ON FRACTIONAL ANGULAR DERIVATIVE

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0. Introduction.

Let $f(z)$ be an analytic function regular and with positive real part in the half-plane $\Re z > 0$. It is well known by the main theorem of Julia [2] and Wolff [4] on angular derivative that there exists a non-negative real constant c such that

$$\frac{f(z)}{z} \to c \quad \text{and} \quad f'(z) \to c$$

uniformly as $z \to \infty$ through any Stolz angle $|\arg z| \leqq \alpha < \pi/2$. In a recent paper [3] this theorem was supplemented by the corresponding behavior of the derivative of higher order which states that $f(z)$ satisfies further limit relations

$$z^{n-1} f^{(n)}(z) \to 0 \qquad (n = 2, 3, \cdots)$$

and here the power $n-1$ of z is best possible.

On the other hand, let $F(z)$ be an analytic function regular and satisfying $|F(z)| < 1$ in the unit circle $|z| < 1$. The main theorem of Carathéodory [1] on angular derivative states that there exists a positive real constant D, eventually equal to $+\infty$, such that

$$\frac{1-F(z)}{1-z} \to D \quad \text{and} \quad F'(z) \to D$$

uniformly as $z \to 1$ through any Stolz angle in $|z| < 1$ with the vertex at $z = 1$, where in the second relation D is supposed finite. In [3] this theorem was supplemented by the statement that $F(z)$ further satisfies

$$(1-z)^{n-1} F^{(n)}(z) \to 0 \qquad (n = 2, 3, \cdots)$$

and here the power $n-1$ of $1-z$ is best possible.

A glance calls, however, our attention to a remarkable distinction between the limit values of $z^{n-1} f^{(n)}(z)$ or $(1-z)^{n-1} d^n(F(z)-1)/dz^n$ with $n = 0, 1$ and $n = 2, 3, \cdots$. In the present paper we shall interpolate these limit relations by introducing a notion of fractional calculus. By means of doing so, every set of these results will be unified and further it will become naturally clear why there is an apparent distinction mentioned above.

Now, we explain the fractional calculus which will be used below. Let, in

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general, $g(z)$ be an analytic function regular in a domain G of the nature that any point $z \in G$ can be connected with a fixed point z_0 by a rectilinear segment in G except possibly the end-point z_0. We first consider the fractional integration. Let q be any positive real number. The integral of order q of $g(z)$ is then defined by

$$\mathfrak{D}^{-q}g(z) = \frac{1}{\Gamma(q)}\int_{z_0}^{z}(z-\zeta)^{q-1}g(\zeta)\,d\zeta$$

where the branch of $(z-\zeta)^{q-1} \equiv \exp((q-1)\lg(z-\zeta))$ is determined by taking the principal value of logarithm, i. e. $-\pi < \arg(z-\zeta) \equiv \Im \lg(z-\zeta) \leqq \pi$ and the integration is supposed to be taken along the rectilinear segment connecting z_0 with z. Putting $\zeta = z_0 + t(z-z_0)$, we get

$$\mathfrak{D}^{-q}g(z) = \frac{(z-z_0)^q}{\Gamma(q)}\int_0^1(1-t)^{q-1}g(z_0+t(z-z_0))\,dt.$$

This may be also regarded as the derivative of (negative) order $-q$. For $q=0$, the operation $\mathfrak{D}^0 \equiv \mathfrak{D}^{-0}$ is understood, of course, to reduce the identy:

$$\mathfrak{D}^0 g(z) \equiv \lim_{q\to+0}\mathfrak{D}^{-q}g(z) = g(z).$$

We next consider the fractional differentiation. Let p be any positive real number and put

$$p = n - s; \qquad n = -[-p] \text{ and } 0 \leqq s < 1;$$

we may write here $n = [p]+1$ unless p is an integer while $n = p$ for an integer p. The derivative of order p of $g(z)$ is then, by definition, given by

$$\mathfrak{D}^p g(z) = \mathfrak{D}^n \mathfrak{D}^{-s} g(z), \qquad \mathfrak{D}^n \equiv \frac{d^n}{dz^n}.$$

If p is an integer, this coincides, of course, with the ordinary derivative of pth order. Otherwise, it is written in the form

$$\mathfrak{D}^p g(z) = \frac{d^n}{dz^n}\frac{1}{\Gamma(s)}\int_{z_0}^{z}(z-\zeta)^{s-1}g(\zeta)\,d\zeta,$$

and, putting again $\zeta = z_0 + t(z-z_0)$, it becomes

$$\mathfrak{D}^p g(z) = \frac{d^n}{dz^n}\frac{(z-z_0)^s}{\Gamma(s)}\int_0^1(1-t)^{s-1}g(z_0+t(z-z_0))\,dt.$$

For the sake of brevity, we shall take $z_0 = 1$ throughout the following discussions. The point z_0 is then an interior point for $\Re z > 0$ while it is a boundary point for $|z| < 1$.

1. Functions with positive real part in a half-plane.

We begin with proving the following theorem on $f(z)$.

THEOREM 1. *Let $f(z)$ be an analytic function regular and with positive real part in the half-plane $\Re z > 0$. Then there exists a non-negative real constant c such that the derivative of $f(z)$ of any real order p satisfies the limit relation*

$$\lim_{z\to\infty} z^{p-1}\mathscr{D}^p f(z) = \frac{c}{\Gamma(2-p)}$$

valid uniformly as z tends to ∞ through any angular region $|\arg z| \leqq \alpha < \pi/2$.

Proof. Suppose first $p = -q < 0$. Since $f(z)/z \to c$ uniformly as $z \to \infty$ along any Stolz path, we have the relation

$$f(1+t(z-1)) = (ct + o(1))z$$

valid uniformly in the wider sense for $0 < t \leqq 1$. Hence we obtain

$$\begin{aligned}\mathscr{D}^{-q} f(z) &= \frac{(z-1)^q}{\Gamma(q)}\int_0^1 (1-t)^{q-1}(ct+o(1))z\,dt \\ &= \frac{z^{q+1}}{\Gamma(2+q)}(c+o(1)),\end{aligned}$$

i. e.

$$z^{p-1}\mathscr{D}^p f(z) = \frac{c}{\Gamma(2-p)} + o(1).$$

The last relation holds, of course, also for $p = 0$.

Suppose next $p > 0$. If p is an integer, $1/\Gamma(2-p)$ is equal to 1 or 0 for $p = 1$ or $p \geqq 2$ respectively, and hence the result to be shown reduces simply to the previous known one. Consequently, we may assume that p is not an integer. Put, as before, $p = n - s$ with $n = [p] + 1$. Then the defining equation of the pth derivative

$$\mathscr{D}^p f(z) = \frac{d^n}{dz^n}\frac{(z-1)^s}{\Gamma(s)}\int_0^1 (1-t)^{s-1} f(1+t(z-1))\,dt$$

becomes, by performing out the repeated differentiation,

$$\mathscr{D}^p f(z) = s\sum_{\nu=0}^{n}\binom{n}{\nu}\frac{(z-1)^{s-n+\nu}}{\Gamma(s-n+\nu+1)}\int_0^1 (1-t)^{s-1} t^\nu f^{(\nu)}(1+t(z-1))\,dt.$$

Applying the known asymptotic behaviors of $f^{(\nu)}(z)$ with integral ν, we have the relations

$$f^{(\nu)}(1+t(z-1)) = \begin{cases} (ct+o(1))z & (\nu = 0), \\ c + o(1) & (\nu = 1), \\ o(1)z^{1-\nu} & (\nu = 2, \cdots, n) \end{cases}$$

valid uniformly in the wider sense for $0 < t \leqq 1$. Inserting them, it follows that

$$\begin{aligned}\mathcal{D}^p f(z) &= \frac{s(z-1)^{s-n}}{\Gamma(s-n+1)}\int_0^1 (1-t)^{s-1}(ct+o(1))\,z\,dt \\ &\quad + \frac{sn(z-1)^{s-n+1}}{\Gamma(s-n+2)}\int_0^1 (1-t)^{s-1}t(c+o(1))\,dt + \sum_{\nu=2}^{\infty} z^{s-n+\nu}\cdot o(1)\,z^{1-\nu} \\ &= \frac{z^{s-n+1}}{\Gamma(s-n+2)}(c+o(1)),\end{aligned}$$

i. e.

$$z^{p-1}\mathcal{D}^p f(z) = \frac{c}{\Gamma(2-p)} + o(1).$$

This proves the theorem.

2. Functions bounded in a circle.

A theorem similar to theorem 1 holds for $F(z)$. It is remarked that, since $F(z)$ is supposed bounded, its fractional derivative is well defined when the boundary point 1 is taken as the lower bound of the integral involved.

THEOREM 2. *Let $F(z)$ be an analytic function regular and satisfying $|F(z)|<1$ in the unit circle $|z|<1$. Then there exists a positive real constant D, eventually equal to $+\infty$, such that the derivative of $F(z)$ of any real order p satisfies the limit relation*

$$\begin{aligned}\lim_{z\to1}(z-1)^{p-1}\mathcal{D}^p(F(z)-1) \equiv \lim_{z\to1}\left((z-1)^{p-1}\mathcal{D}^p F(z) - \frac{1}{\Gamma(1-p)}\frac{1}{z-1}\right) \\ = \frac{D}{\Gamma(2-p)}\end{aligned}$$

valid uniformly as z tends to 1 through any angular region $|\arg(1-z)|\leqq\alpha<\pi/2$ in $|z|<1$. Here it is supposed that D is finite when p is positive.

Proof. The proof of the present theorem proceeds quite similarly as that of theorem 1. Suppose first $p=-q<0$ and $D\neq+\infty$. Since as $z\to1$ along any Stolz path, we have the relation

$$F(1+t(z-1))-1=(Dt+o(1))(z-1)$$

valid uniformly in the wider sense for $0<t\leqq1$. Hence we get

$$\begin{aligned}\mathcal{D}^{-q}(F(z)-1) &= \frac{(z-1)^q}{\Gamma(q)}\int_0^1(1-t)^{q-1}(Dt+o(1))(z-1)\,dt \\ &= \frac{(z-1)^{q+1}}{\Gamma(2+q)}(D+o(1)),\end{aligned}$$

i. e.

$$(z-1)^{p-1}\mathcal{D}^p(F(z)-1) = \frac{D}{\Gamma(2-p)} + o(1).$$

Let $D = +\infty$. We write

$$\Re((z-1)^{-q-1}\mathcal{D}^{-q}(F(z)-1)) = \frac{1}{\Gamma(q)}\int_0^1 (1-t)^{q-1}t\,\Re\frac{F(1+t(z-1))-1}{t(z-1)}\,dt.$$

Since, as $z \to 1$ along any Stolz path, $\Re((F(1+t(z-1))-1)/(t(z-1)))$ approaches $+\infty$ uniformly in the wider sense for $0 < t \leqq 1$, the right side of the last equation also tends to $+\infty$. Consequently, the required limit relation holds also in this case.

The result for $p = 0$ holds trivially.

Suppose next $p > 0$ and $D \neq +\infty$. Based on the same reason as in the proof of theorem 1, we may suppose that p is not an integer. Put $p = n - s$ with $n = [p]+1$. Then, performing out the repeated differentiation, the defining equation of the pth derivative becomes

$$\mathcal{D}^p(F(z)-1) = \frac{s(z-1)^{s-n}}{\Gamma(s-n+1)}\int_0^1 (1-t)^{s-1}(F(1+t(z-1))-1)\,dt$$
$$+ s\sum_{\nu=0}^{n}\binom{n}{\nu}\frac{(z-1)^{s-n+\nu}}{\Gamma(s-n+\nu+1)}\int_0^1 (1-t)^{s-1}t^{\nu}F^{(\nu)}(1+t(z-1))\,dt.$$

In view of the known asymptotic behaviors of $F^{(\nu)}(z)$ with integral ν, we have

$$F^{(\nu)}(1+t(z-1)) = \begin{cases} 1+(Dt+o(1))(z-1) & (\nu=0), \\ D+o(1) & (\nu=1), \\ o(1)(z-1)^{1-\nu} & (\nu=2,\cdots,n) \end{cases}$$

valid uniformly in the wider sense for $0 < t \leqq 1$, provided $D \neq +\infty$. It follows that

$$\begin{aligned}\mathcal{D}^p(F(z)-1) &= \frac{s(z-1)^{s-n}}{\Gamma(s-n+1)}\int_0^1 (1-t)^{s-1}(Dt+o(1))(z-1)\,dt \\ &\quad + \frac{sn(z-1)^{s-n+1}}{\Gamma(s-n+2)}\int_0^1 (1-t)^{s-1}t(D+o(1))\,dt \\ &\quad + \sum_{\nu=2}^{n}(z-1)^{s-n+\nu}\cdot o(1)(z-1)^{1-\nu} \\ &= \frac{(z-1)^{s-n+1}}{\Gamma(s-n+2)}(D+o(1)),\end{aligned}$$

i.e.

$$(z-1)^{p-1}\mathcal{D}^p(F(z)-1) = \frac{D}{\Gamma(2-p)} + o(1).$$

Since we have $\mathcal{D}^p 1 = (z-1)^{-p}/\Gamma(1-p)$, this proves the theorem.

It now becomes evident why the principal part in the asymptotic formula for $\mathcal{D}^p f(z)$ or $\mathcal{D}^p(F(z)-1)$ with non-integral p and integral $p \leqq 1$ is exactly of order z^{-p+1} or $(z-1)^{-p+1}$, respectively, while that with integral $p \geqq 2$ is of lower order. In fact, $1/\Gamma(2-p)$ is an entire function of p whose zero points coincide just with the integers not less than 2.

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