# RIGIDITY OF PROJECTION MAP AND THE GROWTH OF ANALYTIC FUNCTIONS

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**1.** In our previous paper [3] we determined the exceptional class of Picard's theorem on some Riemann surfaces with conformal automorphisms by the aid of the existence of the fundamental functions and the rigidity of projection map which was made use of in a non-emphasized form. We shall here explain it explicitly in the following form. (We make use of the same notations as in [3].)

Let $W$ belong to $\mathfrak{S}_2 \frown P_{MD}$. Let $f(p)$ be a single-valued meromorphic function on an end $\Omega$ of $W$ satisfying $T(\sigma)=o(e^{2\sigma})$, then we have the representation

$$f(p) = F\circ F_0(p)$$

for a single-valued meromorphic function $F(w)$ in the punctured disc $\sigma_0 < \log|w| < \infty$ satisfying the condition $T_P(\sigma, F)=o(e^{2\sigma})$.

This fact says that $f(p)$ preserves the projection map $F_0$: $W \rightarrow \mathfrak{W}^*$, that is, $f(p_1)=f(p_2)$ if $F_0(p_1)=F_0(p_2)$, when $f(p)$ satisfies the desired growth condition. Such a phenomenon was studied non-systematically by the various authors. Excepting the closed surface case, the first one who explained the phenomenon is Selberg [5]. However his ramification theorem in his celebrated theory [4], that is,

$$N(r;\ \mathfrak{X}) < (2k-2)T(r;\ f)+O(1),$$

can be considered as a representation of the rigidity of projection map in terms of the growth condition. Myrberg [2] constructed a striking example which shows that the analytic theory meets a deep difficulty in the first step. In [1] Heins constructed an interesting example by using the phenomenon and proved an elegant composition theorem, which can be considered as an analytic representation of the rigidity of projection map by any bounded analytic function on an end. Heins' proof of the theorem was based upon the Schur algorithm. In [3] we made use of the Heins composition theorem in order to prove the existence of the fundamental functions.

In the present paper we shall give another proof of Heins' composition theorem. Further we shall give some examples, which show how the rigidity of projection map is important and effective.

**2.** *Heins' composition theorem*: Let $W$ be an open Riemann surface of class $O_G$ with the only one ideal boundary component. Let $f(p)$ be a non-constant bounded

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regular function on an end $\Omega$ of $W$. Then there exist a non-constant bounded regular function $f_0(p)$ in a suitable subend $\Omega_0$ and a suitable bounded regular function $\varphi(z)$ in $|z|<1$ for which $f(p)=\varphi\circ f_0(p)$ in $\Omega_0$.

As a logical foundation we should assume that the existence of $f_0(p)$ is guaranteed by the minimality of the local degree in Heins' sense. Let $n$ be the minimum local degree of the ideal boundary. Then $F(z)=f\circ f_0^{-1}(z)$ is at most $n$-valued and bounded in $0<|z|<1$. Let $F_j(z)$ be the $j$ th branch of $F(z)$. Then any fundamental symmetric polynomials of $\{F_j\}$ are single-valued in $0<|z|<1$ and hence $F(z)$ satisfies an algebraic equation

$$F^n+A_1(z)F^{n-1}+\cdots+A_n(z)=0$$

with the single-valued bounded regular coefficients $A_j$ in $0<|z|<1$ and hence in $|z|<1$. Thus $F$ should be reduced to an algebraic function over the unit disc $|z|<1$. Therefore $F_j-F_k$ is an algebraic function in $|z|<1$ having the origin as a cluster point of an infinite number of zeros, when the $j$ th and the $k$ th sheets have an infinite number of common branch points. This shows that $F_j\equiv F_k$ in the unit disc. By the minimality of the local degree for any fixed sheet we can find the second sheet in such a manner that two sheets have an infinite number of common branch points. Therefore we can repeat the above discussion and then we have the single-valuedness of $F$. Further $F$ is regular at the origin. Thus we have the desired representation: $f(p)=F\circ f_0(p)$.

3. We shall construct an interesting example of a Riemann surface. Let $L_{m,n}$ be the segment connecting two points $2m+ni$ and $2m+1+ni$ and $L_n$ be the segment connecting two points $2ni$ and $(2n+1)i$, where $m$ runs through all the non-zero integers and $n$ runs through all the integers. Let $E$ be the finite plane with an infinite number of slits $L_{m,n}$ and $L_n$. Let $E_1$, $E_2$, $E_3$ and $E_4$ be four copies of $E$. $E_1$, $E_2$, $E_3$, $E_4$ are connected along the corresponding slits $L_{m,n}$ and $L_n$ by the standard identification process, which is shown by the following schema:

$$\begin{tikzcd} E_1 \arrow[r, leftrightarrow, "L_{m,n}"] \arrow[d, leftrightarrow, "L_n"'] & E_2 \arrow[d, leftrightarrow, "L_n"] \\ E_3 \arrow[r, leftrightarrow, "L_{m,n}"'] & E_4 \end{tikzcd}$$

The resulting surface $W$ is a four-sheeted covering surface over the punctured sphere $|z|<\infty$. On $W$ there are two involutory conformal automorphisms $J$ and $K$. $J$ permutes $E_1$, $E_3$ to $E_2$, $E_4$, simultaneously and respectively. $K$ permutes $E_1$, $E_2$ to $E_3$, $E_4$, simultaneously and respectively.

Let $f(p)$ be a single-valued regular function on $W$, then $f(p)$ can be considered as a function of $z$. Indeed, let $R(p)$ be the projection map: $W\to\{|z|<\infty\}$, then

$F(z)=f\circ R^{-1}(z)$ is such a function. Then $F(z)$ is at most four-valued in $|z|<\infty$. Let $F_j$ be the $j$ th branch of $F$. Then any fundamental symmetric polynomials of $\{F_j\}$ are single-valued regular in $|z|<\infty$ and hence $F$ is a four-valued algebroid function in $|z|<\infty$.

In the first place we assume that the order $\rho_F$ of $F$ is less than 2. Then $G\equiv F_1-F_2$ satisfies

$$T(r, G)=2T(r, F)+O(1)=o(r^2)$$

and

$$T(r, G)=T(r, 1/G)+O(1)\geqq N(r, 1/G)\geqq 2\pi r^2.$$

This is a contradiction. Thus $F_1\equiv F_2$. By $K(F_1-F_2)=F_3-F_4$ we have $F_3\equiv F_4$. Thus the proper domain of existence of $F$ is not the original $W$ but a new surface $W$ mod $J$. Therefore $F$ must reduce to a two-valued function in $|z|<\infty$.

In the second place we assume that the order $\rho_F$ of $F$ is less than 1. Then we can prove quite similarly that the proper domain of existence of $F$ is not $W$ mod $J$ but ($W$ mod $J$) mod $K$, that is, the finite plane.

For this surface the number $P$ of Picard's exceptional values satisfies

$$P\leqq\begin{cases} 2 & \text{if} \quad \rho_F>2, \\ 8 & \text{if} \quad \rho_F=2, \\ 2 & \text{if} \quad 2>\rho_F>1, \\ 4 & \text{if} \quad \rho_F=1, \\ 1 & \text{if} \quad 1>\rho_F, \end{cases}$$

with the exception of the rational functions. This result is proved by means of the Selberg theory and a classical theorem.

**4.** We shall construct a Riemann surface on which the Denjoy-Carleman-Ahlfors theorem on the number $D$ of finite asymptotic paths of an integral function of finite order remains true fully or restrictively. Let $W$ be a two-sheeted covering surface over the finite plane. In the first place we assume that the integrated Euler characteristic $N(r, W)$ satisfies

$$\varlimsup_{r\to\infty}\frac{\log N(r, W)}{\log r}=+\infty.$$

Such a surface can be easily constructed. Let $f(p)$ be a regular function on $W$. Let $F(z)$ be $f\circ R^{-1}(z)$, where $R$ is the projection map: $W\to\{|z|<\infty\}$. Let $f(p)$ be of finite order in the sense of the general value distribution theory. Then $F$ is also of finite order in the algebroid sense. Thus $F$ should satisfy the ramification theorem due to Selberg

$$N(r, W)\leqq 2T(r, F)+O(1),$$

when $F$ is two-valued in $|z|<\infty$. This contradicts the finiteness of order $\rho_F$. Thus $F$ must satisfy the single-valuedness in $|z|<\infty$. Then by the classical Denjoy-Carleman-Ahlfors theorem we have

$$D \leqq 2[2\rho_F],$$

if $\rho_F \geqq 1/2$. On this surface $P \leqq 4$ in general by the Selberg theory. However $P \leqq 2$ for any meromorphic function on $W$ satisfying $\rho_F < \infty$. Further we can impose some conditions in such a manner that $P \leqq 2$ for any meromorphic function on $W$. This is due to a special method based on the Schottky theorem.

Next we assume that

$$\varlimsup_{r\to\infty} \frac{\log N(r, W)}{\log r} = \alpha < \infty.$$

This is evidently allowable. Then we have similarly

$$D \leqq 2[2\rho_F],$$

if $1/2 \leqq \rho_F < \alpha$.

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