THE ADJOINT PROCESS OF A DIFFUSION WITH REFLECTING BARRIER

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1. Introduction.

Nelson [8] introduced the notion of the adjoint process and discussed the adjoint process of a recurrent diffusion without boundary.

We shall construct the adjoint process, which is unique by the uniqueness of its invariant measure, of a diffusion with reflecting barrier, a typical recurrent diffusion with boundary and prove that it is again a diffusion with reflecting barrier and that the results proved by Nelson [8] for self-adjoint diffusion remain valid.

Then, we shall determine the adjoint process of "the Markov process on the boundary" of the diffusion with reflecting barrier introduced by Ueno [11] (cf. [3]) in connection with the construction of diffusions with Wentzell's boundary conditions.

A comment will be given about processes with more general boundary conditions.

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2. Preliminaries.

Let a compact metric space S be a state space, W a path space of all right continuous path functions w's from $T=[0, \infty)$ to S, $B(S)$ the topological Borel field of subsets of S, $B(W)$ be the Borel field of subsets of W generated by cylinder sets, and let $\{P_x, x\in S\}$ be a system of probability measures on $B(W)$ satisfying the Markovian property. And let $\boldsymbol{M}=\{W; B(W); P_x, x\in S\}$ be a Markov process.[1] Denote by $\mathscr{B}(S)$ the space of all bounded $B(S)$-measurable functions and by $C(S)$ the space of all continuous functions on S. These are both Banach spaces with sup-norm. We write:

(i) $P(t, x, E)=P_x(x_t(w)\in E)$ for $t\in T$, $x\in S$, and $E\in B(S)$;

(ii) $T_t f(x)=E_x(f(x_t(w)))$ for $f\in\mathscr{B}(S)$;

(iii) $$G_\alpha f(x)=E_x\left(\int_0^\infty e^{-\alpha t} f(x_t(w))\,dt\right)=\int_0^\infty e^{-\alpha t}T_t f(x)dt$$ for $f\in\mathscr{B}(S)$ and $\alpha>0$,

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1) Definitions and notations concerning Markov processes are mainly those of Ito [4].

and call them transition probabilities, semi-groups, and resolvents (or Green operators) of the process, respectively.

We assume in addition that the process $\boldsymbol{M}$ has a property:

(A.1) *G_α carries $C(S)$ into $C(S)$ for each $\alpha > 0$.*

Then, $\boldsymbol{M}$ is *strongly Markovian* by the theorem of Watanabe [12]. In the following G_α will be restricted on $C(S)$. The generator $\boldsymbol{G}$ of the process is defined by

$$\boldsymbol{G}f = (\alpha - G_\alpha^{-1})f \qquad \text{for } f \in \mathscr{D}(\boldsymbol{G}), \tag{2.1}$$

where the domain $\mathscr{D}(\boldsymbol{G})$ of $\boldsymbol{G}$ is the range of G_α. Then we have

$$(\alpha - \boldsymbol{G})G_\alpha f = f \qquad \text{for } f \in C(S). \tag{2.2}$$

If $\boldsymbol{M}$ satisfies an additional condition

$$P_x(x_t(w) \in V \quad \textit{for some } 0 < t < \infty) = 1, \tag{A.2}$$

for any non-null open subset V of S and any $x \in S$, it is said to be *recurrent*. A necessary and sufficient condition of the recurrence of the process $\boldsymbol{M} = \{W; B(W); P_x, x \in S\}$ is given by

THEOREM 2.1. *The process $\boldsymbol{M}$ is recurrent if and only if $P_x(\sigma_V < \infty) > 0$, for any non-null open set V and $x \in S$, where σ_V is the first passage time to V.*[2)]

The necessity is obvious. In order to show the sufficiency, we prove the next stronger

PROPOSITION 2.1. *$E_x(\sigma_V(w)) < \infty$, for any $x \in S$ and any non-null open subset V of S.*

Proof.[3)] Take a non-null open subset V_1 ($\bar{V}_1 \subset V$) and let f be a non-negative continuous function on S which is positive on V_1 and vanishes on V^c.[4)] Then $u = G_\alpha f$ satisfies, for any $x \in V^c$,

$$\boldsymbol{G}u(x) = \alpha G_\alpha f(x) = \alpha E_x\left(\int_0^\infty e^{-\alpha t} f(x_t(w))dt\right) > 0,$$

by the assumption. Noting that V^c is compact and $\boldsymbol{G}u$ belongs to $C(S)$, we find that

$$\boldsymbol{G}u(x) \geqq \varepsilon > 0 \qquad \text{on } V^c.$$

Applying the Dynkin's formula to $\sigma_V \wedge n$ and letting $n \to \infty$, we have

$$\varepsilon E_x(\sigma_V(w)) \leqq 2\,\|u\|,$$

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2) $\sigma_V(w) = \inf\{t: x_t(w) \in V\}$ if such t exists, and $= \infty$ otherwise.

3) This was communicated from K. Satō. The author's original proof was available only for A-diffusions.

4) We denote by V^c the complement of V.

for any $x \in S$ by the conservativity of the process $\boldsymbol{M}$. This concludes the proof.

Let m be a finite *invariant measure* of the process $\boldsymbol{M}$, that is, a finite measure on $B(S)$ satisfying

$$\int_S P(t, x, \cdot)\, m(dx) = m(\cdot) \qquad \text{for each } t > 0. \tag{2.3}$$

As the equivalent condition of the invariance of m, we have

PROPOSITION 2.2. *The following conditions are equivalent to each other; for a finite measure m on $B(S)$,*

(i) *m is a invariant measure of $\boldsymbol{M}$,*

(ii) $\int_S T_t f(x)\, m(dx) = \int_S f(x)\, m(dx)$ *for any $f \in C(S)$,*

(iii) $\int_S \alpha G_\alpha f(x)\, m(dx) = \int_S f(x)\, m(dx)$ *for any $f \in C(S)$,*

(iv) $\int_S G f(x)\, m(dx) = 0$ *for any $f \in \mathscr{D}(G)$.*

Proof. The equivalence of (i) and (ii) is obvious. We have (ii)$\to$(iii) by Fubini's theorem. According to right continuity of the path, the left hand side of (ii) is right continuous in t, so (iii)$\to$(ii) is obtained by the uniqueness of Laplace transform. The equivalence of (iii) and (iv) is implied by (2.2), that is, by the equality

$$\int_S G G_\alpha f(x)\, m(dx) = \int_S \alpha G_\alpha f(x)\, m(dx) - \int_S f(x)\, m(dx),$$

for any $f \in C(S)$, completing the proof.

We define now the adjointness of Markov processes. Let $\boldsymbol{M}^i = \{W; B(W); P_x^i, x \in S\}$ $(i = 1, 2)$ be Markov processes with the common path space W and a common finite invariant measure m.

Then, the processes $\boldsymbol{M}^1$ and $\boldsymbol{M}^2$ are said to be *adjoint to one another*, if they satisfy, letting T_t^1 and T_t^2 be semi-groups of $\boldsymbol{M}^1$ and $\boldsymbol{M}^2$ respectively,

$$\int_S T_t^1 f(x)\, g(x)\, m(dx) = \int_S f(x)\, T_t^2 g(x)\, m(dx), \tag{2.4}$$

for any $f, g \in C(S)$, and we write $(\boldsymbol{M}^1)^* = \boldsymbol{M}^2$ (or $\boldsymbol{M}^1 = (\boldsymbol{M}^2)^*$). If $\boldsymbol{M} = \boldsymbol{M}^*$, it is said to be *self-adjoint* (*Umkehrbarkeit* in Kolmogoroff's terminology [6]).

REMARK 2.1. The adjointness can be defined with respect to a sub-invariant (or excessive) measure. But, if the process is conservative, a sub-invariant measure is an invariant measure. Therefore, we shall need not consult with sub-invariant measure in what follows.

PROPOSITION 2.3. (2.4) *is equivalent to the following conditions:*

(2.5) $$\int_S G_\alpha^1 f(x)\, g(x)\, m(dx) = \int_S f(x)\, G_\alpha^2 g(x)\, m(dx),$$

for any $f, g \in C(S)$ *and* $\alpha > 0$; *and*

(2.6) $$\int_S \boldsymbol{G}^1 f(x)\, g(x)\, m(dx) = \int_S f(x)\, \boldsymbol{G}^2 g(x)\, m(dx),$$

for any $f \in \mathfrak{D}(\boldsymbol{G}^1)$ *and* $g \in \mathfrak{D}(\boldsymbol{G}^2)$, *where* G_α^i *and* $\boldsymbol{G}^i$ *are the resolvents and the generators of* $\boldsymbol{M}^i$ $(i=1,2)$, *respectively.*

The proof is completed by the same way as proposition 2.2.

3. The adjoint process of A-diffusion with reflecting barrier.

Let D be a connected domain with compact closure $\bar{D}$ in an N-dimensional orientable manifold of class C^∞, and the boundary ∂D consists of a finite number of $N-1$-dimensional hypersurface of class C^3. Let $\boldsymbol{M} = \{W_c;\ B(W_c);\ P_x,\ x \in \bar{D}\}$ be an A-diffusion with reflicting barrier, where W_c is the path space of all continuous path functions w's from $T=[0,\infty)$ to $\bar{D}$. To be precise, given a diffusion equation with boundary condition

(3.1) $$\begin{aligned} &\frac{\partial u(t,x)}{\partial t} = Au(t,x) \qquad \text{for } x \in D,\ t>0, \\ &Lu(x) = 0 \qquad\qquad\quad\ \ \text{for } x \in \partial D, \end{aligned}$$

where

(3.2) $$\begin{aligned} Au(x) &= \Delta u(x) + b^i(x) \frac{\partial u(x)}{\partial x^i}, \\ Lu(x) &= \frac{\partial u(x)}{\partial n} \equiv a^{ij}(x)\, n_j(x) \frac{\partial u(x)}{\partial x^i}, \end{aligned}$$

and

$$\begin{aligned} \Delta u(x) &= \frac{1}{\sqrt{a(x)}} \frac{\partial}{\partial x^i}\left(\sqrt{a(x)}\, a^{ij}(x) \frac{\partial u(x)}{\partial x^j}\right), \\ n_j(x) &= \left(a^{ij}(x) \frac{\partial \psi(x)}{\partial x^i} \frac{\partial \psi(x)}{\partial x^j}\right)^{-1/2} \frac{\partial \psi(x)}{\partial x^j} \end{aligned}$$

are differential operators satisfying Ito's regularity conditions,[5] denote by $p(t,x,y)$ the fundamental solution of (3.1) (cf. Ito [5]), and define a system of transition probabilities $\{P(t,x,E)\}$ by

5) $a^{ij}(x)$ and $b^i(x)$ are contravariant tensors on $\bar{D}$, $a^{ij}(x)$ is strictly positive definite on $\bar{D}$, $\partial^2 a^{ij}/\partial x^k \partial x^l$, $\partial b^i/\partial x^k$ are uniformly Hölder continuous, and $a(x) = \det(a_{ij}(x))$ where $a_{ij}(x)$ is the conjugate covariant tensor of $a^{ij}(x)$. The boundary ∂D is represented by $\psi(x)=0$ for $x \in \partial D$ and $\psi(x)>0$ for $x \in D$ in a neighborhood of any $x_0 \in \partial D$.

6) We denote, taking $a^{ij}(x)$ as fundamental tensor, the volume element of $\bar{D}$ by dx and the element of surface area on ∂D by $d\tilde{x}$.

(3.3) $$P(t, x, E) = \int_E p(t, x, y)\, dy.$$[6]

Then, $\boldsymbol{M}$ is Markov process with $\{P(t, x, E)\}$ as its transition probabilities.

We shall state some known properties of the process $\boldsymbol{M}$ which we need.

(B.1) For any $\varepsilon > 0$ and any $x \in \bar{D}$,

$$\lim_{t \downarrow 0} \sup_{x \in \bar{D}} (1 - P(t, x, U_\varepsilon(x)) = 0,$$

where $U_\varepsilon(x)$ is ε-neighborhood of x;

(B.2) $P(t, x, U) > 0$, for any $x \in \bar{D}$, $t > 0$ and non-null open subset U of S, and $P(t, x, \bar{D}) = 1$;

(B.3) $\boldsymbol{M}$ is strong Feller process, that is, the semi-group T_t maps $\mathscr{B}(\bar{D})$ into $C(D)$;

(B.4) The semi-group T_t is strongly continuous in $t \geqq 0$ as an operator on $C(\bar{D})$,[7] and the generator $\boldsymbol{G}$ of T_t is given by

(3.4) $$\boldsymbol{G} = \bar{A}(\mathscr{F}),$$

where

(3.5) $$\mathscr{F} = \{f\colon\ f \in C^2(\bar{D}),\ Lf(x) = 0 \text{ on } \partial D\},$$

and $\bar{A}(\mathscr{F})$ denotes the closure of the restriction of A on $\mathscr{F}$;

(B.5) $\boldsymbol{M}$ is recurrent.

For (B.1)$\sim$(B.4), see Ikeda, Sato, Tanaka and Ueno [3]. The recurrence of $\boldsymbol{M}$ is implied by (B.2) and theorem 2.1. The process $\boldsymbol{M}$ has, therefore, the unique[8] finite invariant measure m by the theorem of Ueno [10], Maruyama and Tanaka [7] and Hasminsky [2].

Denote by A° and L° the formal adjoint of A and L, respectively, that is,

(3.6)
$$A^\circ u(x) = \Delta u(x) - \frac{1}{\sqrt{a(x)}} \frac{\partial}{\partial x^i} (b^i(x)\sqrt{a(x)}\, u(x)) \qquad \text{for } x \in D,$$
$$L^\circ u(x) = \frac{\partial u(x)}{\partial n} - b_n(x)\, u(x) \qquad \text{for } x \in \partial D,$$

where $b_n(x) = b^i(x)\, n_i(x)$, $b^i(x)$ and $n_i(x)$ are those of (3.2), and define a family $\mathscr{F}^\circ$ of functions by

(3.7) $$\mathscr{F}^\circ = \{f\colon\ f \in C^2(\bar{D}),\ L^\circ f(x) = 0 \text{ on } \partial D\}.$$

Then, we have

PROPOSITION 3.1. *The invariant measure m of the process $\boldsymbol{M}$ has a positive density function φ with respect to dx such that* (i) $\varphi \in \mathscr{F}^\circ$ *and* (ii) $A^\circ \varphi(x) = 0$ *on D. Conversely, if there exists such φ, then it is the density function of the invariant measure of $\boldsymbol{M}$.*

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7) See for example Yosida [13].

8) The uniqueness means "up to constant multiples".

Proof. It readily follows from (B.2) and (B.3) that the measure m has a density function φ_1 with respect to dx, and hence we have

$$\int_{\bar{D}} Gu(x)\varphi_1(x)\,dx = 0,$$

for any $u \in \mathcal{D}(G)$ by proposition 2.2. Noting that $G = \bar{A}(\mathcal{F})$, we have

$$\int_D Af(x)\varphi_1(x)\,dx = 0,$$

for any $f \in \mathcal{F}$. Therefore, there exists $\varphi > 0$ such that $\varphi \in \mathcal{F}^\circ$, $A^\circ\varphi(x) = 0$ on D and $\varphi_1(x) = \varphi(x)$ almost everywhere by theorem 5 of Ito [5] and by (B.2). Conversely, let (i) and (ii) be satisfied by φ, then we have

$$0 = \int_D f(x)A^\circ\varphi(x)\,dx - \int_D Af(x)\varphi(x)\,dx,$$

for any $f \in \mathcal{F}$ by Green's formula. φ is, therefore, a density function of the invariant measure of $\boldsymbol{M}$ by proposition 2.2, completing the proof.

In the following we shall use φ as the density function of m.

We now construct the adjoint process of the A-diffusion $\boldsymbol{M}$ with reflecting barrier, which is unique by the uniqueness of the invariant measure of $\boldsymbol{M}$. We first define A^* and L^* by

$$\begin{aligned} A^*u(x) &= \Delta u(x) - b^i(x)\frac{\partial u(x)}{\partial x^i} + 2a^{ij}(x)\frac{\partial \log \varphi(x)}{\partial x^i}\frac{\partial u(x)}{\partial x^j}, \qquad x \in D, \\ L^*u(x) &= Lu(x), \qquad x \in \partial D. \end{aligned} \tag{3.8}$$

Then, it will be shown that an A^*-diffusion with reflecting barrier $\boldsymbol{M}^* = \{W_c;\ B(W_c);\ P_x^*,\ x \in \bar{D}\}$ can be constructed in the analogous way that $\boldsymbol{M}$ is done. Therefore $\boldsymbol{M}^*$ has the same properties (B.1)~(B.5) of $\boldsymbol{M}$, replacing P_x, $P(t, x, E)$ and A by P_x^*, $P^*(t, x, E)$ and A^*, respectively.

Define $P^*(t, x, E)$ by

$$P^*(t, x, E) = \int_E p(t, y, x)\frac{\varphi(y)}{\varphi(x)}\,dy, \tag{3.9}$$

where $p(t, y, x)$ is the fundamental solution of (3.1). Then $\{P^*(t, x, E)\}$ is a transition probability.

THEOREM 3.1. *There exists the A^*-diffusion with reflecting barrier $\boldsymbol{M}^* = \{W_c;\ B(W_c);\ P_x^*,\ x \in \bar{D}\}$ with $P^*(t, x, E)$ as its transition probability whose generator G^* is given by*

$$G^* = \bar{A}^*(\mathcal{F}).$$

The A-diffusion $\boldsymbol{M}$ and the A^*-diffusion $\boldsymbol{M}^*$ are adjoint to one another.

We first state two lemmas.

LEMMA 3.1. *(i) $\mathscr{F}=\varphi^{-1}\mathscr{F}^\circ$, and (ii) $A^\circ(\varphi u)(x)=\varphi(x)A^*u(x)$, for any $u \in C^2(\bar{D})$ and $x\in D$.*

Proof. (i) For any $u\in\mathscr{F}$, we have

$$\frac{\partial}{\partial n}\varphi u=\frac{\partial\varphi}{\partial n}u+\varphi\frac{\partial u}{\partial n}=\frac{\partial\varphi}{\partial n}u=b_n\varphi u,$$

and for any $v\in\mathscr{F}^\circ$,

$$\frac{\partial}{\partial n}\frac{v}{\varphi}=\frac{1}{\varphi^2}\left(\frac{\partial v}{\partial n}\varphi-\frac{\partial\varphi}{\partial n}v\right)=\frac{1}{\varphi^2}(b_n v\varphi-b_n\varphi v)=0.$$

This implies $\mathscr{F}=\varphi^{-1}\mathscr{F}^\circ$.

(ii) Noting that

$$\Delta(\varphi u)=\Delta\varphi\cdot u+2a^{ij}\frac{\partial\varphi}{\partial x^i}\frac{\partial u}{\partial x^j}+\varphi\cdot\Delta u,$$

we have

$$A^\circ(\varphi u)=uA^\circ\varphi+\varphi A^*u=\varphi A^*u,$$

since $A^\circ\varphi(x)=0$ on D, completing the proof.

Next we reformulate the Green's formula as follows,

LEMMA 3.2. *For any $u,v\in C^2(\bar{D})$, we have*

$$\begin{aligned}&\int_D\{u(x)Av(x)-v(x)A^*u(x)\}\varphi(x)\,dx\\ &=-\int_{\partial D}\{u(x)Lv(x)-v(x)L^*u(x)\}\varphi(x)\,d\tilde{x},\end{aligned}\tag{3.10}$$

where $L^*=L=\partial/\partial n$.

Proof. By Green's formula and lemma 3.1, we have

$$\begin{aligned}&\int_D\{\varphi(x)u(x)Av(x)-v(x)\varphi(x)A^*u(x)\}\,dx\\ &=-\int_{\partial D}\left\{\varphi(x)u(x)\frac{\partial v(x)}{\partial n}-v(x)\frac{\partial\varphi u(x)}{\partial n}+b_n(x)v(x)\varphi(x)u(x)\right\}d\tilde{x}\end{aligned}$$

but

$$v(x)\frac{\partial\varphi u(x)}{\partial n}-b_n(x)v(x)\varphi(x)u(x)=v(x)\varphi(x)\frac{\partial u(x)}{\partial n},$$

for $x\in\partial D$, since $\varphi\in\mathscr{F}^\circ$. Thus (3.10) is obtained.

Proof of Theorem 3.1. It follows from lemma 3.1 (i) that $\mathscr{F}^\circ$ is dense in $C(\bar{D})$. Using Ito's results [5] which state the existence, uniqueness and boundedness of the solution $f\in\mathscr{F}^\circ$ of $(\alpha-A^\circ)f=g$ for any uniformly Hölder continuous function g on $\bar{D}$, we may conclude that $\bar{A}^\circ(\mathscr{F}^\circ)$ is the generator of a strongly continuous semi-group T_t° on $C(\bar{D})$. Further, T_t° is given by

$$T_t{}^\circ f(x)=\int_D p(t,y,x)f(y)\,dy,$$

which follows from the uniqueness of Laplace transform of $T_t{}^\circ f(x)$.

We now define a semi-group T_t^* by

$$T_t^*f=\frac{1}{\varphi}T_t(\varphi f), \tag{3.11}$$

then T_t^* is strongly continuous in $t\geqq 0$ with the generator $\boldsymbol{G}^*$ and with the norm $\|T_t^*\|\leqq 1$.

Let $f\in\mathscr{F}$, then $\varphi f\in\mathscr{F}^\circ$ by lemma 3.1, and hence $\varphi f\in\mathscr{D}(\boldsymbol{G}^\circ)$ implying

$$\boldsymbol{G}^\circ(\varphi f)(x)=A^\circ(\varphi f)(x)=\varphi(x)A^*f(x) \qquad \text{for } x\in D.$$

Thus we have

$$\boldsymbol{G}^*f(x)=\frac{1}{\varphi(x)}\boldsymbol{G}^\circ(\varphi f)(x)=A^*f(x) \qquad \text{for } x\in D,$$

which implies $\bar{A}^*(\mathscr{F})\subset\boldsymbol{G}^*$, therefore $\boldsymbol{G}^*=\bar{A}^*(\mathscr{F})$. This permits us to construct the A^*-diffusion $\boldsymbol{M}^*$ with reflecting barrier in the same way as $\boldsymbol{M}$ is done.

Next, we have by lemma 3.2 that

$$\int_D f(x)Ag(x)\varphi(x)\,dx-\int_D g(x)A^*f(x)\varphi(x)\,dx=0, \tag{3.12}$$

for any $f,g\in\mathscr{F}$. Since $\boldsymbol{G}=\bar{A}(\mathscr{F})$ and $\boldsymbol{G}^*=\bar{A}^*(\mathscr{F})$, (3.12) implies

$$\int_{\bar{D}}u(x)\boldsymbol{G}v(x)\varphi(x)\,dx=\int_{\bar{D}}\boldsymbol{G}^*u(x)v(x)\varphi(x)\,dx,$$

for any $v\in\mathscr{D}(\boldsymbol{G})$ and $u\in\mathscr{D}(\boldsymbol{G}^*)$. Thus $\boldsymbol{M}$ and $\boldsymbol{M}^*$ are adjoint to one another by proposition 2.3. This completes the proof.

By a formal modification of Nelson's proof for our case ([8], [6]), we have

THEOREM 3.2. *The A-diffusion $\boldsymbol{M}$ with reflecting barrier is self-adjoint if and only if there exists $g\in C^1(\bar{D})$ such that*

$$b^i(x)=a^{ij}(x)\frac{\partial g(x)}{\partial x^j}. \tag{3.13}$$

In this case, the density function φ of the invariant measure m is given by

$$\varphi(x)=\exp(g(x)). \tag{3.14}$$

Proof. If $\boldsymbol{M}$ is self-adjoint, $Af(x)=A^*f(x)$ for $f\in\mathscr{F}$. Thus we have (3.13). Since an equality

$$\int_D \Delta u\cdot v\,dx+\int_D a^{ij}\frac{\partial u}{\partial x^i}\frac{\partial v}{\partial x^j}dx=-\int_{\partial D}v\frac{\partial u}{\partial n}\,d\tilde{x}$$

holds for $u,v\in C^2(\bar{D})$, we have

$$\int_D \Delta f(x)\varphi(x)\,dx = -\int_D a^{ij}(x)\frac{\partial f(x)}{\partial x^i}\frac{\partial \varphi(x)}{\partial x^j}dx \tag{3.15}$$

for any $f \in \mathcal{F}$. It follows from (3.15) and

$$\frac{\partial \varphi}{\partial x^i} = \varphi \frac{\partial g}{\partial x^i},$$

that

$$\int_D Af(x)\varphi(x)\,dx = \int_D \Delta f(x)\varphi(x)\,dx + \int_D a^{ij}(x)\frac{\partial g(x)}{\partial x^i}\frac{\partial f(x)}{\partial x^j}\varphi(x)\,dx = 0,$$

for any $f \in \mathcal{F}$, therefore,

$$\int_{\bar{D}} Gu(x)\varphi(x)\,dx = 0,$$

for any $u \in \mathcal{D}(G)$, and hence φ is a density function of the invariant measure of $\boldsymbol{M}$ by proposition 2.2. Finally we have

$$A^* f(x) = Af(x),$$

by theorem 3.1, completing the proof.

REMARK 3.1. Above discussions remain valid when the boundary ∂D is null, that is, D is a connected compact orientable manifold of class C^∞, which is just the case discussed by Nelson [8] when the coefficients of A are of class C^∞.

4. The adjoint process of "the Markov process on the boundary".

In connection with the construction of diffusions with Wentzell's boundary conditions, Ueno introduced the notion of "the Markov process on the boundary" [11] (cf. [3]). We may conclude the existence of the process on the boundary ∂D of the A-diffusion with reflecting barrier by Ueno [11]. According to Satō's probabilistic construction, it is the Markov process $\boldsymbol{M}^{\partial D} = \{\widetilde{W};\ B(\widetilde{W});\ \widetilde{P}_x,\ x \in \partial D\}$ whose path space $\widetilde{W}$ is the space of all right continuous path functions w's from $[0, \infty)$ to ∂D, whose semi-group $\widetilde{T}_t$ is strongly continuous in $t \geqq 0$.

For any $f \in C(\partial D)$, the equation

$$Au(x) = 0 \text{ for } x \in D \quad \text{and} \quad u(x) = f(x) \text{ for } x \in \partial D \tag{4.1}$$

has the unique solution $u \in C(\bar{D})$. We write $u = Hf$, and define $(\partial/\partial n)H$ by

$$\left(\frac{\partial}{\partial n}H\right)f = \frac{\partial}{\partial n}(Hf) \qquad \text{for } f \in \mathcal{D}, \tag{4.2}$$

where

$$\mathscr{D}=\{f:\ f\in C(\partial D),\ Hf\in C^1(\bar{D})\}.^{9)} \tag{4.3}$$

Then the generator $\tilde{G}$ of $\tilde{T}_t$ is the closure of $(\partial/\partial n)H$ on $\mathscr{D}$, that is,

$$\tilde{G}=\overline{\frac{\partial}{\partial n}H(\mathscr{D})}, \tag{4.4}$$

(cf. Satō [9] or [3]).

We also denote by $(M^*)^{\partial D}$ the process on the boundary ∂D of the adjoint process M^* of M. The generator $\tilde{G}^*$ of the semi-group $\tilde{T}_t^*$ of $(M^*)^{\partial D}$ is given by

$$\tilde{G}^*=\overline{\frac{\partial}{\partial n}H^*(\mathscr{D}^*)}, \tag{4.5}$$

where H^*f is defined by $\varphi^{-1}H^\circ(\varphi f)$ with the solution $H^\circ(\varphi f)$ of $A^\circ u=0$ in D and $u=\varphi f$ on ∂D, and $\mathscr{D}^*$ is defined by (4.3), replacing H by H^*.

PROPOSITION 4.1. *The process $M^{\partial D}$ is recurrent and strong Feller. And it has a finite unique invariant measure.*

Proof. From the construction of $M^{\partial D}$ (cf. [9], [3]), we see that the transition probability $\tilde{P}(t,x,E)$ of $M^{\partial D}$ is given by

$$\tilde{P}(t,x,E)=\int_E p(t,x,y)\,d\tilde{y}, \tag{4.6}$$

for each $x\in\partial D$ and $E\in B(\partial D)$, where $p(t,x,y)$ is the fundamental solution of (3.1) which is a continuous function of (t,x,y) in $(0,\infty)\times\bar{D}\times\bar{D}$ (see [3]). This implies that $M^{\partial D}$ is strongly Feller process and that $\tilde{P}(t,x,U)>0$ for any non-null open subset U of ∂D. Further, $M^{\partial D}$ is recurrent by theorem 2.1, and hence has a finite unique invariant measure $\tilde{m}$.[10] This completes the proof.

The above proposition is available for the process $(M^*)^{\partial D}$.

THEOREM 4.1. *The processes $M^{\partial D}$ and $(M^*)^{\partial D}$ have the common unique invariant measure $\tilde{m}$ which is represented by the density φ of the invariant measure m of M (and M^*) as*

$$\tilde{m}(E)=\int_E \varphi(x)\,d\tilde{x} \qquad \textit{for } E\in B(\partial D). \tag{4.7}$$

The adjoint process $(M^{\partial D})^*$ of $M^{\partial D}$ is $(M^*)^{\partial D}$, symbolically we write

$$(M^{\partial D})^*=(M^*)^{\partial D}. \tag{4.8}$$

Proof. It follows from lemma 3.2 that

9) The set $\mathscr{D}$ contains $C^3(\partial D)$.

10) Cf. corollary A in appendix.

$$\int_{\partial D}\left\{g(x)\frac{\partial}{\partial n}Hf(x)-f(x)\frac{\partial}{\partial n}H^*g(x)\right\}\varphi(x)\,d\tilde{x}$$
$$=\int_{\partial D}\left\{H^*g(x)\frac{\partial}{\partial n}Hf(x)-Hf(x)\frac{\partial}{\partial n}H^*g(x)\right\}\varphi(x)\,d\tilde{x}$$
$$=-\int_{D}\{H^*g(x)AHf(x)-Hf(x)A^*H^*g(x)\}\varphi(x)\,dx=0,$$

for any $f\in\mathscr{D}$ and $g\in\mathscr{D}^*$. Therefore, noting (4.4) and (4.5), we have

$$\int_{\partial D}\tilde{G}f(x)g(x)\varphi(x)\,d\tilde{x}=\int_{\partial D}f(x)\tilde{G}^*g(x)\varphi(x)\,d\tilde{x}, \tag{4.9}$$

for any $f\in\mathscr{D}(\tilde{G})$ and $g\in\mathscr{D}(\tilde{G}^*)$.

Setting $g\equiv 1$ in (4.9), we have

$$\int_{\partial D}\tilde{G}f(x)\varphi(x)\,d\tilde{x}=0 \qquad \text{for any } f\in\mathscr{D}(\tilde{G}), \tag{4.10}$$

and also we have

$$\int_{\partial D}\tilde{G}^*g(x)\varphi(x)\,d\tilde{x}=0 \qquad \text{for any } g\in\mathscr{D}(\tilde{G}^*). \tag{4.11}$$

(4.10) and (4.11) imply that φ is the density founction of the invariant measure of $\boldsymbol{M}^{\partial D}$ and $(\boldsymbol{M}^*)^{\partial D}$ with respect to $d\tilde{x}$, and hence (4.9) implies that $\boldsymbol{M}^{\partial D}$ and $(\boldsymbol{M}^*)^{\partial D}$ are adjoint to one another, completing the proof.

5. A comment to process with more general boundary conditions.[11]

In the proof of theorem 3.1, lemma 3.2 plays an essential role. We shall note that lemma 3.2 can be extended to more general boundary conditions.

Let A and A° be given by (3.2) and (3.6), respectively. L and L° be defined by

$$Lf(x)=\frac{\partial f(x)}{\partial n}+Bf(x) \qquad \text{for } x\in\partial D, \tag{5.1}$$

and

$$L^\circ f(x)=\frac{\partial f(x)}{\partial n}-b_n(x)f(x)+B^\circ f(x) \qquad \text{for } x\in\partial D, \tag{5.2}$$

where

$$Bf(x)=\tilde{\varDelta}f(x)+\beta^i(x)\frac{\partial f(x)}{\partial x^i} \qquad (i=1,2,\cdots,N-1),$$

and

$$B^\circ f(x)=\tilde{\varDelta}f(x)-\frac{1}{\alpha(x)}\frac{\partial}{\partial x^i}(\alpha\beta^i f)(x) \qquad (i=1,2,\cdots,N-1),$$

11) This owes to a discussion with K. Satō.

respectively, and $\tilde{\Delta} f(x)$ and $\alpha(x)$ is given by

$$\tilde{\Delta} f(x) = \frac{1}{\alpha(x)} \frac{\partial}{\partial x^i}\left(\alpha(x)\alpha^{ij}(x)\frac{\partial f(x)}{\partial x^j}\right) \qquad (i, j = 1, 2, \cdots, N-1) \tag{5.3}$$

and $d\tilde{x} = \alpha(x)\, dx^1 dx^2 \cdots dx^{N-1}$. $\{\alpha^{ij}(x)\}$ is a sufficiently smooth (not necessarily strictly) positive definite contravariant tensor on ∂D.

Further, we define $\mathscr{F}$ and $\mathscr{F}^\circ$ by

$$\mathscr{F} = \{f:\ f \in C^2(\bar{D}),\ Lf(x) = 0 \text{ on } \partial D\}, \tag{5.4}$$

and

$$\mathscr{F}^\circ = \{f:\ f \in C^2(\bar{D}),\ L^\circ f(x) = 0 \text{ on } \partial D\}. \tag{5.5}$$

If there exists such positive $\varphi \in \mathscr{F}^\circ$ that $A^\circ \varphi(x) = 0$ for $x \in D$, we can define A^* by (3.8), L^* by

$$L^* f(x) = \frac{\partial f(x)}{\partial n} + B^* f(x) \qquad \text{for } x \in \partial D, \tag{5.6}$$

where

$$B^* f(x) = \tilde{\Delta} f(x) - \beta^i(x)\frac{\partial f(x)}{\partial x^i} + 2\alpha^{ij}(x)\frac{\partial \log \varphi(x)}{\partial x^i}\frac{\partial f(x)}{\partial x^j},$$

and define $\mathscr{F}^*$ by

$$\mathscr{F}^* = \{f:\ f \in C^2(\bar{D}),\ L^* f(x) = 0 \text{ on } \partial D\}. \tag{5.7}$$

Then lemma 3.2 may be generalized as follows:

LEMMA 5.1. *For any $u, v \in C^2(\bar{D})$, we have*

$$\begin{aligned} &\int_D \{u(x)Av(x) - v(x)A^*u(x)\}\varphi(x)\,dx \\ &= -\int_{\partial D} \{u(x)Lv(x) - v(x)L^*u(x)\}\varphi(x)\,d\tilde{x}. \end{aligned} \tag{5.8}$$

Proof. It follows from Green's formula and lemma 3.1 that

$$\begin{aligned} &\int_D \{u(x)Av(x) - v(x)A^*u(x)\}\varphi(x)\,dx \\ &= -\int_{\partial D}\left\{\varphi u\frac{\partial v}{\partial n} - v\varphi\frac{\partial u}{\partial n} - uv\left(\frac{\partial \varphi}{\partial n} - b_n\varphi\right)\right\}d\tilde{x} \\ &= -\int_{\partial D}\left\{u\left(\frac{\partial v}{\partial n} + Bv\right) - v\left(\frac{\partial u}{\partial n} + B^*u\right)\right\}\varphi d\tilde{x} \\ &\quad - \int_{\partial D}\{uvB^\circ\varphi - (Bv)u\varphi + (B^*u)v\varphi\}\,d\tilde{x} \\ &\equiv \mathrm{I} + \mathrm{II}, \qquad \text{say.} \end{aligned}$$

But, we have

$$-\mathrm{II}=\int_{\partial D}\{uvB^\circ\varphi - vB^\circ(u\varphi)+(B^*u)v\varphi\}\,d\tilde{x}=0,$$

since $B^\circ(u\varphi)=\varphi B^*u+uB^\circ\varphi$, completing the proof.

We can state some results which follow directly from lemma 5.1, for example,

1. *If $\bar{A}(\mathscr{F})$ and $\bar{A}^*(\mathscr{F}^*)$ are generators of strongly continuous semi-groups, then the A-diffusion* $\boldsymbol{M}$ *and the A^*-diffusion* $\boldsymbol{M}^*$ *determined by $\bar{A}(\mathscr{F})$ and $\bar{A}^*(\mathscr{F}^*)$, respectively, are adjoint to one another;*

2. *If the boundary condition is given by*

$$Lf=\frac{\partial f}{\partial n}+\beta^i\frac{\partial f}{\partial x^i},$$

and if the A-diffusion $\boldsymbol{M}$ *determined by (A, L) is self-adjoint, then it is the diffusion with reflecting barrier.*

APPENDIX. **Notes on the invariant measure of Markov processes.**

Maruyama and Tanaka [7], Ueno [10] and Hasiminsky [2] proved the existence of the unique σ-finite invariant measure of Markov processes under some different conditions each other. But an observation of their proofs permits us to state the theorem under slightly weaker conditions.

Let $\boldsymbol{M}=\{W;\ B(W);\ P_x,\ x\in S\}$ be a Markov process, where S is locally compact metric space and W is the right continuous path space. The conditions are

(1°) M is recurrent;

(2°) $h^U f(\cdot)=E_\cdot(f(x_{\sigma_U}))$ is continuous in $S-\bar{U}$ for any $f\in C(\bar{U})$, where U is non-null open subset of S and σ_U is the first passage time to U;

(3°) G_α maps $C(S)$ into $C(S)$.

Then we have

THEOREM A. *The Markov process satisfying the conditions (1°), (2°) and (3°) has the unique σ-finite invariant measure m of the form*

$$m(A)=\int_{\bar{U}_1}\mu(dx)\,E_x\left(\int_0^{\tau(w)}\chi_A(x_t(w))\,dt\right), \qquad (A^\circ.1)$$

where $\tau(w)=\sigma_{U_2}(w)+\sigma_{U_1}(w^+_{\sigma_{U_2}})$,[12] *$U_1$ and U_2 are non-null open subsets of S with compact closures $\bar{U}_1$ and $\bar{U}_2$ such that $\bar{U}_1\cap\bar{U}_2=\phi$. σ_{U_1} and σ_{U_2} are the first passage time to U_1 and U_2, respectively.*

The proof of the existence of m had been obtained by Ueno [10][13] under the conditions (1°), (2°) and (3°) (cf. Hasiminsky [2] for the proof of the invariance of m). The uniqueness follows from Maruyama and Tanaka [7].

12) $x_t(w^+_{\sigma(w)})=x_{t+\sigma(w)}(w)$.

13) Ueno's expression of the invariant measure is different from (A°.1). Under his "maximal principle", however, it may be reduced to this form.

Hence we have

COROLLARY A. *Let* $M=\{W;\ B(W);\ P_x,\ x\in S\}$ *be a recurrent Markov process, where* S *is a compact metric space and* W *is the right continuous path space. If* M *is strongly Feller, then it has the unique finite invariant measure.*

This follows from

LEMMA A. *Let* M *be the Markov process in the above corollary. Then* M *has the Property* (2°).

The lemma was first proved by Girsanov (cf. lemma 4.3 in [1]) for the process with continuous paths. But the proof is also applicable for the present case with a little change. We need only to note that the process is conservative and that the right continuity of paths is equivalent, if the state space S is compact, to the validity of $\sup_{x\in S}(1-P(t,x,U_\varepsilon(x)))=O(t)$, where $U_\varepsilon(x)$ is ε-neighborhood of x for any $\varepsilon>0$.

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