ON AN APPLICATION OF L. EHRENPREIS' METHOD TO ORDINARY DIFFERENTIAL EQUATIONS

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Introduction.

In 1956 Ehrenpreis [3] considered an application of the sheaf theory to differential equations and gave a criterion for the existence of global solutions of differential equations where the existence of local solutions are assured.

We shall apply this method to systems of ordinary and linear differential equations with coefficients meromorphic in a domain D on the plane C of one complex variable z.

Let $\mathfrak{O}$ and $\mathfrak{M}$ be the sheaves of all germs of functions holomorphic and meromorphic in D respectively. Let a_{jk} $(j, k=1, 2, \cdots, p)$ be functions meromorphic in D.

For any element $f=(f^1, f^2, \cdots, f^p)$ of $\mathfrak{M}^p$, we define

$$Tf=\left(\frac{df^1}{dz}+\sum_{k=1}^{p} a_{1k}f^k, \frac{df^2}{dz}+\sum_{k=1}^{p} a_{2k}, \cdots f^k, \frac{df^p}{dz}+\sum_{k=1}^{p} a_{pk}f^k\right).$$

Then T is a homomorphism of $\mathfrak{M}^p$ into itself.

Let $\mathfrak{A}$ be the sheaf of all germs $f\in\mathfrak{M}^p$ which satisfy the homogeneous equation $Tf=0$, and $T\mathfrak{M}^p$ be the sheaf of all germs $g\in\mathfrak{M}^p$ for each of which there exists $f\in\mathfrak{M}^p$ such that $g=Tf$.

Then we have the exact sequence $0\rightarrow\mathfrak{A}\rightarrow\mathfrak{M}^p\overset{T}{\rightarrow}T\mathfrak{M}^p\rightarrow 0$. Therefore we have also the exact sequence of cohomology groups

$$H^0(D, \mathfrak{A})\rightarrow H^0(D, \mathfrak{M}^p)\overset{T}{\rightarrow}H^0(D, T\mathfrak{M}^p)\rightarrow H^1(D, \mathfrak{A})\rightarrow H^1(D, \mathfrak{M}^p)\overset{T}{\rightarrow}H^1(D, T\mathfrak{M}^p)\rightarrow\cdots.$$

Since $H^1(D, \mathfrak{M}^p)=0$ by Theorem 1 of the present paper, we have $H^1(D, \mathfrak{A})=H^0(D, T\mathfrak{M}^p)/TH^0(D, \mathfrak{M}^p)$. Therefore $H^1(D, \mathfrak{A})$ measures how many $g\in H^0(D, \mathfrak{M}^p)$, which are locally of the form $g=Tf$ for any point of D, are not globally of the form $g=Tf$ for $f\in H^0(D, \mathfrak{M}^p)$.

Calculating the cohomology group $H^1(D, \mathfrak{A})$ we have the following theorem:

If $H^0(D, T\mathfrak{M}^p)=TH^0(D, \mathfrak{M}^p)$, then D is simply or doubly connected.

If D is simply connected, then the necessary and sufficient condition for $H^0(D, T\mathfrak{M}^p)=TH^0(D, \mathfrak{M}^p)$ is that there exist linearly independent solutions $f_1, f_2, \cdots$ and

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f_p of the homogeneous equation $Tf=0$ each of which is meromorphically continued in each point of D except in one of the poles of a_{jk}'s.

If D is doubly connected, then the necessary and sufficient condition for $H^0(D, T\mathfrak{M}^p)=TH^0(D, \mathfrak{M}^p)$ is that any non trivial solution of the homogeneous equation $Tf=0$ is meromorphic in D but is not uniform.

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§1. Cohomology groups with meromorphic coefficients.

Our proof of the following theorem is similar to Hitotumatu-Kôta [6] in using integralizators.

THEOREM 1. *Let D be a non compact Riemann surface and $\mathfrak{M}$ be the sheaf of all germs of meromorphic functions on D. Then $H^n(D, \mathfrak{M}^p)=0$ for $n=1, 2, 3, \cdots$ and $p=1, 2, 3, \cdots$.*

Proof. Let $\mathfrak{U}=\{U_j;\ j\in J\}$ be a locally finite open covering of D and let

$$\phi=\{\phi_{i_0i_1\cdots i_n}=(\phi^1{}_{i_0i_1\cdots i_n}, \phi^2{}_{i_0i_1\cdots i_n}, \cdots, \phi^p{}_{i_0i_1\cdots i_n});\ i_0, i_1, \cdots \text{ and } i_n\in J\}\in Z^n(\mathfrak{U}, \mathfrak{M}^p)$$

where $Z^n(\mathfrak{U}, \mathfrak{M}^p)$ is the set of all n-cocycles of $\mathfrak{U}$ with value in $\mathfrak{M}^p$. There exists a locally finite open covering $\mathfrak{V}=\{V_\lambda;\ \lambda\in\Lambda\}$ of D such that for any $V_\lambda\in\mathfrak{V}$ we find $U_i\in\mathfrak{U}$ satisfying $\overline{V}_\lambda\subset U_i$. Obviously $\mathfrak{V}$ is finer than $\mathfrak{U}$. Let

$$\psi=\{\psi_{\lambda_0\lambda_1\cdots\lambda_n}=(\psi^1{}_{\lambda_0\lambda_1\cdots\lambda_n}, \psi^2{}_{\lambda_0\lambda_1\cdots\lambda_n}, \cdots, \psi^p{}_{\lambda_0\lambda_1\cdots\lambda_n});\ \lambda_0, \lambda_1, \cdots \text{ and } \lambda_n\in\Lambda\}=\rho^{\mathfrak{U}}_{\mathfrak{V}}\phi$$

where $\rho^{\mathfrak{U}}_{\mathfrak{V}}$ is the canonical projection. Then the numbers of poles of $\psi^1{}_{\lambda_0\lambda_1\cdots\lambda_n}, \psi^2{}_{\lambda_0\lambda_1\cdots\lambda_n}, \cdots$ and $\psi^p{}_{\lambda_0\lambda_1\cdots\lambda_n}$ in $V_{\lambda_0}\cap V_{\lambda_1}\cap\cdots\cap V_{\lambda_n}$ is finite. Since $\mathfrak{V}$ is locally finite, the set of all poles of $\psi^1{}_{\lambda_0\lambda_1\cdots\lambda_n}, \psi^2{}_{\lambda_0\lambda_1\cdots\lambda_n}, \cdots$ and $\psi^p{}_{\lambda_0\lambda_1\cdots\lambda_n}$ in $V_{\lambda_0}\cap V_{\lambda_1}\cap\cdots\cap V_{\lambda_n}$ for $\lambda_0, \lambda_1, \cdots$ and $\lambda_n\in\Lambda$ has no accumulation points in D. From Florack's theorem [5] there exists a holomorphic function $f\not\equiv 0$ in D such that $f\psi^1{}_{\lambda_0\lambda_1\cdots\lambda_n}, f\psi^2{}_{\lambda_0\lambda_1\cdots\lambda_n}, \cdots$ and $f\psi^p{}_{\lambda_0\lambda_1\cdots\lambda_n}$ are holomorphic in $V_{\lambda_0}\cap V_{\lambda_1}\cap\cdots\cap V_{\lambda_n}$. Therefore

$$f\psi=\{f\psi_{\lambda_0\lambda_1\cdots\lambda_n}=(f\psi^1{}_{\lambda_0\lambda_1\cdots\lambda_n}, f\psi^2{}_{\lambda_0\lambda_1\cdots\lambda_n}, \cdots, f\psi^p{}_{\lambda_0\lambda_1\cdots\lambda_n});\ \lambda_0, \lambda_1, \cdots \text{ and } \lambda_n\in\Lambda\}\in Z^n(\mathfrak{V}, \mathfrak{O}^p),$$

where $\mathfrak{O}$ is the sheaf of all germs of holomorphic functions in D.

Since $H^n(D, \mathfrak{O}^p)=0$ as D is a Stein manifold by [1] and [5], there exists an open covering $\mathfrak{W}=\{W_\alpha;\ \alpha\in A\}$ of D finer than $\mathfrak{V}$ and

$$\xi=\{\xi_{\alpha_0\alpha_1\cdots\alpha_n}=(\xi^1{}_{\alpha_0\alpha_1\cdots\alpha_n}, \xi^2{}_{\alpha_0\alpha_1\cdots\alpha_n}, \cdots, \xi^p{}_{\alpha_0\alpha_1\cdots\alpha_n});\ \alpha_0, \alpha_1, \cdots \text{ and } \alpha_n\in A\}\in B^n(\mathfrak{W}, \mathfrak{O}^p),$$

where $B^n(\mathfrak{W}, \mathfrak{O}^p)$ is the set of all n-coboundaries of $\mathfrak{W}$ with value in $\mathfrak{O}^p$, such that $\xi=\rho^{\mathfrak{V}}_{\mathfrak{W}}(f\psi)$. Then

$$\xi/f=\{\xi_{\alpha_0\alpha_1\cdots\alpha_n}/f=(\xi^1{}_{\alpha_0\alpha_1\cdots\alpha_n}/f,\ \xi^2{}_{\alpha_0\alpha_1\cdots\alpha_n}/f,\ \cdots,\ \xi^p{}_{\alpha_0\alpha_1\cdots\alpha_n}/f);$$

$$\alpha_0,\ \alpha_1,\ \cdots\ \alpha_n\in A\}\in B^n(\mathfrak{W},\ \mathfrak{M}^p)$$

and $\xi/f=\rho^{\mathfrak{U}}_{\mathfrak{W}}\ \phi$. Since the set of all locally finite open coverings of D is cofinal in the set of all open coverings of D, we have $H^n(D, \mathfrak{M}^p)=0$ that is to be proved.

§2. Application of L. Ehrenpreis' method to ordinary differential equations.

Let D be a domain on the plane C of one complex variable z, $\mathfrak{O}$ and $\mathfrak{M}$ be the sheaves of all germs of functions holomorphic and meromorphic in D, respectively.

We consider in D a differential operator T with meromorphic coefficients.

Let $a_{jk}=a_{jk}(z)$ $(j, k=1, 2, \cdots, p)$ be meromorphic functions in D. Let G be any subdomain of D. For any $f=(f^1, f^2, \cdots, f^p)\in H^0(G, \mathfrak{M}^p)$ we define

$$Tf=\left(\frac{df^1}{dz}+\sum_{k=1}^{p}a_{1k}f^k,\ \frac{df^2}{dz}+\sum_{k=1}^{p}a_{2k}f^k, \cdots,\ \frac{df^p}{dz}+\sum_{k=1}^{p}a_{n_k}f^k\right).$$

Then T is a homomorphism of the sheaf $\mathfrak{M}^p$ of abelian groups into itself.

Let $\mathfrak{A}$ be the sheaf of all $f\in\mathfrak{M}^p$ which satisfy the homogeneous equation $Tf=0$ and $T\mathfrak{M}^p$ be the sheaf of all $g\in\mathfrak{M}^p$ such that $g=Tf$ for some $f\in\mathfrak{M}^p$.

We have easily the exact sequence $0\to\mathfrak{A}\to\mathfrak{M}^p\xrightarrow{T}T\mathfrak{M}^p\to 0$ where $\mathfrak{A}\to\mathfrak{M}^p$ is a canonical inclusion.

Therefore from [10] we have the exact sequence of cohomology groups

$$H^0(D, \mathfrak{A})\to H^0(D, \mathfrak{M}^p)\xrightarrow{T}H^0(D,\ T\mathfrak{M}^p)\to H^1(D, \mathfrak{A})\to H^1(D, \mathfrak{M}^p)\to H^1(D,\ T\mathfrak{M}^p)\to\cdots.$$

From Theorem 1 we have $H^1(D, \mathfrak{M}^p)=0$, hence there holds $H^1(D, \mathfrak{A})=H^0(D, T\mathfrak{M}^p)/TH^0(D, \mathfrak{M}^p)$. $H^0(D, T\mathfrak{M}^p)$ is the set of all $g\in H^0(D, \mathfrak{M}^p)$ for which there exists $f\in H^0(U, \mathfrak{M}^p)$ such that $g=Tf$ in some neighbourhood U of each point of D. $TH^0(D, \mathfrak{M}^p)$ is the set of all $g\in H^0(D, \mathfrak{M}^p)$ for which there exists $f\in H^0(D, \mathfrak{M}^p)$ such that $g=Tf$.

In other words, $H^0(D, T\mathfrak{M}^p)$ is the set of all $g\in H^0(D, \mathfrak{M}^p)$ for each of which the system $g=Tf$ of differential equations has locally a solution f at each point of D and $TH^0(D, \mathfrak{M}^p)$ is the set of all $g\in H^0(D, \mathfrak{M}^p)$ for each of which the system $g=Tf$ of differential equations has a global solution f.

THEOREM 2. *Let D be a domain in C, $\mathfrak{M}$ be the sheaf of all germs of meromorphic functions in D, $a_{jk}(z)$ $(j, k=1, 2, \cdots, p)$ be meromorphic functions in D, T be a differential operator in $\mathfrak{M}^p$ defined by*

$$Tf=\left(\frac{df^1}{dz}+\sum_{k=1}^{p}a_{1k}(z)f^k,\ \frac{df^2}{dz}+\sum_{k=1}^{p}a_{2k}(z)f^k,\ \cdots,\ \frac{df^p}{dz}+\sum_{k=1}^{p}a_{pk}(z)f^k\right)$$

for $f=(f^1, f^2, \cdots, f^p)\in\mathfrak{M}^p$ *and* $\mathfrak{A}$ *be the sheaf of all germs* $f\in\mathfrak{M}^p$ *which satisfy the homogeneous equation* $Tf=0$. *Then*

$$H^1(D, \mathfrak{A})=H^0(D, T\mathfrak{M}^p)/TH^0(D, \mathfrak{M}^p) \text{ for } p=1, 2, 3, \cdots.$$

§ 3. Necessary condition for $H^1(D, \mathfrak{A})=0$.

We intend to discuss the behaviour of the solutions of the homogeneous equation $Tf=0$ at a pole z_1 of the coefficients a_{jk} under the assumption $H^1(D, \mathfrak{A})=0$, that is $H^0(D, T\mathfrak{M}^p)=TH^0(D, \mathfrak{M}^p)$.

Then there exist subdomains D_1 and D_2 of D satisfying the following conditions:

(1) $D_1\cap D_2$ is simply connected and contains no poles of a_{jk}'s;

(2) $z_1\in D_1$, $z_1\notin D_2$. D_1 contains no poles of a_{jk}'s except z_1;

(3) $D=D_1\cup D_2$.

If we put $\mathfrak{U}=\{D_1\ D_2\}$, then $\mathfrak{U}$ is an open covering of D. Since $H^1(D, \mathfrak{A})=0$, we have $H^1(\mathfrak{U}, \mathfrak{A})=0$ from [9]. Hence for any $h\in H^0(D_1\cap D_2, \mathfrak{A})$ there exists $h_1 \in H^0(D_1, \mathfrak{A})$ and $h_2\in H^0(D_2, \mathfrak{A})$ such that $h_1-h_2=h$ in $D_1\cap D_2$. h_2 can be meromorphically continued in each point of $D-\{z_1\}$.

Now we suppose that there exists a solution h of the homogeneous equation $Tf=0$ which can not be meromorphically continued in z_1. Then there exist solutions h_1 and h_2 of the homogeneous equation $Tf=0$ satisfying the following conditions: h_2 can not be meromorphically continued in z_1 but can be meromorphically continued in each point of $D-\{z_1\}$. h_1 can be meromorphically continued in z_1. It holds that $h=h_1-h_2$. We can take a simply connected domain as D_1. Then it is worth remarking that h_2 is uniform in D_2.

Let z_i be a pole of the coefficients a_{jk} such that a solution of the homogeneous equation $Tf=0$ is not meromorphically continued in z_i $(i=1, 2, 3, \cdots)$. Then for any $i(=1, 2, 3, \cdots)$ there exists a solution h_i of the homogeneous equation $Tf=0$ such that h_i cannot be meromorphically continued in z_i but can be meromorphically continued in each point of $D-\{z_i\}$ and is uniform on any Jordan closed curve which does not contain z_i in its interior. Since h_i's are linearly independent, the number q of such poles z_i does not exceed p.

Now we shall prove by induction with respect to $m(\leqq q)$ that any solution f of the homogeneous equation $Tf=0$, which cannot be meromorphically continued in $z_1, z_2, \cdots$ and z_m but can be meromorphically continued in each point of $D-\{z_1, z_2, \cdots z_m\}$, can be represented as a linear combination of the solutions $f_0, f_1, \cdots$ and f_m of the homogeneous equation $Tf=0$ such that f_0 can be meromorphically continued in each point of D and f_i cannot be meromorphically continued in z_i but can be meromorphically continued in each point of $D-\{z_i\}$ for $i=1, 2, \cdots$ and m.

From the preceding result there exist solutions h_1 and h_2 of the homogeneous equation $Tf=0$ such that h_1 can be meromorphically continued in each point of $D-\{z_1, z_2, \cdots, z_{m-1}\}$ and h_2 cannot be meromorphically continued in z_m but can be meromorphically continued in each point of $D-\{z_m\}$ and $f=h_1-h_2$. From the assumption of our induction h_1 can be represented as the linear combination of $f_0, f_1, \cdots$ and f_{m-1} satisfying our requests. If we put $f_m=h_2$, then our proof by induction is completed.

Thus we have the following theorem.

THEOREM 3. *Under the assumptions of Theorem 2, if $H^1(D, \mathfrak{A})=0$, then there exist linearly independent solutions $f_1, f_2, \cdots$ and f_p of the homogeneous equation $Tf=0$ each of which can be meromorphically continued in each point of D except in one of the poles of the coefficients a_{jk}.*

§ 4. Necessary condition concerning the connectivity of *D*.

We shall consider the connectivity of D under the assumption $H^1(D, \mathfrak{A})=0$.

Suppose that the connectivity of D is larger than 2.

Then there exist Jordan closed curves $K_1=\{z=k_1(t);\ 0\leqq t\leqq 1\}$ and $K_2=\{z=k_2(t);\ 0\leqq t\leqq 1\}$ in D and subdomains D_1 and D_2 of D which satisfy the following conditions:

(1) $D_1\cup D_2=D$. $D_1\cap D_2$ contains no poles of a_{jk}'s. K_1 and K_2 pass no poles of a_{jk}'s;

(2) $k_1(1/2)=k_2(1/2)$. The direction of K_1 and K_2 with increasing t is counter clockwise;

(3) The connected components Δ_0, Δ_1 and Δ_2 of $D_1\cap D_2$, which contain $k_1(1/2)=k_2(1/2)$, $k_1(0)$ and $k_2(0)$ respectively, are disjoint each other and simply connected.

Then $\mathfrak{U}=\{D_1, D_2\}$ is an open covering of D. Since $H^1(D, \mathfrak{A})=0$, we have $H^1(\mathfrak{U}, \mathfrak{A})=0$ from [9]. Hence for any $h\in H^0(D_1\cap D_2, \mathfrak{A})$ there exist $h_1\in H^0(D_1, \mathfrak{A})$ and $h_2\in H^0(D_2, \mathfrak{A})$ such that $h_1-h_2=h$ in $D_1\cap D_2$. Let $f_1, f_2, \cdots$ and f_p be linearly independent element of $H^0(\Delta_1, \mathfrak{A})$. Since all a_{jk}'s are holomorphic on K_1, $f_1, f_2, \cdots$ and f_p are analytically continued to $g_1, g_2, \cdots$ and g_p of $H^0(\Delta_1, \mathfrak{A})$ along K_1. Since $f_1, f_2, \cdots$ and f_p are linearly independent, for some complex numbers c_{jk} $(j, k=1, 2, \cdots, p)$ there holds $g_j=\sum_{k=1}^{p} c_{jk}f_k$, $j=1, 2, \cdots, p$.

Let $a_1, a_2, \cdots$ and a_p be any complex numbers. We define $h\in H^0(D_1\cap D_2, \mathfrak{A})$ by putting $h=\sum_{k=1}^{p} a_k f_k$ in Δ_1, $h=0$ in $D_1\cap D_2-\Delta_1$. There exist $h_1\in H^0(D_1, \mathfrak{A})$ and $h_2\in H^0(D_2, \mathfrak{A})$ such that $h=h_1-h_2$ in $D_1\cap D_2$. Since $h_1=h_2$ in $D_1\cap D_2-\Delta_1$, h_2 is an analytic continuation of h_1 along K_1. Let $h_1=\sum_{k=1}^{p} b_k f_k$ in Δ_1. Then $h_2=\sum_{j,k=1}^{p} b_j c_{jk} f_k$ in Δ_1. Therefore we have $\sum_{k=1}^{p}(b_k-\sum_{j=1}^{p} b_j c_{jk})f_k=\sum_{k=1}^{p} a_k f_k$ in Δ_1. Since $f_1, f_2, \cdots$ and f_p are linearly independent we have $b_k-\sum_{j=1}^{p} b_j c_{jk}=a_k$ $(k=1, 2, \cdots, p)$. Since a_k's are arbitrary, we have $\det(\delta_{jk}-c_{jk})\neq 0$ where δ_{jk}'s are Kronecker's δ.

Thus we have the following results:

(1) For any $h\in H^0(\Delta_1, \mathfrak{A})$ there exists $f_1\in H^0(\Delta_1, \mathfrak{A})$ which is analytically continued to $f_2=f_1-h\in H^0(\Delta_1, \mathfrak{A})$ along K_1;

(2) Any element h of $H^0(\Delta_2, \mathfrak{A})$ can be uniformly and analytically continued along K_2.

The same argument concerning K_1 leads us up to

(3) Any element h of $H^0(\Delta_1, \mathfrak{A})$ can be uniformly and analytically continued along K_1.

(1) and (3) contradict each other.

Thus we have the following theorem.

THEOREM 4. *Under the assumptions of Theorem 2, if $H^1(D, \mathfrak{A})=0$, then D is simply or doubly connected.*

§ 5. Necessary and sufficient condition for $H^1(D, \mathfrak{A})=0$ where D is simply connected.

We shall prove that the converse of Theorem 3 is true when D is simply connected. We denote by $q(\leqq p)$ the number of the poles of a_{jk}'s in which some solutions of the homogeneous equation $Tf=0$ cannot be meromorphically continued. We shall prove our proposition by induction with respect to q.

A. In the case that $q=0$. $\mathfrak{A}$ is a constant sheaf over D isomorphic with C^p. Therefore we have $H^1(D, \mathfrak{A})=H^1(D, C^p)=0$.

B. In the case that $q=1$. The set of all open coverings $\mathfrak{U}$ of D with the following properties is cofinal in the directed set of all open coverings of D:

1. $\mathfrak{U}=\{U_\lambda, \lambda\in\Lambda\}$ is locally finite. Each open set of $\mathfrak{U}$ is simply connected. The intersection of two open sets of $\mathfrak{U}$ is empty or simply connected.

2. If we denote by z_1 the pole of a_{jk} in which a solution of homogeneous equation $Tf=0$ cannot be meromorphically continued, then z_1 is contained only in U_{λ_0}.

Let $\mathfrak{U}_0$ be the set consisting of only U_{λ_0}. If for $j=1, 2, 3, \cdots$ we define $\mathfrak{U}_j=\{U_{\lambda_j}, U_{\mu_j}, \cdots\}$ by induction as the set of all open sets of $\mathfrak{U}$ which have the common points with at least one of the open sets of $\mathfrak{U}_{j-1}$ and does not belong to $\mathfrak{U}_0\cup\mathfrak{U}_1\cup\cdots\cup\mathfrak{U}_{j-1}$, then the following conditions are satisfied:

3. If $U_{\lambda_j}\cap U_{\mu_j}\neq\phi$ for U_{λ_j} and $U_{\mu_j}\in\mathfrak{U}_j$, then there exists $U_{\lambda_{j-1}}\in\mathfrak{U}_{j-1}$ such that $U_{\lambda_j}\cap U_{\mu_j}\cap U_{\lambda_{j-1}}\neq\phi$. An open set of $\mathfrak{U}$ which has the common points with at least one of the open sets of $\mathfrak{U}_j$ belongs to $\mathfrak{U}_{j-1}$, $\mathfrak{U}_j$ or $\mathfrak{U}_{j+1}$.

4. If $U_{\lambda_{j-1}}\cap U_{\lambda_j}\neq\phi$ and $U_{\mu_{j-1}}\cap U_{\lambda_j}\neq\phi$ for $U_{\lambda_{j-1}}$ and $U_{\mu_{j-1}}\in\mathfrak{U}_{j-1}$ and $U_{\lambda_j}\in\mathfrak{U}_j$, then there exist $U_{\lambda^0_{j-1}}=U_{\lambda_{j-1}}, U_{\lambda^1_{j-1}}, \cdots, U_{\lambda^{s-1}_{j-1}}, U_{\lambda^s_{j-1}}=U_{\mu_{j-1}}\in\mathfrak{U}_{j-1}$ such that $U_{\lambda^p_{j-1}}\cap U_{\lambda^{p+1}_{j-1}}\cap U_{\lambda_j}\neq\phi$ $(p=0, 1, 2, \cdots, s-1)$.

Of course it holds that $\mathfrak{U}=\mathfrak{U}_0\cup\mathfrak{U}_1\cup\mathfrak{U}_2\cup\cdots$. Let $\{h_{\lambda_j\mu_k}; h_{\lambda_j\mu_k}\in H^0(U_{\lambda_j}\cap U_{\mu_k}, \mathfrak{A})$ with $U_{\lambda_j}\cap U_{\mu_k}\neq\phi\}$ be any 1-cocycle of $\mathfrak{U}$ with value in $\mathfrak{A}$. We shall construct a 0-cochain $\{h_{\lambda_j}; h_{\lambda_j}\in H^0(U_{\lambda_j}, \mathfrak{A})\}$ whose coboundary is the above cocycle as follows.

We put $h_{\lambda_0}=0$. For any $U_{\lambda_1}\in U_1$ we put $h_{\lambda_1}=h_{\lambda_1\lambda_0}$. Then it holds that $h_{\lambda_1}-h_{\lambda_0}=h_{\lambda_1\lambda_0}$ and $h_{\lambda_1}-h_{\mu_1}=h_{\lambda_1\lambda_0}-h_{\mu_1\lambda_0}=h_{\lambda_1\mu_1}$ from Condition 3 for U_{λ_1} and U_{μ_1} of $\mathfrak{U}_1$ such that $U_{\lambda_1}\cap U_{\mu_1}\neq\phi$. We shall assume that for any $U_{\lambda_k}\in\mathfrak{U}_k$ $h_{\lambda_k}\in H^0(U_{\lambda_k}, \mathfrak{A})$ $(k=1, 2, \cdots,$

$j-1$) can be defined such that they satisfy $h_{\lambda_k}-h_{\mu_m}=h_{\lambda_k\mu_m}$ for $U_{\lambda_k}\in\mathfrak{U}_k$ and $U_{\mu_m}\in\mathfrak{U}_m$ ($k, m=1, 2, \cdots, j-1$) with $U_{\lambda_k}\cap U_{\mu_m}\neq\phi$. For any $U_{\lambda_j}\in\mathfrak{U}_j$, there exists $U_{\lambda_{j-1}}\in\mathfrak{U}_{j-1}$ such that $U_{\lambda_j}\cap U_{\lambda_{j-1}}\neq\phi$. We put $h_{\lambda_j}=h_{\lambda_{j-1}}+h_{\lambda_j\lambda_{j-1}}$. Then for other $U_{\mu_{j-1}}\in\mathfrak{U}_{j-1}$ such that $U_{\lambda_j}\cap U_{\lambda_{j-1}}\cap U_{\mu_{j-1}}\neq\phi$ we have

$$(h_{\lambda_{j-1}}+h_{\lambda_j\lambda_{j-1}})-(h_{\mu_{j-1}}+h_{\lambda_j\mu_{j-1}})=h_{\lambda_{j-1}\mu_{j-1}}+h_{\lambda_j\lambda_{j-1}}-h_{\lambda_j\mu_{j-1}}=0.$$

Therefore h_{λ_j} does not depend on the special choice of $U_{\lambda_{j-1}}$ from Condition 4. For any U_{λ_j} and U_{μ_j} such that $U_{\lambda_j}\cap U_{\mu_j}\neq\phi$, from Condition 3 there exists $U_{\lambda_{j-1}}\in\mathfrak{U}_{j-1}$ such that $U_{\lambda_{j-1}}\cap U_{\lambda_j}\cap U_{\mu_j}\neq\phi$. Then we have

$$h_{\lambda_j}-h_{\mu_j}=(h_{\lambda_{j-1}}+h_{\lambda_j\lambda_{j-1}})-(h_{\lambda_{j-1}}+h_{\mu_j\lambda_{j-1}})=h_{\lambda_j\lambda_{j-1}}-h_{\mu_j\lambda_{j-1}}=h_{\lambda_j\mu_j}.$$

Hence we can construct the cochain $\{h_{\lambda_j};\ h_{\lambda_j}\in H^0(U_{\lambda_j}, \mathfrak{A})\}$ of $\mathfrak{U}$ with value in $\mathfrak{A}$ such that its coboundary is the given cocycle $\{h_{\lambda_j\mu_k};\ h_{\lambda_j\mu_k}\in H^0(U_{\lambda_j}\cap U_{\mu_k}, \mathfrak{A})\}$.

Thus we have proved $H^1(\mathfrak{U}, \mathfrak{A})=0$. Hence it holds that $H^1(D, \mathfrak{A})=0$.

C. In the case that $q=m$. Suppose that there holds $H^1(D, \mathfrak{A})=0$ in the case that $q=m-1$. Let $z_1, z_2, \cdots$ and z_m be the poles of a_{jk}'s in each of which some solution of the homogeneous equation $Tf=0$ can not be meromorphically continued. There exist subdomains D_1 and D_2 of D satisfying the following conditions:

(1) D_1, D_2 and $D_1\cap D_2$ are simply connected;

(2) $z_1, z_2, \cdots$ and $z_{m-1}\in D_1-D_1\cap D_2$ and $z_m\in D_2-D_1\cap D_2$;

(3) $D=D_1\cup D_2$.

If we put $\mathfrak{U}=\{D_1, D_2\}$, then $\mathfrak{U}$ is an open covering of D. Since $H^1(D_1, \mathfrak{A})=H^1(D_2, \mathfrak{A})=0$ from B and the assumption of our induction, we have $H^1(D, \mathfrak{A})=H^1(\mathfrak{U}, \mathfrak{A})$ from [9]. Any $h\in H^0(D_1\cap D_2, \mathfrak{A})$ can be represented as a linear combination $\sum_{j=1}^{m} a_j f_j$ of solutions f_j of the homogeneous equation $Tf=0$ in $D_1\cap D_2$ such that f_j can be meromorphically continued in each point of D except in z_j. If we put $h_1=a_m f_m$ and $h_2=-\sum_{j=1}^{m-1} a_j f_j$, then we have $h_1\in H^0(D_1, \mathfrak{A})$, $h_2\in H^0(D_2, \mathfrak{A})$ and $h=h_1-h_2$ in $D_1\cap D_2$. Hence we have $H^1(D, \mathfrak{A})=H^1(\mathfrak{U}, \mathfrak{A})=0$.

Thus our proof by induction is completed.

THEOREM 5. *Suppose that D is simply connected. Then under the assumptions of Theorem 2 the necessary and sufficient condition for $H^1(D, \mathfrak{A})=0$ is that there exist linearly independent solutions $f_1, f_2, \cdots$ and f_p of the homogeneous equation $Tf=0$ each of which can be meromorphically continued in each point of D except in one of the poles of the coefficients a_{jk}.*

§6. Another proof of Theorem 5.

We shall give another proof of the sufficiency of Theorem 5 without cohomology theory. At first we shall give the following lemma.

LEMMA 1. *Under the assumptions of Theorem 2, if $g \in H^0(D, T\mathfrak{M}^p)$, then any solution h of $Th=g$ can be meromorphically continued in any point of D in which all solutions of the homogeneous equation $Tf=0$ can be meromorphically continued.*

Proof. h can be analytically continued in any point of D except in the poles of g and a_{jk}'s. Let z_0 be any point in which any solution of the homogeneous equation $Tf=0$ can be meromorphically continued. Since $g \in H^0(D, T\mathfrak{M}^p)$, there exists a meromorphic function h_0 at z_0 such that $Th_0=g$. Then $h-h_0$ is a homogeneous solution of $Tf=0$. Therefore $h-h_0$ can be meromorphically continued in z_0. Hence h can be meromorphically continued in z_0. q.e.d.

Suppose that D is simply connected and that there exist linealy independent solutions $f_1, f_2, \cdots$ and f_p of the homogeneous equation $Tf=0$ each of which can be meromorphically continued in each point of D except in one of the poles of the coefficients a_{jk}. Let f_j be able to be meromorphically continued in each point of D except in z_j ($j=1, 2, \cdots, p$). For any $g \in H^0(D, T\mathfrak{M}^p)$, there exist meromorphic functions h_j at z_j such that $Th_j=g$ ($j=1, 2, \cdots, p$).

Suppose that there exists a solution h of $Th=g$ which can be meromorphically continued in $z_1, z_2, \cdots$ and z_{m-1}. Since $h-h_m$ is a solution of the homogeneous equation $Tf=0$, it holds that

$$h-h_m=\sum_{j=1}^{p} a_j f_j.$$

Then

$$h'=h-\sum_{j=m}^{p} a_j f_j=h_m+\sum_{j=1}^{m-1} a_j f_j$$

can be meromorphically continued in $z_1, z_2, \cdots$ and z_m and satisfies $Th'=g$. Thus we have proved by induction that there exists a solution h of $Th=g$ which is meromorphic in D. Hence we have $H^0(D, T\mathfrak{M}^p)=TH^0(D, \mathfrak{M}^p)$.

§7. Necessary and sufficient condition for a doubly connected domain.

Next we shall consider the case that D is doubly connected and there exist linearly independent solutions $f_1, f_2, \cdots$ and f_p of the homogeneous equation $Tf=0$ satisfying the conditions of Theorem 3.

Let a Jordan closed curve $K=\{z=k(t);\ 0 \leqq t \leqq 1\}$ be the homology base of D which does not pass poles of a_{jk}'s. There exist subdomains D_1 and D_2 of D such that the connected components of $D_1 \cap D_2$ are two simply connected domains Δ_0 and Δ_1 which contain no poles of a_{jk}'s, D_1 and D_2 are also simply connected $D=D_1 \cup D_2$, $\{z=k(t);\ 0 \leqq t \leqq 1/2\} \subset D_1$, $\{z=k(t);\ 1/2 \leqq t \leqq 1\} \subset D_2$, $k(0) \in \Delta_0$ and $k(1/2) \in \Delta_1$. Then $\mathfrak{U}=\{D_1, D_2\}$ is an open covering of D. Since $H^1(D_1, \mathfrak{A})=H^1(D_2, \mathfrak{A})=0$ from Theorem 5, we have $H^1(\mathfrak{U}, \mathfrak{A})=H^1(D, \mathfrak{A})$ from [9].

Suppose that $H^1(D, \mathfrak{A})=0$. Then for any $h \in H^0(D_1 \cap D_2, \mathfrak{A})$ there exist $h_1 \in H^0$ $D_1, \mathfrak{A})$ and $h_2 \in H^0(D_2, \mathfrak{A})$ such that $h=h_1-h_2$ in $D_1 \cap D_2$. Let $f_1, f_2, \cdots$ and f_p be linearly independent elements of $H^0(\varDelta_0, \mathfrak{A})$. From our assumption, all $f_1, f_2, \cdots$ and f_p are analytically continued to $g_1, g_2, \cdots$ and $g_p \in H^0(\varDelta_0, \mathfrak{A})$ along K. Then we can put

$$g_j=\sum_{k=1}^{p} c_{jk} f_k \quad (j=1, 2, \cdots, p)$$

in $\varDelta_0$. Let $a_1, a_2, \cdots$ and a_p be any complex numbers. Let

$$h=\sum_{k=1}^{p} a_k f_k$$

in $\varDelta_0$ and $h=0$ in $\varDelta_1$. Then there exist $h_1 \in H^0(D_1, \mathfrak{A})$ and $h_2 \in H^0(D_2, \mathfrak{A})$ such that $h=h_1-h_2$ in $D_1 \cap D_2$. Then since $h_1=h_2$ in $\varDelta_1$, h_2 is the analytic continuation of h_1 along K. Putting

$$h_1=\sum_{k=1}^{p} b_k f_k$$

in $\varDelta_0$, then we have

$$b_j-\sum_{k=1}^{p} b_k c_{kj}=a_j \quad (j=1, 2, \cdots, p).$$

Since $a_1, a_2, \cdots$ and a_p are arbitrary complex numbers, we have $\det(\delta_{jk}-c_{jk}) \neq 0$.

In this case $b_1, b_2, \cdots$ and b_p can be taken arbitrarily as $a_1, a_2, \cdots$ and a_p are arbitrary. Therefore any $h \in H^0(\varDelta_0, \mathfrak{A})$ can be meromorphically continued in each point of D, that is, all non trivial solutions of the homogeneous equation $Tf=0$ are meromorphic in D, but are not uniform in D.

Conversely, if all non trivial solutions of the homogeneous equation $Tf=0$ can be meromorphically continued in each point of D and are not uniform, we can easily prove that $H^1(D, \mathfrak{A})=H^1(\mathfrak{U}, \mathfrak{A})=0$.

We can summarize these facts in the following theorem.

THEOREM 6. *Suppose that D is doubly connected. Under the assumptions of Theorem 2 the necessary and sufficient condition for $H^1(D, \mathfrak{A})=0$ is that any non trivial solution of the homogeneous equation $Tf=0$ is meromorphic in D but is not uniform.*

§ 8. Main results.

We shall summarize the preceding results in the following theorem.

MAIN THEOREM. *Let D be a domain on the complex plane C and a_{jk} $(j, k=1, 2 \cdots, p)$ be meromorphic functions in D. We define in the sheaf $\mathfrak{M}^p$ where $\mathfrak{M}$ is the sheaf of all germs of meromorphic functions in D, the differential operator T by putting*

$$T(f^1, f^2, \cdots, f^p)=\left(\frac{df^1}{dz}+\sum_{k=1}^{p} a_{1k}f^k, \frac{df^2}{dz}+\sum_{k=1}^{p} a_{2k}f^k, \cdots, \frac{df^p}{dz}+\sum_{k=1}^{p} a_{pk}f^k\right)$$

for any $f=(f^1, f^2, \cdots, f^p)\in\mathfrak{M}^p$.

If $H^0(D, T\mathfrak{M}^p)=TH^0(D, \mathfrak{M}^p)$, then D is simply or doubly connected.

If D is simply connected, the necessary and sufficient condition for $H^0(D, T\mathfrak{M}^p)=TH^0(D, \mathfrak{M}^p)$ is that there exist linearly independent solutions $f_1, f_2 \cdots$ and f_p of the homogeneous equation $Tf=0$ each of which can be meromorphically continued in each point of D except in one pole of the coefficients a_{jk}.

If D is doubly connected, then the necessary and sufficient condition for $H^0(D, T\mathfrak{M}^p)=TH^0(D, \mathfrak{M}^p)$ is that any non trivial solution of the homogeneous equation $Tf=0$ is meromorphic in D but is not uniform.

§ 9. Examples.

We consider the case that $p=1$ and $T=d/dz-\lambda z^{-1}$.

EXAMPLE 1. Let $D=C$.

For any $g\in H^0(D, T\mathfrak{M})$, there exists a meromorphic function h in $z=0$ such that $Th=g$. Then from Lemma 1, h can be meromorphically continued in each point of D. Since D is simply connected, $h\in H^0(D, \mathfrak{M})$. Therefore for any λ we have $H^0(D, T\mathfrak{M})=TH^0(D, \mathfrak{M})$. In this case all the homogeneous solution az^λ can be meromorphically continued in any point of D except in $z=0$.

EXAMPLE 2. Let $D=D-\{0\}$.

For any integer λ let $g=z^{\lambda-1}\in H^0(D, T\mathfrak{M})$. If $h\in H^0(D, \mathfrak{M})$ satisfies $Th=g$ then h must be of the form $h(z)=z^\lambda(\log z)+az^\lambda$. Therefore $g\in H^0(D, T\mathfrak{M})-TH^0(D, \mathfrak{M})$. In this case all the homogeneous solutions az^λ are uniform and meromorphic in D and it holds that $H^0(D, T\mathfrak{M})\neq TH^0(D, \mathfrak{M})$.

In the case that λ is not an integer, for any $g\in H^0(D, T\mathfrak{M})$, there exists a solution h of $Th=g$ in $z=1$. Then h can be meromorphically continued in any point of D from Lemma 1. We denote by h' the meromorphic continuation of h along the Jordan curve $K=\{z=e^{2\pi it}; 0\leqq t\leqq 1\}$ in $z=1$. Then $h'-h$ is a solution of the homogeneous equation $Tf=0$ and, therefore, $h'-h=az^\lambda$ in a suitable neighbourhood of $z=1$ for any fixed branch of z^λ. We consider $h+bz^\lambda$ in a suitable neighbourhood of $z=1$, then its meromorphic continuation along K is $h+(a+be^{2\pi i\lambda})z^\lambda$ in $z=1$. If we determine b such that $b=a+be^{2\pi i\lambda}$, that is, $b=a(1-e^{2\pi i\lambda})^{-1}$, then it holds that $h+bz^\lambda\in H^0(D, \mathfrak{M})$ and $T(h+bz^\lambda)=g$. Therefore we have $H^0(D, T\mathfrak{M})=TH^0(D, \mathfrak{M})$. In this case all non trivial homogeneous solutions az^λ are not uniform but meromorphic in D.

In the same way we can give the proof of the sufficiency of Theorem 6 similarly to §6.

EXAMPLE 3. Let $D=C-\{1\}$. In the case that λ is not a positive integer, let $g(z)=(1-z)^{-1}\in H^0(D, \mathfrak{M})$. Then

$$h(z)=\sum_{n=1}^{\infty}(n-\lambda)^{-1}z^n$$

satisfies $Th=g$ in $|z|<1$. Therefore $g\in H^0(D, T\mathfrak{M})$. Suppose that $g\in TH^0(D, \mathfrak{M})$. Then there exists $f\in H^0(D, \mathfrak{M})$ such that $Tf=g$. Then it holds that

$$f(z)=z^{\lambda}\int^{z} z^{-\lambda}g(z)dz+az^{\lambda}.$$

Since z^{λ} and $z^{-\lambda}$ are holomorphic in $z=1$, the residue of $z^{-\lambda}g(z)$ in $z=1$ must be zero. But this a contradiction. Therefore we have $g\in H^0(D, T\mathfrak{M})$. Hence it holds that $H^0(D, T\mathfrak{M})\neq TH^0(D, \mathfrak{M})$.

In the case that λ is a positive integer, let $g(z)=z^{\lambda}(z-1)^{-1}\in H^0(D, T\mathfrak{M})$. If $h\in H^0(D, \mathfrak{M})$ satisfies $Th=g$, then h must be of the form $h(z)=z^{\lambda}\log(z-1)+az^{\lambda}$. Therefore it holds that $g\in H^0(D, T\mathfrak{M})-TH^0(D, \mathfrak{M})$. Hence we have $H^0(D, T\mathfrak{M})\neq TH^0(D, \mathfrak{M})$.

In the case that λ is not an integer a homogeneous solution z^{λ} is not meromorphic in D. In the case that λ is an integer all homogeneous solutions az^{λ} are uniform and meromorphic in D.

Examples 1, 2 and 3 illustrate our Main Theorem.

REFERENCES

[1] BEHNKE, H., UND K. STEIN, Entwicklung analytischer Funktionen auf Riemannschen Flächen. Math. Ann. **120** (1948), 430–461.

[2] Colloque sur les fonctions de plusieurs variables à Bruxelles. Masson & Cie, Paris (1953).

[3] EHRENPREIS, L., Sheaves and differential equations, Proc. Amer. Math. Soc., **7** (1956), 1131–1139.

[4] EILENBERG, S., AND N. STEENROD, Foundations of Algebraic topology. Princeton Math. series **15** (1952).

[5] FLORACK, H., Reguläre und meromorphe Funktionen auf nichtgeschlossenen Riemannschen Flächen, Dissertation, Münster (1948).

[6] HITOTUMATU, S., AND O. KÔTA, Ideals of meromorphic functions of several complex variables. Math. Ann. **125** (1952), 119–128.

[7] HUKUHARA, M., Ordinary differential equations. Iwanami Zensho (1950). (in Japanese)

[8] KAJIWARA, J., On the equivalence of Hitotumatu's conjecture and the decomposition theorem. Mem. Fac. Sci., Kyushu Univ. **12** (1958), 113–135.
[9] SCHEJA, G., Riemannsche Hebbarkeitssätze für Cohomologie-klassen. Math. Ann., **144** (1961), 345–360.
[10] SERRE, J. P., Faisceaux algébrique cohérentes. Ann. Math. **61** (1955), 197–278.

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