ON THE APPROXIMATIONS TO SOME LIMITING DISTRIBUTIONS WITH SOME APPLICATIONS

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§ 1. Introduction

The Poisson approximations to the Poisson binomial distribution are recently discussed by Hodges and Le Cam [5] and Le Cam [7]. The purposes of this paper are to evaluate the error term more precisely than those obtained in [5] and [7] by making use of the theory of characteristic functions and also to remark some of its applications to statistics. This paper is the continuation of our previous paper [12] and may be regarded as the Part II of it.

Let X_k $(k=1, 2, \cdots)$ be independently distributed random variables such that $P(X_k=1)=p_k$, $P(X_k=0)=1-p_k$, we then call the distribution of $S=\sum_{k=1}^{n} X_k$ the Poisson binomial distribution $\boldsymbol{Q}$. The applications and the probability theoretical meaning of the Poisson approximation to the Poisson binomial distribution are described in [5] and [7].

Put

$$\lambda=\sum_{k=1}^{n} p_k, \quad \alpha=\max_k \{p_k\}, \quad \mu=\sum_{k=1}^{n} p_k^2, \quad \varpi=\frac{\mu}{\lambda},$$

and

$$P(X=k; \lambda)=P(k; \lambda)=\frac{\lambda^k}{k!}e^{-\lambda}.$$

We can state the previous results as follows:

(1) Kolmogorov [6]:

$$D=\sup_l \left| \sum_{k=0}^{l} P(S=k)-\sum_{k=0}^{l} P(k; \lambda) \right| \leqq C\alpha^{1/5}, \tag{1.1}$$

where C is a constant independent of n and p_k $(k=1, 2, \cdots)$.

(2) Hodges and Le Cam [5]:

$$D \leqq 3\alpha^{1/3}. \tag{1.2}$$

(3) Le Cam [7]:

$$D \leqq 9\alpha \tag{1.3}$$

and more strongly

$$D'=\sum_{k=0}^{\infty} | P(S=k)-P(k: \lambda) | = \| Q-P \| \leqq 9\alpha \tag{1.4}$$

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and also

(1.4′) $$D' \leqq 16\varpi.$$

As pointed out in [7], the above evaluations of the error term D of the approximation have an only theoretical meaning that it depends only on α and *not* on n and p_k or λ in dispite of our experience in the case as usual Poisson approximation. It is meaningful that the power of α in the above evaluations can be made 1, but in [7] in order to deduce the excellent result (1.4), one must use the semi-group theoretical method. While in [5] only an elementary probability theoretical discussions are used to deduce (1.2).

In the proceeding paper, it is shown that those results can be improved for not necessarily small α: $\alpha \leqq 1/4$. For the proof of (1.4), in [7] the condition $\alpha \leqq 1/4$ was necessary and it was remarked that the constant factor 9 can be reduced for more small α. And also Le Cam [7] suggested the inquiry about the existence of the bound lower than 9α or 16ϖ. This paper was written motivated by Hodges and Le Cam's work. Using the similar method as we have adopted in [11], [8] and [12] and Esseen [1], we can prove that

(1.5) $$D \leqq 3.7\alpha \quad \text{or} \quad 3.7\varpi$$

and

$$D \leqq 0.5\alpha + O(\alpha^2)$$

or

$$\leqq 0.5\varpi + 0.25\varpi^2 \Big/ \left(\frac{2}{5} - \varpi\right)^2 + 1.7\alpha\varpi \Big/ \left(1 - 2\alpha - \frac{5}{2}\varpi + \frac{5}{3}\alpha\varpi\right)$$

where in the first term, we can say that 0.5α and 0.5ϖ are the best possible ones in this form.

§ 2. Poisson approximations to Poisson binomial distribution.

In this parapraph, we can state the theorems in our previous paper [12], substituting the word "Poisson binomial" in the place of the word "binomial", and then express the absolute error of the approximation in term of α or ϖ.

Characteristic function (c.f.) of the Poisson binomial distributed random variable S is

(2.1) $$f_n(t) = \prod_{k=1}^{n} (p_k e^{it} + q_k)$$

and c.f. of X is

(2.2) $$g_n(t) = e^{\lambda(e^{it}-1)}.$$

For simplicity we divide the proof in several steps.

1°. $f_n(t)$ in (2.1) can be calculated as follows:

(2.3) $$f_n(t) = \prod_{k=1}^{n} (p_k e^{it} + q_k) = \prod_{k=1}^{n} \{1 + p_k(e^{it} - 1)\}.$$

Taking logarithm of the both sides of (2.3), we get

$$\log f_n(t) = \sum_{k=1}^{n} \log \{1 + p_k(e^{it} - 1)\}$$

(2.4)
$$= \sum_{k=1}^{n} p_k(e^{it} - 1) - \frac{1}{2} \sum_{k=1}^{n} p_k^2(e^{it} - 1)^2 + \Theta$$

where

$$\Theta = \sum_{k=1}^{n} \int_0^{p_k(e^{it}-1)} \frac{z^2}{1+z} dz$$

and

$$|\Theta| \leqq \sum_{k=1}^{n} \frac{1}{1-2p_k} \frac{p_k^3}{3} |e^{it} - 1|^3.$$

Hence it follows that

$$f_n(t) = \exp \{ \log f_n(t)\}$$

(2.5)
$$= e^{(\Sigma_{k=1}^{n} p_k)(e^{it}-1)} \cdot e^{-(1/2)(\Sigma_{k=1}^{n} p_k^2)(e^{it}-1)^2} \cdot e^{\Theta}$$

$$= e^{\lambda(e^{it}-1)} \cdot e^{-(1/2)\mu(e^{it}-1)^2} \cdot e^{\Theta}$$

where

$$\lambda = \sum_{k=1}^{n} p_k, \quad \mu = \sum_{k=1}^{n} p_k^2.$$

From (2.5), final expression for $f_n(t)$ follows:

$$f_n(t) = e^{\lambda(e^{it}-1)} - \frac{1}{2}\mu\{e^{\lambda(e^{it}-1)+2it} - 2e^{\lambda(e^{it}-1)+it} + e^{\lambda(e^{it}-1)}\}$$

$$+ \vartheta \frac{1}{2}\left(\frac{1}{2}\mu\right)^2 |e^{it}-1|^4 e^{(1/2)\mu|e^{it}-1|^2} e^{\lambda'e^{it}-1)}$$

(2.6)
$$+ \vartheta |\Theta| \cdot e^{|\Theta|} \cdot e^{-(1/2)\mu(e^{it}-1)^2} \cdot e^{\lambda(e^{it}-1)}$$

$$= I + I_1 + J_1 + J_2 = I + I_1 + R, \quad \text{say,}$$

where ϑ are unspecified complex-valued quantities such that $|\vartheta| \leqq 1$.

2°.

THEOREM 1. *For random variable S of Poisson binomial distribution and X of Poisson distribution defined in* §1, *we have the following evaluations*

(2.7)
$$\left| \sum_{k=l+1}^{l'} P(S = k) - \sum_{k=l+1}^{l'} P(k; \lambda) + \frac{1}{2}\mu\{\Delta^2 P(l') - \Delta^2 P(l)\} \right| \leqq R_1 + R_2$$

where

$$\Delta^2 P(k) = \{(P(k+2; \lambda) - P(k+1; \lambda)) - (P(k+1; \lambda) - P(k; \lambda))\}$$

and R_1, R_2 are in the following expressions (2.10) *and* (2.11).

Proof. We now proceed to the proof of the above inequality following the same way as we have taken for the proof of the Theorem 2 of our previous paper [12].

Partial sum of Poisson binomial probabilities can be expressed as follows:

$$\begin{aligned}\sum_{k=l+1}^{l'} P(S=k) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f_n(t)\cdot\left\{\sum_{k=l+1}^{l'} e^{-ikt}\right\}dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f_n(t)\, e^{-i(l+l'+1)t/2}\frac{\sin(l'-l)t/2}{\sin t/2}dt.\end{aligned} \tag{2.8}$$

Substitute $I+I_1+J_1+J_2$ obtained in (2.6) in the place of $f_n(t)$ in (2.8), we get then

$$\begin{aligned}\sum_{k=l+1}^{l'} P(S=k) &= \sum_{k=l+1}^{l'} \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\lambda(e^{it}-1)}e^{-ikt}dt \\ &\quad -\frac{1}{2}\mu\left[\sum_{k=l+1}^{l'}\frac{1}{2\pi}\int_{-\pi}^{\pi}\{e^{\lambda(e^{it}-1)+2it}-2e^{\lambda(e^{it}-1)+it}+e^{\lambda(e^{it}-1)}e^{-ikt}dt\right] \\ &\quad +\frac{1}{2\pi}\int_{-\pi}^{\pi} Re^{-i(l+l'+1)t/2}\frac{\sin(l-l')t/2}{\sin t/2} \\ &= \sum_{k=l+1}^{l'} P(k;\lambda)-\frac{1}{2}\mu(\Delta^2P(l';\lambda)-\Delta^2P(l;\lambda))+\boldsymbol{R}.\end{aligned} \tag{2.9}$$

The proof of our Theorem 1 can be accomplished by the evaluation of $\boldsymbol{R}$ in (2.9):

$$\begin{aligned}|\boldsymbol{R}| &= \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} Re^{-i(l-l'+1)t/2}\frac{\sin(l-l')t/2}{\sin t/2}dt\right| \\ &\leqq \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} J_1\frac{1}{|\sin t/2|}dt\right| + \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} J_2\frac{1}{|\sin t/2|}dt\right| \\ &= \boldsymbol{R}_1+\boldsymbol{R}_2, \quad \text{say.}\end{aligned}$$

For $\boldsymbol{R}_1$, we have

$$\begin{aligned}\boldsymbol{R}_1 &\leqq \frac{1}{8}\mu^2\frac{1}{2\pi}\int_{-\pi}^{\pi}|e^{it}-1|^4 e^{(1/2)\mu|e^{it}-1|^2}|e^{\lambda(e^{it}-1)}\frac{dt}{|\sin t/2|} \\ &\leqq \frac{1}{8}\mu^2\frac{1}{\pi}\int_0^{\pi} t^4 e^{(1/2)\mu t^2}e^{-2\lambda\sin t/2}\frac{dt}{t/\pi} \\ &\leqq \frac{\mu^2}{8\lambda^2}\int_0^{\infty}(\sqrt{\lambda}\,t)^3 e^{-(\sqrt{\lambda}\,t)^2/2A}d(\sqrt{\lambda}\,t) \\ &= \frac{1}{4}\left(\frac{\mu}{\lambda}\right)^2 A^2 = \frac{1}{4}\varpi^2A^2\end{aligned} \tag{2.10}$$

where

$$A=\left(\frac{4}{\pi^2}-\frac{\mu}{\lambda}\right)^{-1}\leqq\left(\frac{2}{5}-\varpi\right)^{-1}$$

We next evaluate the second error term $\boldsymbol{R}_2$ as follows: we have

$$\begin{aligned}\boldsymbol{R}_2 &\leqq \frac{1}{2\pi}\left|\int_{-\pi}^{\pi}|\Theta|e^{|\Theta|}\cdot e^{-1/2\mu(e^{it}-1)^2}\cdot e^{\lambda(e^{it}-1)}\frac{dt}{|t|/\pi}\right| \\ &\leqq \frac{2}{2\pi}\left(\sum_{k=1}^{n}\frac{1}{1-2p_k}\frac{p_k^3}{3}\right)\cdot\int_{-\pi}^{\pi}|t|^2\exp\left(\sum_{k=1}^{n}\frac{1}{1-2p_k}\frac{p_k^3}{3}\right)(|e^{it}-1|^3)\cdot e^{-(\sqrt{\lambda}\,t)^2/2A}\frac{dt}{|t|/\pi}\end{aligned}$$

(2.11)

$$\leqq \frac{B2}{\lambda}\int_0^\infty (\sqrt{\lambda}\,t)e^{-(\sqrt{\lambda}\,t)^2/2B'}d(\sqrt{\lambda}\,t)$$

$$= \frac{BB'2}{\lambda} = R_2$$

where

$$B = \sum_{k=1}^{n}\frac{1}{1-2p_k}\frac{p_k^3}{3} \leqq \frac{\alpha}{1-2\alpha}\cdot\frac{\mu}{3} = \frac{\alpha}{3(1-2\alpha)}\cdot\lambda\varpi$$

and

$$B' = \left(\frac{4}{\pi^2} - \frac{\mu}{\lambda} - \frac{4B}{\lambda}\right)^{-1} \leqq \left(\frac{2}{5} - \varpi - \frac{4}{3}\frac{\alpha}{1-2\alpha}\varpi\right)^{-1}.$$

Thus Theorem 1 is proved.

3°. Now we prove the following theorem which can be seen as a corollary of Theorem 1.

THEOREM 2. *If* $\alpha = \max_k p_k$ *is less than* 1/4, *we have*

$$\left|\sum_{k=l+1}^{l'} P(S=k) - \sum_{k=l+1}^{l'} P(X=k) + \frac{1}{2}\mu(\Delta^2 P(l') - \Delta^2 P(l))\right| \leqq D_1\alpha^2 \tag{2.12}$$

where

$$D_1 = \frac{1}{4}\left(\frac{4}{\pi^2} - \alpha\right)^{-2} + \frac{2}{3}(1-2\alpha)^{-1}\left(\frac{4}{\pi^2} - \alpha - \frac{4\alpha^2}{3(1-2\alpha)}\right)^{-1}. \tag{2.13}$$

Proof. For our proof, we need to evaluate $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ in terms of α. $\boldsymbol{R}_1$ can be majorated easily: Noting

$$\frac{\mu}{\lambda} = \sum_{k=1}^{n} p_k^2 \Big/ \sum_{k=1}^{n} p_k \leqq \max_k p_k = \alpha$$

we have

$$R_1 \leqq \frac{1}{4}\alpha^2\left(\frac{4}{\pi^2} - \alpha\right)^{-2}$$

and next as for $\boldsymbol{R}_2$ term, we have from (2.11)

$$R_2 = BB'\frac{2}{\lambda}$$

$$\leqq \frac{2}{3}\alpha\varpi\frac{1}{(1-2\alpha)\left(\frac{4}{\pi^2} - \varpi - \frac{4B}{\lambda}\right)}$$

$$\leqq \frac{2}{3}\alpha\varpi\frac{1}{(1-2\alpha)\left(\frac{4}{\pi^2} - \varpi - \frac{4}{3}\frac{\alpha}{1-2\alpha}\varpi\right)}$$

$$\leqq \frac{2}{3}(1-2\alpha)^{-1}\left(\frac{4}{\pi^2} - \alpha - \frac{4\alpha^2}{3(1-2\alpha)}\right)^{-1}\alpha^2.$$

Complements to theorem 2.

The evaluation of R_2 given by the above inequality can be deformed as follows:

$$\frac{2}{3}\alpha^2\frac{1}{(1-2\alpha)\left(\frac{4}{\pi^2}-\alpha-\frac{4\alpha^2}{3(1-2\alpha)}\right)}\leqq\frac{2}{3}\cdot\frac{\pi^2}{4}\cdot\frac{\alpha^2}{1-2\alpha-\frac{\pi^2}{4}\alpha(1-2\alpha)-\frac{\pi^2}{4}\cdot\frac{4}{3}\alpha^2}$$

(2.14)

$$\leqq\frac{5}{3}\frac{\alpha^2}{1-\frac{9}{2}\alpha+\frac{5}{3}\alpha^2}\qquad\left(\alpha<\frac{1}{4.2}\right)$$

and for $\alpha\leqq 1/5$ one may take

$$R_2\leqq 10\alpha^2 \tag{2.15}$$

while by the evaluation of R_1, we can take as

$$R_1\leqq\frac{\frac{1}{4}\alpha^2}{\left(\frac{4}{\pi^2}-\alpha\right)^2}\leqq\frac{25}{4}\alpha^2.$$

Here the constant factor D_1 in (2.12) is majorated by 16 that is right hand side of (2.12), $D_1\alpha^2$ can be replaced by $16\alpha^2$ if $\alpha\leqq 1/5$.

Concerning the evaluation in term of ϖ, one may find another discussion in §3.

4°.

THEOREM 3. *Under the same condition as in the above, we have*

$$\left|\sum_{k=l+1}^{l'}P(S=k)-\sum_{k=l+1}^{l'}P(X=k;\lambda)\right|$$

$$\leqq 0.5\alpha+D_1\alpha^2,\quad 0.5\varpi+0.25\varpi^2\cdot\frac{1}{\left(\frac{2}{5}-\varpi\right)^2}+R_2 \tag{2.17}$$

or

$$=0.5\alpha+O(\alpha^2),\quad 0.5\varpi+0.25\varpi^2\cdot\frac{1}{\left(\frac{2}{5}-\varpi\right)^2}+O(\alpha\cdot\varpi). \tag{2.17'}$$

For the *proof* we need the following

LEMMA.

$$\frac{1}{2}\mu\,|\,\Delta^2P(X=k:\lambda)\,|\leqq 0.5\alpha\quad\text{or}\quad 0.5\varpi$$

for ***all*** *k and* p_i.

Proof. From the fact $\mu\leqq\lambda\alpha$, we have

$$\frac{1}{2}\mu\,|\,\Delta^2P(k:\lambda)\,|\leqq\frac{1}{2}\alpha\lambda\,|\,\Delta^2P(k:\lambda)\,|$$

$$= \alpha \left| \lambda\left(\frac{\lambda^{k+1}}{(k+1)!} - \frac{\lambda^k}{k!}\right)e^{-\lambda} \right| \cdot \frac{1}{2}$$

or

$$= \alpha \cdot |\lambda - (k+1)| \frac{\lambda^{k+1}}{(k+1)!} e^{-\lambda} \cdot \frac{1}{2}.$$

This value attains its maximum value for fixed k when

$$\lambda = k + 1 + \left(\frac{1}{2} \pm \sqrt{k + 5/4}\right), \quad k \geqq 0,$$

hence we have

$$\frac{1}{2}\mu \,| \Delta^2 P(k\colon \lambda) |$$

$$\leqq \alpha \cdot \left(\frac{1}{2} \pm \sqrt{k + 5/4}\right) \cdot \max_{\lambda} \frac{\lambda^{k+1}}{(k+1)!} e^{-\lambda} \cdot \frac{1}{2}$$

$$= \alpha \cdot \left(\frac{1}{2} \pm \sqrt{k + 5/4}\right) \cdot \frac{(k+1)^{k+1}}{(k+1)!} e^{-(k+1)} \cdot \frac{1}{2}.$$

To evaluate $(k+1)!$ from the below, use the generalization of the Stirling's formula (c.f. Feller [2]), we then easily see that the maximum term is majorated by 0.5α. Or otherwise one may use the way of Hodges and Le Cam [5] who have evaluated the maximum term of the Poisson distribution $P(k; \lambda)$.

This completes the proof of our lemma. Thus the inequalities (2.17) and (2.17′) are proved.

5°. For the proof of the following Theorem which is analogue to the Theorem 3, we need only by far simpler calculations in sacrisfy of the fineness of the result of evaluation. I think one may utilize this one page proof as an example for the text book in probability theory.

THEOREM 4. *For $\alpha < 1/5$, we have*

$$\left| \sum_{k=l+1}^{l'} P(S = k) - \sum_{k=l+1}^{l'} P(k\colon \lambda) \right|$$

$$\leqq 1.3\alpha(1-2\alpha)^{-1}\left(1 - \frac{\alpha\pi^2}{4(1-2\alpha)}\right)^{-1} \leqq \frac{1.3\alpha}{1-4.5\alpha} \tag{2.18}$$

or

$$\leqq \frac{5}{4} \cdot \frac{\varpi}{1-2\alpha}\left(1 - \frac{5}{2} \cdot \frac{\varpi}{1-2\alpha}\right)^{-1}. \tag{2.18′}$$

Proof. We quote the formula for the expression of $f_n(t)$ as in the following simpler form:

$$f_n(t) = e^{\lambda(e^{it}-1) + \Theta''}$$

where

$$\Theta'' = \sum_{k=1}^{n} \frac{1}{1-2p_k} p_k^2 \frac{|e^{it}-1|^2}{2} \cdot \vartheta.$$

Hence we have

$$f_n(t)=e^{\lambda(e^{it}-1)}\{1+\vartheta\,|\,\Theta''\,|\,e^{|\Theta''|}\}$$

and from the inversion formula

$$\sum_{k=l+1}^{l'}P(S=k)=\sum_{k=l+1}^{l'}\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{\lambda(e^{it}-1)}e^{-ikt}dt+R$$

where

$$\begin{aligned}|R|&\leqq\frac{1}{2\pi}\left|\int_{-\pi}^{\pi}|\Theta''|\,e^{|\Theta''|}\cdot e^{\lambda(e^{it}-1)}\cdot e^{-i(l+l'+1)t/2}\frac{\sin(l'-l)t/2}{\sin t/2}\vartheta\,dt\right|\\&\leqq\frac{1}{2\pi}\frac{\mu}{2(1-2\alpha)}\cdot 2\int_0^{\pi}\exp\left(\frac{\mu}{1-2\alpha}\frac{t^2}{2}\right)\cdot t^2\exp\left(-\frac{4\lambda}{\pi^2}\frac{t^2}{2}\right)\frac{dt}{t/\pi}\\&\leqq\frac{\mu/\lambda}{2(1-2\alpha)}\int_0^{\infty}\exp\left(-\left(\frac{4}{\pi^2}-\frac{\mu/\lambda}{1-2\alpha}\right)\frac{\lambda t^2}{2}\right)d\left(\frac{\lambda t^2}{2}\right)\\&\leqq\frac{\varpi}{2}(1-2\alpha)^{-1}\left(\frac{4}{\pi^2}-\frac{\varpi}{1-2\alpha}\right)^{-1}\\&\leqq\frac{\pi^2}{4}\frac{\varpi}{2}(1-2\alpha)^{-1}\left(1-\frac{\varpi\pi^2}{4(1-2\alpha)}\right)^{-1}\\&\leqq\frac{5}{4}\varpi(1-2\alpha)^{-1}\left(1-\frac{5}{2}\frac{\varpi}{1-2\alpha}\right)^{-1}.\end{aligned}$$

Thus our theorem has been proved.

§ 3. On further considerations of the Theorems in the previous sections.

As a complements to the above Theorems we mention the following propositions.

1) The fact that the results (2.17) and (2.17′) can be replaced by the following estimates will be shown easily from the above discussions.

$$\begin{aligned}D&\leqq 0.5\varpi+0.25\left(\frac{\varpi}{(2/5)-\varpi}\right)^2+1.7\alpha\varpi\Big/\left(1-2\alpha-\frac{5}{2}\varpi+\frac{5}{3}\alpha\varpi\right)\\&=0.5\varpi+0.25\left(\frac{\varpi}{(2/5)-\varpi}\right)^2+1.7\alpha\varpi\Big/\left((1-2\alpha)-\frac{5}{6}(3-2\alpha)\right)\end{aligned}\tag{3.1}$$

where

$$\alpha<\frac{1}{2},\quad \varpi<\frac{2}{5}\quad\text{and}\quad 1\geqq 2\alpha+\frac{5}{2}\varpi-\frac{5}{3}\alpha\varpi.\tag{3.2}$$

One may wish to have the form

$$D\leqq 0.5\varpi+C\varpi^2\tag{3.3}$$

for this estimation of the error term, but by our method this try was failed. We have only the following propositions.

2) For $\alpha\leqq 1/5$, the evaluations so far obtained give the results:

$$\begin{aligned}D&\leqq 2.5\varpi+0.25\varpi^2\Big/\left(\frac{2}{5}-\varpi\right)^2\\&\leqq 2.5\varpi+6.3\varpi^2.\end{aligned}$$

This fact may be replaced in terms of α:

$$(3.4)\qquad \begin{aligned} D &\leqq 0.5\alpha + 16\alpha^2 \\ &\leqq 3.7\alpha \quad \text{or} \quad 3.7\varpi . \end{aligned}$$

This evaluation is a partial improvement of Le Cam's result [7].

3) In proposition 1), the form of the right hand side of inequality (3.1) is complicated. But only by this evaluation we can give the refined upper bound for smaller α or ϖ, and if it is necessary, we can easily reach to simpler form from (3.1) under the some restrictions on α and ϖ. Here it is also be seen that in leading to (3.1) the conditions were complicated but *not* restrictive.

As an example for this discussion we have the following proposition.

4) Result of Theorem 3 shows when $\alpha = 1/4$ but $\varpi = 1/100$ (the same values which were taken up by Le Cam [7], p. 1156, Remark 1 as an illustration) that we have

$$D \leqq 1.4\varpi$$

and from the result of Theorem 4 we have

$$D \leqq 2.5\varpi$$

while by Le Cam, $D' \leqq 8\varpi$. Note that to compare with the Le Cam's result, one must double the upper bound in general, because Le Cam's upper bound was calculated for D' and not for D.

5) We can say the first term of the evaluation in Theorem 3, 0.5α and 0.5ϖ is best possible in this form. This can also be recognized by Prohorov's result [13], which concerns with the approximation to binomial distribution, as pointed out by Le Cam [7]. We can also say from Theorem 1 and 2, the main terms for the evaluation of $|Q-P|$ are influenced by that of $|B-P|$. That is 0.5α or 0.5ϖ is the main term of evaluations for both $|Q-P|$ and $|B-P|$. This fact was noted by Le Cam [7], and will be seen easily from our results [12], [10]. In the forthcoming paper similar result may be noted [9].

6) The evaluation of the error term in Theorem 4 is inferior to that of the Theorem 3, but the derivation is simpler, we used only Taylor expansion.

§ 4. Applications to some problems.

1°. A production model.

When sampling, each fraction defectives is decided as a sample from some population with mean proportional to the time passed from the first sampling, where population distribution is not indicated. If necessary, we may suppose it is beta-distributed or gamma-distributed as we experience in practice.

To obtain the probability for the occurrence of k defectives (define $X = k$, where X is a random variable) in n-samples, we must calculate the characteristic function of X as follows:

$$\phi_X(t) = \prod_{k=1}^{n} \int_0^1 (p_k e^{it} + q_k) f_k(p_k)\, dp_k = \prod_{k=1}^{n} (\bar{p}_k e^{it} + \bar{q}_k) \tag{4.1}$$

where $\bar{p}_k$ is mean fraction defective which may be assumed to be proportional to k and $\bar{q}_k = 1 - \bar{p}_k$ (in this discussion, proportionality is not essential).

This means that distribution of X is Poisson binomial with p_k substituted by $\bar{p}_k$. Hence i) when $\bar{p}_k$ are all small, we can apply the results of § 2, and ii) when variation of $\bar{p}_k$ is small, results of [9] are useful.

As an example we also consider the other one type of the production model. We assume two values p and $p + \Delta p$ for the fraction defectives (f.d.), and p and Δp are both small as indicated in the following.

At the first sampling the probability that f.d. is p is $1 - \alpha$, and at the second sampling the probability that f.d. is still p is $1 - \alpha$ and so on. Assume also when f.d. become $p + \Delta p$, f.d. continues to be $p + \Delta p$ till sampling ends at n-th sampling. Concering these models, we will give the full discussions in the forthcoming paper [10].

2°. A type of the conditional probability.

Let X_1, X_2 be independently distributed random variables obeying to the binomial distribution:

$$P(X_1 = k) = b(k; n_1, p_1), \quad P(X_2 = k') = b(k'; n_2, p_2)$$

respectively. Problem is to investigate the conditional probability

$$f_k = P(X_1 = k \mid X_1 + X_2 = c). \tag{4.2}$$

Approximation formula for this probability f_k were obtained by Hannan and Harkness [3] under the conditions such as to facilitate the normal approximation to f_k.

In this note, under the conditions $n_i p_i \sim \lambda_i$ for $n_i \to \infty$ $(i = 1, 2)$, it is shown that

$$f_k = \frac{c!}{k!\,(c-k)!} \left(\frac{\lambda_1}{\lambda_1 + \lambda_2} \right)^k \left(\frac{\lambda_2}{\lambda_1 + \lambda_2} \right)^{c-k} + O(p_i). \tag{4.3}$$

Its proof can be proceeded as follows:

$$\begin{aligned} f_k &= (X_1 = k \mid X_1 + X_2 = c) \\ &= \frac{P(X_1 = k) \cdot P(X_2 = c - k)}{P(X_1 + X_2 = c)} \\ &= \frac{e^{-\lambda_1} \lambda_1^k}{k!} \cdot \frac{e^{-\lambda_2} \lambda_2^{c-k}}{(c-k)!} \Big/ \frac{e^{-(\lambda_1 + \lambda_2)} (\lambda_1 + \lambda_2)^c}{c!} + O(p_i) \\ &= \binom{c}{k} \left(\frac{\lambda_1}{\lambda_1 + \lambda_2} \right)^k \left(\frac{\lambda_2}{\lambda_1 + \lambda_2} \right)^{c-k} + O(p_i) \end{aligned} \tag{4.4}$$

and when the random variables were X_1 X_2, X_3, we have under the similar conditions

$$f_{k_1,k_2}=P(X_1=k_1, X_2=k_2 \mid X_1+X_2+X_3=c)$$

(4.5)

$$=\binom{c}{k_1, k_2}\left(\frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3}\right)^{k_1}\left(\frac{\lambda_2}{\lambda_1+\lambda_2+\lambda_3}\right)^{k_2}\left(\frac{\lambda_3}{\lambda_1+\lambda_2+\lambda_3}\right)^{c-(k_1+k_2)}+O(p_i).$$

Hannan and Harkness' object is to apply their results to calculate the power function of the test of the independency in 2×2 contingency tables. In the forthcoming paper [10] we also consider the applicability of our results.

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