu$ in (1), then we have

$$(4) \qquad I(\mu,\mu)=\int_{D^{\mathrm{R}}} r_{\mathrm{PQ}}{}^{\alpha-m}\,d\mu(\mathrm{P})\,d\mu(\mathrm{Q})=K_m(\alpha)\int_{D^{\mathrm{R}}} [U^{(\alpha/2)}(\mathrm{T})]^2\,d\mathrm{T} \quad (\alpha>0).$$

(4) corresponds to the energy integral in the usual potential theory.

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