ON SOME ASYMPTOTIC EXPANSION THEOREMS

By Tatsuo Aonuma

§ 1. We shall give the asymptotic expansion theorems of intergrals of the type

$$\int f\left(x+\frac{\xi}{n}\right)K(\xi)\,d\xi$$

as extensions of the well-known summability theorem (§ 2) and convergence theorems (§ 3) in this paper.

Before proceeding them, we state the following summability theorem and convergence theorems which we are concerned with.

THEOREM A. (Bochner and Chandrasekharan [2]). *If* $K(u)\in L_1(0,\infty)$, $K(u)=o(u^{-1})$ *as* $u\to\infty$, $f(u)\in L_1(0,\infty)$ *and* $K(u)$ *is monotone decreasing in* $0\leqq u<\infty$, *then the condition at a point* x

$$\int_0^h |f(x+\xi)-f(x)|\,d\xi=o(h) \quad as \quad h\to+0$$

implies

$$\lim_{n\to\infty}\int_0^\infty f\left(x+\frac{\xi}{n}\right)K(\xi)\,d\xi=f(x)\int_0^\infty K(\xi)\,d\xi.$$

THEOREM B. (Bochner and Izumi [1]). *If*

$$\int_{-\infty}^\infty \frac{|f(u)|^p}{1+|u|}\,du<\infty \qquad for\ p>1,$$

$K(u)\in L_1(-\infty,\infty)$, $u^{q-1}K(u)^q\in L_1(-\infty,\infty)$ *where* $1/p+1/q=1$, *and* $f(u)$ *is continuous at a point* x, *then we have at a point* x

$$\lim_{n\to\infty}\int_{-\infty}^\infty f\left(x+\frac{\xi}{n}\right)K(\xi)\,d\xi=f(x)\int_{-\infty}^\infty K(\xi)\,d\xi.$$

THEOREM C. *If*

$$\int_{-\infty}^\infty \frac{|f(u)|}{1+|u|}\,du<\infty,$$

$K(u)\in L_1(-\infty,\infty)$, $|uK(u)|<B<\infty$, *and* $f(u)$ *is continuous at a point* x, *then we have the same result as Theorem* B.

§ 2. In this section an asymptotic expansion theorem corresponding to Theorem A as a summability theorem is considered.

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THEOREM 1. *If*

(1) $f(u) \in L_1$,

(2) $(1+u^{r+\alpha})K(u) \in L_1(0, \infty)$, *where* r *is a non-negative integer and* α *is a constant such as* $0 \leqq \alpha < 1$, *and*

(3) $K(u)$ *is monotone decreasing in* $0 \leqq u < \infty$,

then the condition at a point x

$$\int_0^h |f(x+u) - \sum_{k=0}^{r} c_k u^k| \, du = o(h^{r+1+\alpha}) \text{ as } h \to 0, \tag{2.1}$$

where c_i $(i=0, \cdots, r)$ *are some constants, implies*

$$\int_0^\infty f\left(x+\frac{\xi}{n}\right)K(\xi)d\xi = \sum_{k=0}^{r} \frac{c_k}{n^k}\int_0^\infty \xi^k K(\xi)d\xi + o\left(\frac{1}{n^{r+\alpha}}\right) \quad (n\to\infty).$$

In particular, if $f(u)$ *is differentiable* r *times in a neighbourhood of the point* x *and* $f^{(r)}(u)$ *is continuous at the point* x, c_k *is equal to* $f^{(k)}(x)/k!$ *and* (2.1) *is always satisfied.*

Proof. At first, it would be noted that

$$x^{r+1+\alpha}K(x) = o(1) \quad \text{as } x\to\infty \tag{2.2}$$

follows from the assumptions (2) and (3). For, since we have $K(x) \geqq 0$ in $0 \leqq x < \infty$ from the assumption (3) and $K(x) \in L_1(0, \infty)$ in (2),

$$\int_x^{2x} \xi^{r+\alpha}K(\xi)\,d\xi \geqq K(x)\int_x^{2x}\xi^{r+\alpha}\,d\xi = K(x)\,x^{r+1+\alpha}\int_1^2 t^{r+\alpha}\,dt$$

is obtained for $x>0$. So we have

$$x^{r+1+\alpha}K(x) \leqq (r+\alpha+1)/(2^{r+\alpha+1}-1)\int_x^{2x}\xi^{r+\alpha}K(\xi) = o(1) \text{ as } x\to\infty$$

by the assumption (2).

Now to prove our theorem it is sufficient to show that

$$I = n^{r+\alpha}\int_0^\infty \left\{ f\left(x+\frac{\xi}{n}\right) - \sum_{k=0}^{r} c_k \frac{\xi^k}{n^k}\right\}K(\xi)\,d\xi$$

tends to zero as $n\to\infty$. If we put $u=\xi/n$,

$$I = n^{r+1+\alpha}\int_0^\infty \left\{ f(x+u) - \sum_{k=0}^{r} c_k u^k\right\}K(nu)\,du$$

$$= n^{r+1+\alpha}\int_0^t + n^{r+1+\alpha}\int_t^\infty = I_1 + I_2.$$

Putting

$$G(h) = \int_0^h \left| f(x+u) - \sum_{k=0}^{r} c_k u^k \right| du,$$

we have for sufficiently small $t>0$

$$
\begin{aligned}
|I_1| &\leqq \int_0^t n^{r+1+\alpha} K(nu)\, dG(u) \\
&= n^{r+1+\alpha} K(nt)\, G(t) - n^{r+1+\alpha} \int_0^t G(u)\, d_u K(nu) \\
&= (nt)^{r+1+\alpha} K(nt)\, G(t) \frac{1}{t^{r+1+\alpha}} - \varepsilon \int_0^t (nu)^{r+1+\alpha}\, d_u K(nu) \\
&= (nt)^{r+1+\alpha} K(nt)\, G(t) \frac{1}{t^{r+1+\alpha}} \\
&\quad - \varepsilon \left\{ (nt)^{r+1+\alpha} K(nt) - (r+1+\alpha) \int_0^t (nu)^{r+\alpha}\, nK(nu)\, du \right\} \\
&\leqq (nt)^{r+1+\alpha} K(nt)\, G(t) \frac{1}{t^{r+1+\alpha}} \\
&\quad + \varepsilon (r+1+\alpha) \int_0^\infty (nu)^{r+\alpha}\, nK(nu)\, du,
\end{aligned}
$$

where ε may be an arbitrary small positive number, and so

$$
\overline{\lim_{n\to\infty}} |I_1| \leqq \varepsilon (r+1+\alpha) \int_0^\infty \xi^{r+\alpha} K(\xi) d\xi \tag{2.3}
$$

by (2) and (2.2). On the other hand,

$$
\begin{aligned}
I_2 &= n^{r+1+\alpha} \int_t^\infty f(x+u)\, K(nu)\, du - n^{r+1+\alpha} \int_t^\infty \left(\sum_{k=0}^{r} c_k u^k \right) K(nu) du \\
&= I_{2\cdot 1} + I_{2\cdot 2}
\end{aligned}
$$

$$
|I_{2\cdot 1}| \leqq n^{r+1+\alpha} \int_t^\infty |f(x+u) K(nu)|\, du
$$

$$
\leqq n^{r+1+\alpha} K(nt) \int_t^\infty |f(x+u)|\, du \tag{2.4}
$$

$$
\leqq (nt)^{r+1+\alpha} K(nt) t^{-(r+1+\alpha)} \int_0^\infty |f(x+u)|\, du \to 0, \text{ as } n\to\infty, \text{ for fixed } t,
$$

by (1) and (2.2). And

$$
\begin{aligned}
|I_{2\cdot 2}| &\leqq \sum_{k=0}^{r} n^{r+1+\alpha} |c_k| \int_t^\infty |u^k K(nu)|\, du \\
&\leqq \sum_{k=0}^{r} |c_k|\, t^{k-r-\alpha}\, n \int_t^\infty (nu)^{r+\alpha} K(nu)\, du \\
&= \sum_{k=0}^{r} |c_k|\, t^{k-r-\alpha} \int_{nt}^\infty K(\xi) \xi^{r+\alpha} d\xi \\
&= o(1) \quad \text{as} \quad n\to\infty, \quad \text{for fixed } t.
\end{aligned} \tag{2.5}
$$

Thus we have by (2.3)–(2.5)

$$\overline{\lim_{n\to\infty}} |I| \leqq \varepsilon (r+1+\alpha) \int_0^\infty \xi^{r+\alpha} K(\xi)\, d\xi,$$

from which it results that

$$\lim_{n\to\infty} I = 0.$$

§ **3.** In this section we shall prove the asymptotic expansion theorems corresponding to the well-known Izumi-Bochner's theorem (Theorem B) and others. These are stated as the following two types. One is a theorem in which the assumption concerning with $f(x)$ is weaker than that in the other, but the assumption concerning with the kernel function $K(x)$ is stronger. However we can see in the remark below that these two theorems are combined into one and are considered as the special cases of it.

THEOREM 2. *If*

$$\int_{-\infty}^{\infty} \frac{|f(u)|^p}{1+|u|^{(r+\alpha)p+1}}\, du < \infty \tag{1}$$

for some $p>1$, *where* r *and* α *are the same as in Theorem* 1,

$$(1+|u|^{r+\alpha}) K(u) \in L_1(-\infty, \infty), \tag{2}$$

$$\int_{-\infty}^{\infty} |u|^{(r+\alpha+1)q-1} |K(u)|^q\, du < \infty, \ \textit{where}\ \frac{1}{p}+\frac{1}{q}=1, \tag{3}$$

(4) $f^{(r)}(u)$ *exists at a point* x, *and*

$$f(x+u) = \sum_{k=0}^{r} \frac{f^{(k)}(x)}{k\,!} u^k + o(|u|^{r+\alpha}) \tag{5}$$

for small $|u|$, *then we have*

$$\int_{-\infty}^{\infty} f\left(x+\frac{\xi}{n}\right) K(\xi)\, d\xi = \sum_{k=0}^{r} f^{(k)}(x) \frac{1}{k\,!\ n^k} \int_{-\infty}^{\infty} \xi^k K(\xi)\, d\xi + o(n^{-r-\alpha}),$$

as $n\to\infty$.

THEOREM 3. *If, in Theorem* 2, *we assume*

$$\int_{-\infty}^{\infty} \frac{|f(u)|^p}{1+|u|}\, du < \infty, \quad \textit{for some}\ p>1, \textit{ and} \tag{1'}$$

$$\int_{|u|\geqq A} |u|^{q-1} |K(u)|^q du = o(A^{-q(r+\alpha)}), \tag{3'}$$

instead of (1) *and* (3), *then we have the same result.*

In the first place, we prove Theorem 2. It is sufficient to show that

$$I = n^{r+1+\alpha} \int_{-\infty}^{\infty} \left\{ f(x+u) - \sum_{k=0}^{r} \frac{1}{k\,!} f^{(k)}(x) u^k \right\} K(nu)\, du$$

tends to zero as $n\to\infty$. We have

$$I=n^{r+1+\alpha}\int_{|u|\leqq t}\left\{f(x+u)-\sum_{k=0}^{r}\frac{1}{k!}f^{(k)}(x)u^k\right\}K(nu)\,du$$

$$+n^{r+1+\alpha}\int_{|u|\geqq t}\left\{f(x+u)-\sum_{k=0}^{r}\frac{1}{k!}f^{(k)}(x)u^k\right\}K(nu)\,du$$

$$=I_1+I_2,\qquad \text{say.}$$

For an arbitrary small $\varepsilon>0$, we can choose t by (5) such as

$$\left|\frac{1}{u^{r+\alpha}}\left\{f(x+u)-\sum_{k=0}^{r}\frac{1}{k!}f^{(k)}(x)u^k\right\}\right|<\frac{\varepsilon}{A},\ \text{if } |u|<t,$$

where

$$A=\int_{-\infty}^{\infty}|x|^{r+\alpha}|K(x)|\,dx.$$

Then,

$$|I_1|\leqq\int_{|u|\leqq t}\left|\frac{1}{u^{r+\alpha}}\left\{f(x+u)-\sum_{k=0}^{r}\frac{1}{k!}f^{(k)}(x)u^k\right\}\right|n^{r+1+\alpha}|u|^{r+\alpha}|K(nu)|\,du$$

(3.1)

$$\leqq\frac{\varepsilon}{A}\cdot\int_{-\infty}^{\infty}|\xi|^{r+\alpha}|K(\xi)|\,d\xi=\varepsilon.$$

Also

$$I_2=n^{r+1+\alpha}\int_{|u|\geqq t}f(x+u)K(nu)du-n^{r+1+\alpha}\int_{|u|\geqq t}\sum_{k=0}^{r}\frac{1}{k!}f^{(k)}(x)\,u^k K(nu)\,du$$

$$=I_{2\cdot1}+I_{2\cdot2},$$

and

$$|I_{2\cdot1}|\leqq n^{r+1+\alpha}\int_{|u|\geqq t}|f(x+u)K(nu)|\,du$$

$$=n^{r+1+\alpha}\int_{|u|\geqq t}\frac{|f(x+u)|}{|u|^{\{(r+\alpha)p+1\}/p}}|u|^{\{(r+\alpha)p+1\}/p}|K(nu)|\,du.$$

Now by Hölder's inequality, we have

$$|I_{2\cdot1}|\leqq\left\{\int_{|u|\geqq t}\frac{|f(x+u)|^p}{|u|^{(r+\alpha)p+1}}\,du\right\}^{1/p}\left\{\int_{|u|\geqq t}|u|^{(rp+\alpha p+1)q/p}\,n^{(r+\alpha+1)q}|K(nu)|^q\,du\right\}^{1/q}.$$

By (1), we have

$$\left\{\int_{|u|\geqq t}\frac{|f(x+u)|^p}{|u|^{(r+\alpha)p+1}}\,du\right\}^{1/p}<\infty.$$

On the other hand,

$$\int_{|u|\geqq t}|u|^{(rp+\alpha p+1)q/p}\,n^{(r+1+\alpha)q}|K(nu)|^q\,du$$

$$=\int_{|u|\geqq t}|nu|^{(r+\alpha+1)q-1}|K(nu)|^q n\,du$$

$$=\int_{|\xi|\geqq nt}|\xi|^{(r+\alpha+1)q-1}|K(\xi)|^q d\xi=o(1) \text{ as } n\to\infty, \text{ for fixed } t,$$

by (3). Hence we get

$$|I_{2\cdot1}|=o(1) \qquad \text{as } n\to\infty. \tag{3.2}$$

Furthermore,

$$\begin{aligned}|I_{2\cdot2}|&\leqq\sum_{k=0}^{r}\frac{1}{k!}|f^{(k)}(x)|\int_{|u|\geqq t}|u|^k|K(nu)|n^{r+1+\alpha}du\\ &\leqq\int_{|u|\geqq t}|nu|^{r+\alpha}|K(nu)|n\,du\cdot\sum_{k=0}^{r}\frac{1}{k!}|f^{(k)}(x)||t|^{k-r}\\ &\leqq\int_{|\xi|\geqq nt}|\xi|^{r+\alpha}|K(\xi)|d\xi\sum_{k=0}^{r}\frac{1}{k!}|f^{(k)}(x)||t|^{k-r}\\ &=o(1) \qquad \text{as} \quad n\to\infty,\end{aligned} \tag{3.3}$$

by (2). Thus our theorem has been proved.

Next we prove Theorem 3, but the proof except that for $I_{2\cdot1}$ is quite similar to above one. So it will be sufficient to prove $I_{2\cdot1}=o(1)$ an $n\to\infty$.

$$\begin{aligned}|I_{2\cdot1}|&\leqq n^{r+1+\alpha}\int_{|u|\geqq t}|f(x+u)K(nu)|du\\ &=n^{r+1+\alpha}\int_{|u|\geqq t}\left|\frac{f(x+u)}{u^{1/p}}\right||u^{1/p}K(nu)|du\\ &\leqq\left\{\int_{|u|\geqq t}\frac{|f(x+u)|^p}{|u|}du\right\}^{1/p}\left\{\int_{|u|\geqq t}n^{(r+1+\alpha)q}|u|^{q-1}|K(nu)|^q du\right\}^{1/q}\end{aligned}$$

by Hölders inequality. By (1′), we have

$$\left\{\int_{|u|\geqq t}\frac{|f(x+u)|^p}{|u|}du\right\}^{1/p}<\infty.$$

Also

$$\int_{|u|\geqq t}n^{(r+1+\alpha)q}|u|^{q-1}|K(nu)|^q du=n^{(r+\alpha)q}\int_{|u|\geqq t}(nu)^{q-1}|K(nu)|^q n\,du$$

$$=n^{(r+\alpha)q}\int_{|x|\geqq nt}|x|^{q-1}|K(x)|^q dx=o(1) \text{ as } n\to\infty,$$

by (3′). Hence we get the desired result

$$|I_{2\cdot1}|=o(1) \qquad \text{as} \qquad n\to\infty.$$

REMARK. The assumptions (1) and (3) in above two theorems are represented in the more generalized forms as follows;

(1) $$\int_{-\infty}^{\infty} \frac{|f(x)|^p}{1+|x|^s} dx < \infty,$$

(3) $$\int_{|x| \geqq A} |x|^{s(q-1)} |K(x)|^q dx = o(A^{-(r+\alpha+1)-(1-q)s+1}).$$

As the result, the above theorems correspond to the cases of $s=(r+\alpha)p+1$ and $s=1$ respectively.

Next we shall give the asymptotic expansion theorems corresponding to Theorem C. These theorems are stated as follows, where r and α are the same as in the above theorems.

THEOREM 4. *If*

(1) $$\int_{-\infty}^{\infty} \frac{|f(u)|}{1+|u|^{r+1+\alpha}} du < \infty$$

(2) $$(1+|u|^{r+1+\alpha}) K(u) \in L_1(-\infty, \infty),$$

(3) $$|u^{r+1+\alpha} K(u)| < B < \infty,$$

(4) $$f(x+u) = \sum_{k=0}^{r} \frac{f^{(k)}(x)}{k!} u^k + o(|u|^{r+\alpha}) \text{ for small } |u|, \text{ and}$$

(5) *if $f^{(r)}(u)$ exists at a point x, we have*

$$\int_{-\infty}^{\infty} f\left(x+\frac{\xi}{n}\right) K(\xi)\, d\xi = \sum_{k=0}^{r} \frac{1}{k!} f^{(k)}(x) \frac{1}{n^k} \int_{-\infty}^{\infty} \xi^k K(\xi) d\xi + o(n^{-r-\alpha}), \text{ as } n \to \infty.$$

THEOREM 5. *If*

(1) $|f(u)| < B(1+|u|^{r+\alpha})$, *where $f(u)$ is Lebesgue-Stieltjes integrable for measure $|dH(u)|$,*

(2) $$\int_{-\infty}^{\infty} (1+|u|^{r+\alpha}) |dH(u)| < \infty,$$

(3) $$f(x+u) = \sum_{k=0}^{r} \frac{f^{(k)}(x)}{k!} u^k + o(|u|^{r+\alpha}) \text{ for small } |u| \text{ and}$$

(4) *if $f^{(r)}(u)$ exists at a point x,*

then we have the similar result as the previous theorem.

Before proving Theorem 4, we establish the following lemma.

LEMMA ([1]). *If*

$$\int_a^b |F(u)| du < C, \quad |H(u)| < B, \text{ and } \int_{-\infty}^{\infty} \left| \frac{H(u)}{u} \right| du < D,$$

then

$$\int_a^b |F(u)H(nu)|\,du$$

tends to zero, as $n\to\infty$.

We omit the proof of this lemma. Now we shall proceed to prove Theorem 4. If we put

$$I=n^{r+1+\alpha}\int_{-\infty}^{\infty}\left\{f(x+u)-\sum_{k=0}^{r}\frac{f^{(k)}(x)}{k\,!}u^k\right\}K(nu)\,du$$

$$=n^{r+1+\alpha}\int_{|u|\leqq a}+n^{r+1+\alpha}\int_{|u|\geqq a}=I_1+I_2,$$

then it is sufficient to show that I tends to zero as $n\to\infty$. Moreover, if we put

$$I_2=n^{r+1+\alpha}\int_{|u|\geqq a}f(x+u)K(nu)\,du-\sum_{k=0}^{r}\frac{n^{r+1+\alpha}}{k\,!}\int_{|u|\geqq a}f^{(k)}(x)u^kK(nu)\,du$$

$$=I_{2\cdot1}+I_{2\cdot2},$$

it is proved by the method similar to the proofs of the previous theorems that $|I_1|<\varepsilon$ for some fixed $a>0$, and $I_{2\cdot2}=o(1)$, as $n\to\infty$. So we shall only prove $|I_{2\cdot1}|=o(1)$, as $n\to\infty$.

$$|I_{2\cdot1}|\leqq n^{r+1+\alpha}\int_{a\leqq|u|\leqq b}|f(x+u)K(nu)|\,du+n^{r+1+\alpha}\int_{b\leqq|u|}|f(x+u)K(nu)|\,du=J_1+J_2.$$

For sufficiently large $L>0$, choosing $b>L$, we have

$$\int_{b\leqq|u|}\frac{|f(x+u)|}{|u|^{r+1+\alpha}}du<\frac{\varepsilon}{B}\quad\text{for arbitrary }\varepsilon>0$$

by (1), and

$$J_2=n^{r+1+\alpha}\int_{b\leqq|u|}|f(x+u)K(nu)|\,du$$

$$=\int_{b\leqq|u|}|f(x+u)|\frac{1}{|u|^{r+1+\alpha}}|nu|^{r+1+\alpha}|K(nu)|\,du$$

$$\leqq B\int_{b\leqq|u|}|f(x+u)|\frac{1}{|u|^{r+1+\alpha}}du<\varepsilon\quad\text{for }b>L,$$

by (3). If we put

$$F(u)=\frac{f(x+u)}{u^{r+1+\alpha}}\quad\text{and}\quad H(u)=u^{r+1+\alpha}K(u),$$

these $F(u)$ and $H(u)$ satisfy the assumptions of Lemma. Hence we have $J_1=o(1)$ as $n\to\infty$, by our Lemma, because of

$$J_1=\int_{a\leqq|u|\leqq b}|F(u)H(nu)|\,du.$$

Thus we have proved Theorem 4,

Next the proof of Theorem 5 is quite similar to above theorems except that Lebesgue-Stieltjes integral is used. So we do not prove it.

§4. As the application of our expansion theorems we shall show the two examples below.

EXAMPLE 1. Let us consider $K(u)$ in our expansion theorems as

$$K(u)=e^{-u} \quad \text{for} \quad u\geqq 0,$$
$$=0 \quad \text{for} \quad u<0.$$

This $K(u)$ clearly satisfies the assumptions in Theorem 2-4. Particularly, putting $x=0$ and $u=\xi/n$ in the theorems, we have the following asymptotic expansion formula for the integral

$$\int_0^\infty f(u)\,K(nu)\,du,$$

where $f(u)$ must satisfy the assumptions in these theorems:

$$\int_0^\infty f(u)e^{-nu}\,du=\frac{1}{n}f(0)+\frac{1}{n^2}f'(0)+\frac{1}{n^3}f''(0)+\cdots+\frac{1}{n^{r+1}}f^{(r)}(0)+o\left(\frac{1}{n^{r+1+\alpha}}\right).$$

EXAMPLE 2. In the case of

$$K(u)=e^{-u^2} \quad \text{for} \quad u\geqq 0,$$
$$=0 \quad \text{for} \quad u<0,$$

similarly we have the asymptotic expansion formulae as follows:

$$\int_0^\infty f(u)e^{-n^2u^2}\,du=\frac{\sqrt{\pi}}{2n}\left\{f(0)+\frac{1}{2!\,2}\frac{f''(0)}{n^2}+\frac{1\cdot 3}{4!\,2^2}\frac{f^{(IV)}(0)}{n^4}+\cdots\right.$$
$$\left.+\frac{1\cdot 3\cdots(2p-1)}{(2p)!\,2^p}\frac{f^{(2p)}(0)}{n^{2p}}\right\}+\frac{1}{2n}\left\{\frac{f'(0)}{n}+\frac{1!}{3!}\frac{f'''(0)}{n^3}\right.$$
$$\left.+\frac{2!}{5!}\frac{f^{(V)}(0)}{n^5}+\cdots+\frac{p!}{(2p+1)!}\frac{f^{(2p+1)}(0)}{n^{2p+1}}\right\}+o\left(\frac{1}{n^{2p+2+\alpha}}\right),$$

or

$$=\frac{\sqrt{\pi}}{2n}\left\{f(0)+\frac{1}{2!\,2}\frac{f''(0)}{n^2}+\frac{1\cdot 3}{4!\,2^2}\frac{f^{(IV)}(0)}{n^4}+\cdots\right.$$
$$\left.+\frac{1\cdot 3\cdots(2p-1)}{(2p)!\,2^p}\frac{f^{(2p)}(0)}{n^{2p}}\right\}+\frac{1}{2n}\left\{\frac{f'(0)}{n}+\frac{1!}{3!}\frac{f'''(0)}{n^3}\right.$$
$$\left.+\frac{2!}{5!}\frac{f^{(V)}(0)}{n^5}+\cdots+\frac{(p-1)!}{(2p-1)!}\frac{f^{(2p-1)}(0)}{n^{2p-1}}\right\}+o\left(\frac{1}{n^{2p+\alpha}}\right),$$

where α is a constant such as $0\leqq\alpha<1$ and p is any positive integer.

The asymptotic expansion formulae in the above examples have been derived by the other procedure in Willis [4].

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DEPARTMENT OF MANAGEMENT SCIENCE,
KOBE UNIVERSITY OF COMMERCE.