CESÀRO SUMMABILITY OF SUCCESSIVELY DIFFERENTIATED SERIES OF A FOURIER SERIES AND ITS CONJUGATE SERIES

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1. Let $f(x)$ be a Lebesgue-integrable function of period 2π. The Lebesgue-Fourier series of $f(x)$ is

$$f(x) \sim \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt + \sum_{n=1}^{\infty}\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos n(t-x)\,dt, \tag{1.1}$$

and the derived series of (1.1) is

$$\sum_{n=1}^{\infty} n\int_{-\pi}^{\pi} f(x)\sin n(t-x)\,dt. \tag{1.2}$$

It is well-known that if $f(x+t)+f(x-t)-2A \to 0$, as $t\to 0$, then the series (1.1) is summable (C,α) to A, where $\alpha>0$. This condition can be improved to the Lebesgue's condition

$$\int_0^t |f(x+u)+f(x-u)-2A|\,du = o(t), \quad \text{as} \quad t\to 0.$$

Concerning the derived series of (1.1), if

$$f(x+t)-f(x-t)-2At = o(t), \tag{1.3}$$

as $t\to 0$, then the series (1.2) is summable (C,α), $\alpha>1$, to A.[1] The result has been improved by K. K. Chen [4], [5]: If

$$\int_0^t \left|\frac{f(x+u)-f(x-u)}{u} - 2A\right| du = o(t), \tag{1.4}$$

as $t\to 0$, then the series (1.2) is summable (C,α) to A, where $\alpha>1$.

Suppose that a function $f(x)$ is defined in the neighbourhood of a point x and that there exist constants $\alpha_0, \alpha_1, \cdots, \alpha_r$ such that for small $|t|$,

$$f(x+t) = \alpha_0 + \alpha_1 t + \cdots + \alpha_{r-1}\frac{t^{r-1}}{(r-1)!} + (\alpha_r+\varepsilon_t)\frac{t^r}{r!},$$

where ε_t tends to 0 with t. We then say that f has a generalized rth derivative (unsymmetric derivative) $f_{(r)}(x)$ at x and define $f_{(r)}(x)=\alpha_r$. This definition is due to Peano. For applications to trigonometric series a certain modification of it is due to de la Vallée-Poussin. We consider it separately for r even and odd. Write

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1) This was proved by Priwaloff [10] and Young [12], [13].

$$\chi_x(t) = \chi(t) = \frac{1}{2}\{f(x+t) + f(x-t)\},$$

$$\psi_x(t) = \psi(t) = \frac{1}{2}\{f(x+t) - f(x-t)\}.$$

Suppose that r is even. If there are constants $\beta_0, \beta_2, \beta_4, \cdots, \beta_r$ such that

$$\chi(t) = \beta_0 + \beta_2 \frac{t^2}{2!} + \cdots + \beta_{r-2} \frac{t^{r-2}}{(r-2)!} + (\beta_r + \varepsilon_t) \frac{t^r}{r!},$$

where ε_t tends to 0 with t, we call β_r the generalized symmetric derivative — or simply, the rth symmetric derivative of f at x. The definition of the rth symmetric derivative for odd integer r is similar. We denote the symmetric derivative by the same symbol $f_{(r)}(x)$.

THEOREM A. *If the symmetric derivative $f_{(r)}(x)$ of $f(x)$ exists, then the rth derived series of the Fourier series of $f(x)$ is summable (C, α), $\alpha > r$, to the sum $f_{(r)}(x)$.*

Theorem A is due to Gronwall [6], Priwaloff and Zygmund [14].[2] Suppose that f, defined in the neighbourhood of x, has $r-1$ unsymmetric derivatives $\alpha_0, \alpha_1, \cdots, \alpha_{r-1}$, and define $\omega_r(x, t)$ by

$$f(x_0 + t) = \alpha_0 + \alpha_1 t + \cdots + \alpha_{r-1} \frac{t^{r-1}}{(r-1)!} + \omega_r(x, t) \frac{t^r}{r!}. \tag{1.5}$$

If $\omega_r(x, t)$ has a limit as $t \to 0$, f has also an rth derivative $f_{(r)}(x)$. If r is odd, it follows from (1.5) that

$$\chi(t) = \alpha_0 + \alpha_2 \frac{t^2}{2!} + \cdots + \alpha_{r-1} \frac{t^{r-1}}{(r-1)!} + \frac{1}{2} \delta_r(x, t) \frac{t^r}{r!}; \tag{1.6}$$

and if r is even, then we have

$$\psi(t) = \alpha_1 t + \alpha_3 \frac{t^3}{3!} + \cdots + \alpha_{r-1} \frac{t^{r-1}}{(r-1)!} + \frac{1}{2} \delta_r(x, t) \frac{t^r}{r!}. \tag{1.7}$$

Suppose now, without assuming anything about $\omega_r(x, t)$, that there exist constants α_j such that we have (1.6) or (1.7) according to r is odd or even, and that

$$\delta_r(x) = \lim_{t \to +0} \delta_r(x, t) \tag{1.8}$$

exists. Then $\delta_r(x)$ may be thought of as a jump of the rth derivative, even if this derivative does not exist near x.

THEOREM B. *If f satisfies $\delta_r(x) = 0$ and if $r < \alpha \leqq r + 1$, then*

$$\{\tilde{\sigma}_n^\alpha(x)\}^{(r)} - \frac{(-1)}{\pi} \int_{1/n}^{\infty} \frac{\delta_r(x, t)}{t} dt \to 0, \tag{1.9}$$

as $n \to \infty$, where $\tilde{\sigma}_n^\alpha(x)$ is the nth (C, α) mean of the allied series of $f(x)$.

2) Cf. also [15], p. 60.

Theorem B is due to Plessner [8], [9].[3] The purpose of this paper is to generalize Theorem A and Theorem B by K. K. Chen's method.

2. The following theorem is an extension of K. K. Chen's result:

THEOREM 1. *If* $\Delta_t^2 f(x)$ *is the second symmetric difference of* $f(x)$: $2\phi_x(t) = 2\phi(t) = \Delta_t^2 f(x) = f(x+t) - 2f(x) + f(x-t)$, *and if*

$$\int_0^t \left| \frac{\Delta_u^2 f(x)}{u^2} - A \right| du = o(t), \quad as \quad t \to 0, \tag{2.1}$$

then the second differentiated series of the Fourier series (1.1) *of* $f(x)$:

$$\begin{cases} -\sum_{n=1}^{\infty} \frac{n^2}{\pi} \int_{-\pi}^{\pi} f(t) \cos n(t-x)\, dt = -\sum_{n=1}^{\infty} \frac{n^2}{\pi} \int_{-\pi}^{\pi} f(x+t) \cos nt\, dt \\ = -\sum_{n=1}^{\infty} \frac{n^2}{\pi} \int_0^{\pi} \{f(x+t) - 2f(x) + f(x-t)\} \cos nt\, dt \end{cases} \tag{2.2}$$

is summable (C, α), $\alpha > 2$, *to the value* A.

LEMMA 1. *Let* $S_n^\alpha(\sum_0^\infty u_k)$ *denote the* α*th Cesàro sum of* $\sum_{k=0}^\infty u_k$, *i.e.*

$$S_n^\alpha\left(\sum_0^\infty u_k\right) = \sum_{k=0}^{n} A_{n-k}^\alpha u_k = \sum_{k=0}^{n} \binom{n-k+\alpha}{n-k} u_k.$$

If $\alpha \geqq 1$, *then*

$$\begin{cases} \int_0^\pi t^2 S_n^\alpha\left(\sum_1^\infty \nu^2 \cos \nu t\right) dt = -2\int_0^\pi u\, du \int_0^u S_n^\alpha\left(\sum_1^\infty \nu^2 \cos \nu t\right) dt \\ = -2\int_0^\pi t S_n^\alpha\left(\sum_1^\infty \nu \sin \nu t\right) dt \simeq -\frac{\pi n^\alpha}{\Gamma(1+\alpha)} \simeq -\pi A_n^\alpha, \quad as \quad n \to \infty. \end{cases} \tag{2.3}$$

The proof of (2.3) is essentially due to K. K. Chen.[4]

LEMMA 2. *Let* $K_n^\alpha(x)$ *denote the nth* (C, α) *mean of the series* $1/2 + \cos x + \cos 2x + \cdots$. *Then*

$$\left| \frac{d^r}{dt^r} K_n^\alpha(t) \right| \leqq C n^{r+1} \quad (0 \leqq t \leqq \pi), \tag{2.4}$$

$$\left| \frac{d^r}{dt^r} K_n^\alpha(t) \right| \leqq \frac{C}{n^{\alpha-r} t^{\alpha+1}} \quad \left(\frac{1}{n} \leqq t \leqq \pi\right), \tag{2.5}$$

for $-1 \leqq \alpha \leqq r+1$, $n = 1, 2, \cdots$.

Lemma 2 is due to Zygmund.[5]

Proof of Theorem 1. Let us denote by $\sigma_n^\alpha(\sum_0^\infty u_k)$ the nth Cesàro mean of

3) See also [15], p. 63.
4) See [4], p. 83.
5) Cf. [15] pp. 60–61; and also [14], p. 213.

order α of $\sum_{k=0}^{\infty} u_k$. By (2.2) and Lemma 1 (with $\alpha > 2$), we have, as $n \to \infty$, the difference between the nth Cesàro mean of order α of the second differentiated series of the Fourier series of $f(x)$ and A is

$$(2.6)\quad \begin{cases} R_n = \sigma_n^\alpha\left\{-\sum_1^\infty \frac{\nu^2}{\pi}\int_{-\pi}^{\pi} f(t)\cos\nu(t-x)\,dt\right\} - A \\ \quad = -\frac{1}{\pi}\int_0^\pi \{\Delta_t^2 f(x) - At^2\}\sigma_n^\alpha\left(\sum_1^\infty \nu^2\cos\nu t\right)dt + o(1) \\ \quad = -\frac{1}{\pi}\int_0^\pi \{2\varphi(t) - At^2\}\sigma_n^\alpha\left(\sum_1^\infty \nu^2\cos\nu t\right)dt + o(1) \\ \quad = -\frac{1}{\pi}\left(\int_0^{1/n} + \int_{1/n}^{\varepsilon} + \int_\varepsilon^\pi\right) + o(1) = -\frac{1}{\pi}(I_n + J_n + T_n) + o(1), \end{cases}$$

say. Applying Lemma 2 with $r = 2$, we obtain

$$(2.7)\quad \left|-\sigma_n^\alpha\left(\sum_1^\infty \nu^2\cos\nu t\right)\right| = \left|\frac{d^2}{dt^2}K_n^\alpha(t)\right| \leqq \frac{K}{n^{\alpha-2}t^{\alpha+1}} \quad \left(\frac{1}{n} \leqq t \leqq \pi\right),$$

$$(2.8)\quad \left|\frac{d^2}{dt^2}K_n^\alpha(t)\right| \leqq Kn^3 \quad (0 \leqq t \leqq \pi),$$

and therefore

$$(2.9)\quad \begin{cases} |T_n| = \left|\int_\varepsilon^\pi [2\varphi(t) - At^2]\left\{\frac{d^2}{dt^2}K_n^\alpha(t)\right\}dt\right| \leqq K\int_\varepsilon^\pi |2\varphi(t) - At^2|\, n^{2-\alpha}t^{-\alpha-1}\,dt \\ \quad \leqq Kn^{2-\alpha}\varepsilon^{-(1+\alpha)}\int_0^\pi |2\varphi(t) - At^2|\,dt \\ \quad = o(1), \quad \text{as} \quad n \to \infty. \end{cases}$$

By (2.8) and (2.1), we have the estimate:

$$(2.10)\quad \begin{cases} |I_n| = \left|\int_0^{1/n} [2\varphi(t) - At^2]\left\{\frac{d^2}{dt^2}K_n^\alpha(t)\right\}dt\right| \\ \quad \leqq Kn^3\int_0^{1/n} |2\varphi(t) - At^2|\,dt \\ \quad \leqq Kn\int_0^{1/n}\left|\frac{2\varphi(t)}{t^2} - A\right|dt \\ \quad = o(1), \quad \text{as} \quad n \to \infty. \end{cases}$$

It remains to estimate $|J_n|$: From integration by parts and in virtue of (2.7), we find, as $n \to \infty$,

$$(2.11)\quad \begin{cases} |J_n| \leqq Kn^{2-\alpha}\int_{1/n}^{\varepsilon}\left|\frac{2\varphi(t)}{t^2} - A\right| t^{1-\alpha}\,dt \\ \quad \leqq n^{2-\alpha}\{\delta(\varepsilon)\varepsilon^{2-\alpha} + \delta(n)n^{-2+\alpha}\} + Kn^{2-\alpha}\int_{1/n}^{\varepsilon} t^{-\alpha}\,dt\int_0\left|\frac{2\varphi(u)}{u^2} - A\right|du \\ \quad \leqq n^{2-\alpha}\{\delta(\varepsilon)\varepsilon^{2-\alpha} + \delta(n)n^{-2+\alpha}\} + \delta(\varepsilon)\left\{n^{2-\alpha}\int_{1/n}^{\varepsilon} t^{1-\alpha}\,dt\right\} \\ \quad = o(1), \end{cases}$$

as $\varepsilon \to 0$ and $n \to \infty$. Collecting (2.6), (2.9), (2.10), (2.11), we obtain

$$R_n = \sigma_n^\alpha \left\{ -\sum_1^\infty \frac{\nu^2}{\pi} \int_{-\pi}^{\pi} f(t) \cos \nu(t-x)\, dt \right\} - A = o(1), \tag{2.12}$$

as $n \to \infty$. This completes the proof of Theorem 1.

3. We come now to similar results for successive derivatives of higher orders. The result of Theorem 1 cannot be extended to symmetric difference of order 3, and therefore we are imposed to change it slightly to a similar form with the generalized symmetric derivatives. The following results are generalizations of Theorem A:

THEOREM 2. *If r is even and if there exist constants $\beta_0, \beta_2, \cdots, \beta_r$ such that*

$$\frac{1}{t^{r+1}} \int_0^t \left| \chi(u) - \left(\beta_0 + \beta_2 \frac{u^2}{2!} + \cdots + \beta_r \frac{u^r}{r!} \right) \right| du = o(1), \tag{3.1}$$

as $t \to 0$, where $\chi(t) = (1/2)\{f(x+t) + f(x-t)\}$, then the rth derived series of the Fourier series of $f(x)$ is summable (C, α), $\alpha > r$, to β_r.

THEOREM 3. *If r is add and if there exist constants $\beta_1, \beta_3, \cdots, \beta_r$, such that*

$$\frac{1}{t^{r+1}} \int_0^t \left| \psi(u) - \left(\beta_1 u + \beta_3 \frac{u^3}{3!} + \cdots + \beta_r \frac{u^r}{r!} \right) \right| du = o(1), \tag{3.2}$$

as $t \to 0$, where $\psi(t) = (1/2)\{f(x+t) - f(x-t)\}$, then the rth derived series of the Fourier series of $f(x)$ is summable (C, α), $\alpha > r$, to β_r.

The proofs of Theorem 2 and Theorem 3 are practically similar. So let us consider a proof of Theorem 3. The result is obvious if $f(x)$ is a trigonometric polynomial, and therefore we may, without loss of generality, take $\beta_1 = \beta_3 = \cdots = \beta_r = 0$. The nth Cesàro sum of order α of the rth derived series of (1.1) is given by

$$\left\{ \begin{aligned} \{\sigma_n^\alpha(x)\}^{(r)} &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) \{K_n^\alpha(t)\}^{(r)}\, dt = \frac{2}{\pi} \int_0^\pi \chi(t) \{K_n^\alpha(t)\}^{(r)}\, dt \\ &= \frac{2}{\pi} \left[\int_0^{1/n} + \int_{1/n}^{\varepsilon} + \int_{\varepsilon}^{\pi} \right] = \frac{2}{\pi} [I_1 + I_2 + I_3], \quad \text{say.} \end{aligned} \right. \tag{3.3}$$

From Lemma 2 and (3.2) we find

$$|I_1| = O(n^{r+1}) \int_0^{1/n} |\chi(t)|\, dt = o(1), \tag{3.4}$$

as $n \to \infty$. In virtue of (2.5) and by integration by parts we find

$$\left\{ \begin{aligned} |I_2| &\leqq K n^{r-\alpha} \int_{1/n}^{\varepsilon} t^{-1-\alpha} |\chi(t)|\, dt \\ &\leqq n^{r-\alpha} \left\{ \delta(\varepsilon) \varepsilon^{r-\alpha} + \delta(n) n^{-(r-\alpha)} + \delta(\varepsilon) \int_{1/n}^{\varepsilon} t^{r+1-2-\alpha}\, dt \right\}, \end{aligned} \right. \tag{3.5}$$

which tends to zero as $n\to\infty$ and $\varepsilon\to 0$. Then by (2.5) in Lemma 2 we obtain the estimate:

$$(3.6)\qquad \begin{cases} |I_3| \leqq \int_\varepsilon^\pi |\chi(t)|\,|\{K_n^\alpha(t)\}^{(r)}|\,dt \\ \leqq K\int_\varepsilon^\pi |\chi(t)|\, n^{r-\alpha}t^{-\alpha-1}\,dt \\ \leqq Kn^{r-\alpha}\varepsilon^{-\alpha-1}\int_\varepsilon^\pi |\chi(t)|\,dt \\ = o(1), \end{cases}$$

as ε is arbitrarily fixed and $n\to\infty$. Collecting (3.4), (3.5), (3.6), we obtain the required result:

$$(3.7)\qquad \{\sigma_n^\alpha(x)\}^{(r)} = o(1), \quad \text{as} \quad n\to\infty.$$

4. We now consider corresponding results for the repeated differentiated series of the conjugate series of the Fourier series of $f(x)$.

LEMMA 3. *Let $\tilde{K}_n^\alpha(t)$ be the conjugate (C,α) kernel, and let*

$$(4.1)\qquad H_n(t) = \frac{1}{2}\cos\frac{1}{2}t - \tilde{K}_n^\alpha(t).$$

If $0\leqq\alpha\leqq r+1$, we have

$$(4.2)\qquad |\{\tilde{K}_n^\alpha(t)\}^{(r)}| \leqq Cn^{r+1} \quad (0\leqq t\leqq\pi),$$

$$(4.3)\qquad |H_n^{(r)}(t)| \leqq Cn^{r-\alpha}t^{-\alpha-1} \quad \left(\frac{1}{n}\leqq t\leqq\pi\right).$$

Lemma 3 is due to Zygmund [14], [15], p. 64. Corresponding to Theorem 2, we have the following results which are generalizations of Theorem B:

THEOREM 4. *Suppose r is an odd integer and that there exist constants $\alpha_0, \alpha_2, \cdots, \alpha_{r-1}$, such that*

$$(4.4)\qquad \int_0^t u^{-r}\left|\psi(u) - \alpha_0 u - \alpha_2\frac{u^2}{2!} - \cdots - \alpha_{r-1}\frac{u^{r-1}}{(r-1)!}\right| du = o(1),$$

as $t\to 0$, then the rth allied series of the Fourier series of $f(x)$ is summable (C,α), $r<\alpha\leqq r+1$, to the integral:

$$(4.5)\qquad -\frac{1}{\pi}\int_{1/n}^\infty \frac{\delta_r(x,t)}{t}\,dt,$$

where

$$\delta_r(x,t) = 2\left\{\psi(t) - \alpha_0 - \alpha_2\frac{t^2}{2!} - \cdots - \alpha_{r-1}\frac{t^{r-1}}{(r-1)!}\right\},$$

and

$$\psi(t) = \psi_x(t) = \frac{1}{2}\{f(x+t) - f(x-t)\}.$$

THEOREM 5. *Suppose that r is even and that there exist integers $\alpha_1, \alpha_3, \cdots, \alpha_{r-1}$, such that*

$$\int_0^t u^{-r}\left|\varphi(u)-\alpha_1 u-\alpha_3\frac{u^3}{3!}-\cdots-\alpha_{r-1}\frac{u^{r-1}}{(r-1)!}\right|du=o(1), \tag{4.6}$$

as $t\to 0$, then the rth allied series of the Fourier series of $f(x)$ is summable (C,α), $r<\alpha\leqq r+1$, to the integral (4.5).

The proofs of Theorem 4 and Theorem 5 follow in a similar way as in the proof of Theorem 2, except we use the corresponding inequalities in Lemma 3 which we omit here. I also remark that by the method due to Izumi [7], we can further generalize our results. So we have:

THEOREM 1′. *In Theorem* 1, *the condition* (2.1) *may by replaced by*:

$$\int_0^t\left\{\frac{\Delta_u^2 f(x)}{u^2}-A\right\}du=o(t), \tag{2.1a}$$

$$\int_0^t\left|\frac{\Delta_u^2 f(x)}{u^2}-A\right|du=O(t), \tag{2.1b}$$

as $t\to 0$.

THEOREM 2′. *In Theorem* 2, *the condition* (3.1) *may be replaced by*:

$$\frac{1}{t^{r+1}}\int_0^t\left\{\chi(u)-\left(\beta_0+\beta_2\frac{u^2}{2!}+\cdots+\beta_r\frac{u^r}{r!}\right)\right\}du=o(1), \tag{3.1a}$$

$$\frac{1}{t^{r+1}}\int_0^t\left|\chi(u)-\left(\beta_0+\beta_2\frac{u^2}{2!}+\cdots+\beta_r\frac{u^r}{r!}\right)\right|du=O(1), \tag{3.1b}$$

as $t\to 0$.

Similar results for Theorem 3, Theorem 4, and Theorem 5 also hold; and we omit the details here. The proofs of these results follow in a similar way. For simplicity, let us consider a sketch of the proof of Theorem 1′: We only need to change slightly the proof of Theorem 1. We change (2.6) to:

$$R_n=-\frac{1}{\pi}\left(\int_0^{m/n}+\int_{m/n}^{\varepsilon}+\int_{\varepsilon}^{\pi}\right)+o(1)=-\frac{1}{\pi}(I_n'+J_n'+T_n)+o(1). \tag{2.6′}$$

It is easy to see that $T_n=o(1)$. To estimate I_n', we set

$$\Phi(t)=\int_0^t\left\{\frac{\varphi(u)}{u^2}-A\right\}du. \tag{4.7}$$

Then we have, by (2.4) and integration by parts:

$$\left\{\begin{aligned} I_n'&=\int_0^{m/n}\left\{\frac{2\varphi(t)}{t^2}-A\right\}\cdot t^2\frac{d^2}{dt^2}\{K_n^\alpha(t)\}\,dt\\ &=\left[\Phi(t)t^2\frac{d^2}{dt^2}\{K_n^\alpha(t)\}\right]_0^{m/n}+o\left\{\int_0^{m/n}[t^2n^3+t^3n^4]\,dt\right\}\\ &=o(1),\end{aligned}\right. \tag{4.8}$$

as m is arbitrarily fixed and $n \to \infty$. It remains to consider J_n'. In fact, similar to (2.11), we have

$$
(4.9)\quad \begin{cases} |J_n'| \leqq Kn^{2-\alpha}\displaystyle\int_{m/n}^{\varepsilon} \left| \frac{2\varphi(t)}{t^2} - A \right| t^{1-\alpha}\,dt \\ \leqq Kn^{2-\alpha}\left\{\varepsilon^{2-\alpha} + \left(\dfrac{m}{n}\right)^{2-\alpha}\right\} + Kn^{2-\alpha}\displaystyle\int_{m/n}^{\varepsilon} t^{-\alpha}\,dt \int_0^t \left| \frac{2\varphi(u)}{u^2} - A \right| du \\ \leqq Km^{2-\alpha}, \end{cases}
$$

which tends to zero as $m \to \infty$. Hence the result.

Finally it should be remarked that Bosanquet [1], [2], [3] has found a necessary and sufficient condition that the r-times differentiated Fourier series of $f(x)$ should be summable $(C, \alpha + r)$, for $\alpha \geqq 0$. He has also found a necessary and sufficient condition for the Cesàro summability $(C, \alpha + r)$, $\alpha \geqq 0$, of the successively derived allied series of a Fourier series. But his results are related to Cesàro-Lebesgue integrability of a certain function and Cesàro summability (C, α) of its Fourier series.

Added in proof: I just learnt from Prof. Kenji Yano that his paper: On Fejér kernels. Proc. Japan Acad. 35 (1959), 59–64, also contains some detailed estimates of the kernels $K_n^\alpha(x)$ and $\widetilde{K}_n^\alpha(x)$ which are related to results in Lemma 2 and Lemma 3.

REFERENCES

[1] BOSANQUET, L. S., Note on differentiated Fourier series. Quart. J. Math. **10** (1939), 67–74.

[2] BOSANQUET, L. S., A solution of the Cesàro summability problem for successively derived Fourier series. Proc. London Math. Soc. **46** (1940), 270–289.

[3] BOSANQUET, L. S., The Cesàro summability of the successively derived allied series of a Fourier series. Proc. London Math. Soc. **49** (1941), 63–76.

[4] CHEN, K. K., On the summability of the derived series of Fourier's series. Tôhoku Math. J. 31 (1929), 62–67.

[5] CHEN, K. K., Summation of the Fourier Series of Orthogonal Functions. Peking, 1957, pp. 82–86.

[6] GRONWALL, T. H., Über eine Summationsmethode und ihre Anwendung auf die Fourierreihe. J. f. Math. **147** (1916), 16–35.

[7] IZUMI, S., A theorem on the summability of the derived Fourier series. Tôhoku Math. J. **31** (1929), 68–71.

[8] PLESSNER, A., Über die Verhalten analytischer Funktionen am Rande ihres Definitionsbereiches. J. f. Math. **159** (1927), 219–227.

[9] PLESSNER, A., Trigonometrische Reihen, in Pascal's Repertorium der Höheren Analysis I, 3, pp. 1325–1396 (especially p. 1361).

[10] PRIWALOFF, I., Sur la dérivation des séries de Fourier. Rend. Circ. Math. Palermo 41 (1916), 202–206.

[11] WEISS, M., AND A. Zygmund, On the existence of conjugate functions of higher order. Fund. Math. 48 (1960), 175–187.

[12] YOUNG, W. H., On the usual convergence of a class of trigonometric series.

Proc. London Math. Soc. **13** (1914), 13–28.
[13] YOUNG, W. H., On the convergence of the derived series of Fourier series. Proc. London Math. Soc. **17** (1918), 195–236.
[14] ZYGMUND, A., Sur une série de M. Gronwall. Bull. de l'Académie Polonaise (1925), pp. 207–217.
[15] ZYGMUND, A., Trigonometric Series. Vol. 2, 2nd. ed. Cambridge, 1959.

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