NOTE ON SOME GENERALIZATIONS OF QUASI-FROBENIUS RINGS

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Let A be a ring satisfying the minimum condition for right and left ideals (by a ring we shall always understand such one). Let A have a unit element. Then owing to Ikeda [3] we know that A is a quasi-Frobenius ring if and only if it satisfies the following condition:

(a) Every homomorphism between two left (right) ideals of A is given by the right (left) multiplication of an element of A.

Recently Kawada [4] discussed the following condition which is a weaker one than the above:

(*) Every left (right) ideal A-isomorphic to a given left ideal $\mathfrak{l}$ (right ideal $\mathfrak{r}$) can be expressed as $\mathfrak{l}a$ ($a\mathfrak{r}$) by the right (left) multiplication of a regular element of A.

In the present note we shall deal with rings (and algebras) which satisfy the condition (*) for simple left (and right) ideals. Besides, we shall give a remark on the duality relations of two-sided ideals in a ring.

1. Remarks on division algebras.

Let D be a (finite dimensional) division algebra over a field F; let $(u_1, u_2, \cdots, u_n)$ be a basis of D over F. Let $\xi_1, \xi_2, \cdots, \xi_n$ be n independent variables and put

$$S(\xi) = \left\| \sum_{i=1}^{n} a_{ijk}\xi_i \right\|_{kj},$$

where a_{ijk} $(1 \leqq i, j, k \leqq n)$ are the coefficients of the multiplication table

$$u_i u_j = \sum_{k=1}^{n} a_{ijk} u_k.$$

The matrix $S(\xi)$ is called the group matrix of D with parameters $\xi_1, \xi_2, \cdots, \xi_n$ (defined by the basis $(u_1, u_2, \cdots, u_n)$). In this section we shall prove the following proposition.

PROPOSITION 1. *Let D be a division algebra over a field F; let $S(\xi)$ be the group matrix of D, defined by a basis $(u_1, u_2, \cdots, u_n)$ of D over F, with parameters $\xi_1, \xi_2, \cdots, \xi_n$. Then any minor determinant of $S(\xi)$ does not vanish identically. More generally, let P and Q be two non-singular $(n \times n)$ matrices with coefficients in F. Then any minor determinant of $PS(\xi)Q$ does not vanish identically.*

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For the proof of this proposition, we have only to consider the case where F is an infinite field. In fact, assume that our assertion is valid for a infinite underlying field. Let η be a variable over F and consider $D_{F(\eta)}$; as is well known, this algebra is a division algebra over $F(\eta)$. The group matrix $S(\xi)$ of D may be regarded also as the group matrix of $D_{F(\eta)}$, defined by the basis $(u_1, u_2, \cdots, u_n)$. Our assumption will imply then, since $F(\eta)$ is an infinite field, that the proposition is valid for any F.

We can see now straightforwardly that the above prop. 1 is equivalent to the following one, to which we shall give a proof.

PROPOSITION 2. *Let D be a division algebra over a field F with finite rank n; let F have at least $[n/2]$ elements.*[1] *Let $(x_1, x_2, \cdots, x_r)$ and $(y_1, y_2, \cdots, y_{n-r})$ be two sets of elements of D and let the elements of each set be linearly independent over F. Then there exists at least one element a in D such that the set $(x_1, x_2, \cdots, x_r, y_1a, y_2a, \cdots, y_{n-r}a)$ constitutes a basis of D over F.*

Proof. At the outset we may assume, without loss of generality, that $r \geqq n/2$. For the sake of brevity we write

$$S_0 = (x_1, x_2, \cdots, x_r).$$

As D is a division algebra, we can take an element a_1 of D such that (S_0, y_1a_1) is a set of linearly independent elements over F; then we choose as many elements $y_{i_2}, y_{i_3}, \cdots, y_{i_{\alpha_1}}$ as possible from $(y_2, y_3, \cdots, y_{n-r})$ for which the elements of the set $(S_0, y_1a_1, y_{i_2}a_1, \cdots, y_{i_{\alpha_1}}a_1)$ are linearly independent over F. After suitable reordering, we may set $i_2 = 2, \cdots, i_{\alpha_1} = \alpha_1$; we write

$$S_1 = (S_0, y_1a_1, y_2a_1, \cdots, y_{\alpha_1}a_1).$$

By the definition of S_1 the elements $y_{\alpha_1+1}a_1, y_{\alpha_1+2}a_1, \cdots, y_{n-r}a_1$ are linearly dependent over F to S_1; therefore, if we put

$$y_i^{(1)} = y_{\alpha_1+i} - \sum_{j=1}^{\alpha_1} \gamma_{ij} y_j$$

with suitably chosen coefficients γ_{ij} $(1 \leqq i \leqq n-(r+\alpha_1), 1 \leqq j \leqq \alpha_1)$ in F, the elements $y_i^{(1)}a_1$ $(1 \leqq i \leqq n-(r+\alpha_1))$ are linearly dependent over F to S_0. The two sets $(y_1, y_2, \cdots, y_{n-r})$ and $(y_1, y_2, \cdots, y_{\alpha_1}, y_1^{(1)}, y_2^{(1)}, \cdots, y_{n-(r+\alpha_1)}^{(1)})$ are linearly dependent over F to each other. Similarly proceeding, we obtain the sets of linearly independent elements over F:

$$\begin{aligned} S_2 &= (S_1, y_1^{(1)}a_2, y_2^{(1)}a_2, \cdots, y_{\alpha_2}^{(1)}a_2), \\ S_2 &= (S_2, y_1^{(2)}a_3, y_2^{(2)}a_3, \cdots, y_{\alpha_3}^{(2)}a_3), \\ &\cdots\cdots\cdots\cdots\cdots\cdots, \\ S_t &= (S_{t-1}, y_1^{(t-1)}a_t, y_2^{(t-1)}a_t, \cdots, y_{\alpha_t}^{(t-1)}a_t). \end{aligned}$$

Here, the last set S_t constitutes a basis of D over F; the two sets $(y_1, y_2, \cdots,$

1) $[n/2]$ means the largest integer not exceeding $n/2$.

y_{n-r}) and $(y_1, y_2, \cdots, y_{\alpha_1}, y_1^{(1)}, y_2^{(1)}, \cdots, y_{\alpha_2}^{(1)}, \cdots, y_{\alpha_t}^{(t-1)})$ are linearly dependent over F to each other; $y_\mu^{(\lambda)} a_\nu$ is dependent to $S_{\nu-1}$ if $\lambda \geqq \nu$. Obviously we have only to prove our assertion when $(y_1, y_2, \cdots, y_{n-r}) = (y_1, y_2, \cdots, y_{\alpha_1}, y_1^{(1)}, y_2^{(1)}, \cdots, y_{\alpha_2}^{(1)}, \cdots, y_{\alpha_t}^{(t-1)})$.

Let $(u) = (u_1, u_2, \cdots, u_n)$ be a basis of D over F and let the expressions of the elements x_ι $(1 \leqq i \leqq r)$ and $y_j a_k$ $(1 \leqq j \leqq n-r, 1 \leqq k \leqq t)$ be

$$x_\iota = (u) X_\iota \qquad \text{and} \qquad y_j a_k = (u) Y_{jk},$$

respectively, where X_ι and Y_{jk} are $(n \times 1)$ matrices with coefficients in F. Now let $\xi_1, \xi_2, \cdots, \xi_t$ be t independent variables over F and put $a(\xi) = \sum_{k=1}^t a_k \xi_k$. Then the products $y_j a(\xi)$ are the elements of $D_{F(\xi)} = D_{F(\xi_1, \xi_2, \cdots, \xi_t)}$ and we have $y_j a(\xi) = (u) \sum_{k=1}^t Y_{jk} \xi_k$. Consider the derminant

$$d(\xi) = \det \left\| X_1\, X_2 \cdots X_r\; \sum_k Y_{1k}\xi_k\; \sum_k Y_{2k}\xi_k \cdots \sum_k Y_{n-r,k}\xi_k \right\|;$$

by the definitions, the coefficients of $\xi_1^{\alpha_1} \xi_2^{\alpha_2} \cdots \xi_t^{\alpha_t}$ of $d(\xi)$ does not vanish, and hence $d(\xi)$ does not vanish identically. But, as F has at least $[n/2]$ elements and as $n - r \leqq [n/2]$, there exists at least one set of values $(\gamma) = (\gamma_1, \gamma_2, \cdots, \gamma_t)$ of (ξ) in F such that $d(\gamma) \neq 0$. This means that the set of elements $(x_1, x_2, \cdots, x_r, y_1 a(\gamma), y_2 a(\gamma), \cdots, y_{n-r} a(\gamma))$ constitutes a basis of D over F. This completes the proof.

The above prop. 1 will be used in the subsequent section 2. (It should be observed that the same fact as in prop. 1 is also valid for the antistrophic group matrix of a division algebra. Cf. section 2.)

2. Lemmas on simple (A, A) double modules.

Let A be a ring with a unit element and let N be its radical. Let $\bar{A} = A/N = \bar{A}_1 + \bar{A}_2 + \cdots + \bar{A}_k$ be the direct decomposition of $\bar{A}$ into simple two-sided ideals. The unit element E of A is expressible as $E = E_1 + E_2 + \cdots + E_k$, $E_\kappa = e_{\kappa,1} + e_{\kappa,2} + \cdots + e_{\kappa, f(\kappa)}$ $(1 \leqq \kappa \leqq k)$, where $e_{\kappa,\iota}$ $(1 \leqq \kappa \leqq k, 1 \leqq i \leqq f(\kappa))$ are mutually orthogonal primitive idempotent elements and $A e_{\kappa,\iota} \cong A e_{\lambda,\jmath}$ if and only if $\kappa = \lambda$. Moreover, for each κ there exists a system of $f(\kappa)^2$ elements $c_{\kappa,ij}$ $(1 \leqq i, j \leqq f(\kappa))$ such that $c_{\kappa,ii} = e_{\kappa,\iota}$, $c_{\kappa,\iota j} c_{\kappa,kl} = \delta_{jk} c_{\kappa,il}$. We set $e_{\kappa,1} = e_\kappa$ $(1 \leqq \kappa \leqq k)$ for the sake of brevity.

Let M be a simple (A, A) double module; moreover we shall always assume that M is finitely generated when it is considered as a left (right) A-module. For some E_κ and E_λ we have $E_\kappa M E_\lambda = M$; when that is so, M is called to be of type (κ, λ). We now introduce the following two conditions corresponding to (a) and (*), respectively:

(a)′ Every homomorphism between two left submodules of M is given by the right multiplication of an element of A;

(*)′ Every left submodule A-isomorphic to a given left submodule $\mathfrak{l}$ of M can be expressed as $\mathfrak{l}a$ by the right multiplication of a regular element of A.

We shall assume in the rest of this section, without loss of generality, that A is semisimple.

LEMMA 1. *Let M be a simple (A, A) double module of type (κ, λ). Then M satisfies* (a)$'$ *for simple left submodules if and only if $e_\kappa M$ is a simple right submodule. Moreover, M satisfies* (a)$'$ *for every left submodule if and only if $e_\kappa M$ and Me_λ are simple right and left submodules, respectively.*

The first assertion is a restatement of prop. 1 of [1]. The proof for the second assertion is similar to that of Ikeda [3], prop. 2.

LEMMA 2. *Let M be a simple (A, A) double module of type (κ, λ). Let M satisfy* $(*)'$ *for simple left submodules. Then either $e_\kappa M$ is a simple right submodule or we have $f(\lambda)=1$.*

Proof. Assume that $e_\kappa M$ is not simple; let m be a non-zero element of $e_\kappa Me_\lambda$. Then, as $e_\kappa Me_\lambda$ is a simple $(e_\kappa Ae_\kappa, e_\lambda Ae_\lambda)$ module, we have $e_\kappa Me_\lambda = e_\kappa Ae_\kappa \cdot m \cdot e_\lambda Ae_\lambda$; but, it follows from our assumption that $m \cdot e_\lambda Ae_\lambda \subsetneqq e_\kappa Me_\lambda$. Therefore we can choose an element x of $e_\kappa Ae_\kappa$ such that xm does not lie in $m \cdot e_\lambda Ae_\lambda$. Suppose now that $f(\lambda)>1$. Put $\mathfrak{l}=A(m+xmc_{\lambda,12})$; $\mathfrak{l}$ is a simple left submodule and so A-isomorphic to $\mathfrak{l}_0=Am$. Then we have $\mathfrak{l}=\mathfrak{l}_0 z$ for a suitable regular element z of $E_\lambda AE_\lambda$, and hence there exists an element y $(\neq 0)$ of A such that $m+xmc_{\lambda,12}=ymz$; here y may be assumed to be contained in $e_\kappa Ae_\kappa$. This implies $m=ymze_\lambda$ and $xm=ymzc_{\lambda,21}$, which show that ym $(\neq 0)$ is contained in $m\cdot e_\lambda Ae_\lambda$ as well as in $xm \cdot e_\lambda Ae_\lambda$. On the other hand, however, we have by the definition of x that $m\cdot e_\lambda Ae_\lambda \frown xm\cdot e_\lambda Ae_\lambda=0$. Thus we are led to a contradiction and this completes the proof.

Let the notations and assumptions be as in the above lemma; let $e_\kappa M$ be not a simple right submodule. By the lemma we have $f(\lambda)=1$; $e_\kappa M=e_\kappa Me_\lambda$ is a simple $(e_\kappa Ae_\kappa, e_\lambda Ae_\lambda)$ double module and satisfies $(*)'$ for simple left submodules. For the sake of brevity we write M, K and L in place of $e_\kappa Me_\lambda$, $e_\kappa Ae_\kappa$ and $e_\lambda Ae_\lambda$, respectively. We now prove the following

LEMMA 3. *Let K and L be two (finite dimensional) division algebras over a field F. Let M be a simple (K, L) double module over F and let M satisfy* $(*)'$ *for simple left submodules. Then either M is a simple right L-module or M is a simple left K-module.*

Proof. We first prove our assertion in the case where the underlying field F is an infinite field. Let m be an arbitrary non-zero element of M. By our assumptions it follows easily that every element z of M is expressible in the form xmy, where $x\in K$ and $y \in L$. Put $(M:F)=n$, $(K:F)=r$ and $(L:F)=s$; let $(u_1, u_2, \cdots, u_n)$ $[(v_1, v_2, \cdots, v_r), (w_1, w_2, \cdots, w_s)]$ be a F-basis of M $[K, L]$. Further we take a system of $n+r+s$ indeterminates (z_i, x_j, y_k) $(1\leqq i\leqq n$, $1\leqq j \leqq r$, $1\leqq k\leqq s)$. Then a equation $(\sum_j x_jv_j)(\sum_i z_iu_i)=m(\sum_k y_kw_k)$ must have a non-trivial solution in (x_j, y_k) for every values of (z_i) in F. This equation is equivalent to a system of linear equations

$$(\alpha) \qquad \sum_{j=1}^{r}\sum_{i=1}^{n} c_{i\nu}^{j} z_i x_j - \sum_{k=1}^{s} d_{k\nu} y_k = 0 \qquad (1 \leqq \nu \leqq n),$$

where $c^j_{i\nu}$ $(1 \leqq i,\ \nu \leqq n,\ 1 \leqq j \leqq r)$, $d_{k\nu}$ $(1 \leqq k \leqq s,\ 1 \leqq \nu \leqq n)$ are coefficients of multiplication tables: $v_j u_i = \sum_\nu c^j_{i\nu} u_\nu$, $m w_k = \sum_\nu d_{k\nu} u_\nu$. Suppose now that M is not simple as left K-module and (at the same time) as right L-module. Then we have evidently $n \geqq r + s$. Let $M = Km\tilde{w}_1 + Km\tilde{w}_2 + \cdots + Km\tilde{w}_\sigma$, where $\tilde{w}_1, \tilde{w}_2, \cdots, \tilde{w}_\sigma$ are elements of L, be a decomposition of M into direct sum of simple left K-modules; according to the decomposition we take a basis of M: $(v_1 m\tilde{w}_1, v_2 m\tilde{w}_1, \cdots, v_r m\tilde{w}_1, v_1 m\tilde{w}_2, v_2 m\tilde{w}_2, \cdots, v_r m\tilde{w}_2, \cdots, v_r m\tilde{w}_\sigma)$, moreover, we write for simplicity $z^1_1, z^1_2, \cdots, z^1_r, z^2_1, z^2_2, \cdots, z^2_r, \cdots z^\sigma_r)$ instead of $(z_1, z_2, \cdots, z_n)$. The matrix of the coefficients of (α) is then of the following form:

$$C(z) = \begin{bmatrix} R(z^1)\ R(z^2) \cdots R(z^\sigma) \\ -D \end{bmatrix},$$

where $R(z^i)$ denotes the first regular representation of the general element $z^i_1 v_1 + z^i_2 v_2 + \cdots + z^i_r v_r$ of K (i. e. the transposed matrix of the antistrophic group matrix of K with parameters $z^i_1, z^i_2, \cdots, z^i_r$), and $D = \| d_{k\nu} \|$. Next we take another basis $(mw_1, mw_2, \cdots, mw_s, *) = (v_1 m\tilde{w}_1, \cdots, v_r m\tilde{w}_1, v_1 m\tilde{w}_2, \cdots, v_r m\tilde{w}_\sigma)T$ of M, and consider the correspondingly transformed matrix $C(z)T$ of $C(z)$; by definitions D is transformed into

$$DT = (E_s\ \underbrace{0\ 0 \cdots 0}_{n-s})$$

where E_s denotes the unit matrix of order s. On the other hand, $(R(z^1)\, R(z^2) \cdots R(z^\sigma))$ is transformed into $(R(z^1)\, R(z^2) \cdots R(z^\sigma))T = (A_1(z)\, A_2(z) \cdots A_s(z), B_1(z)\, B_2(z) \cdots B_{n-s}(z))$, say, where $A_i(z)$ $(1 \leqq i \leqq s)$ and $B_j(z)$ $(1 \leqq j \leqq n - s)$ are $(r \times 1)$ matrices. But, by prop. 1 we can see straightforwardly that for a suitable set of values (γ_i) of (z_i) in F we have $\operatorname{rank}(B_1(z)\, B_2(z) \cdots B_r(z)) = r$ (observe that $n - s \geqq r$ and that F is an infinite field); so we must have for the same values of (z_i) that $\operatorname{rank} C(z)T = \operatorname{rank} C(z) = r + s$, and hence the sysem of linear equations (α) has no non-trivial solution in (x_j, y_k) for $(z_j) = (\gamma_i)$. This is a contradiction and therefore proves our assertion.

We now consider the second case where the underlying field F is a finite field. The division algebras K and L must be then commutative. Denote by K_0 the set of all x's in K satisfying $xm = my$ for some y in L; similarly denote by L_0 the set of all y's in L satisfying $my = xm$ for some x in K. Since K_0 and L_0 are isomorphic fields, we may identify them and regard K and L as (commutative) division algebras over $K_0 = L_0$. From this point of view we assume without loss of generality that $K_0 = L_0 = F$; moreover, we may set $v_1 = w_1 = \tilde{w}_1 = 1$, the unit element of F (the notations be the same as before). The proof of our assertion in this case is now analogous to the above case; we have only to observe that $Km \frown mL = mF$.

3. Rings with the condition (*).

Let A be a ring and let N be its radical. If A satisfies (*) for simple left ideals, A has a right unit element. (This fact can be proved in the same way as the proof of Ikeda [3], lemma 1.) For a subset S of A we denote by

$l(S)$ $[r(S)]$ the totality of left [right] annihilators of S.

LEMMA 4. *Let A have a left unit element and let A satisfy* (*) *for simple left ideals. Then A has a unit element and there exists a permutation π of* $(1, 2, \cdots, k)$ *such that the largest completely reducible left subideal of Ae_κ is a direct sum of simple left ideals which are isomorphic to* $Ae_{\pi(\kappa)}/Ne_{\pi(\kappa)}$.

The proof is similar to that of Ikeda [3], lemma 2. If the assumptions of this lemma are satisfied, we have $r(N) \subseteqq l(N)$, $E_{\pi(\kappa)}r(N) = r(N)E_\kappa$ and that each $r(N)E_\kappa$ is a non-zero simple two-sided ideal of A.

THEOREM 1. *Let A be a ring satisfying* (*) *for simple left ideals and for simple right ideals. Then*: (i) *A has a unit element.* (ii) *There exists a permutation π of* $(1, 2, \cdots, k)$ *such that for each κ the largest completely reducible left subideal of Ae_κ is a sum of simple left subideals of the form $\mathfrak{l}x$, where $\mathfrak{l}$ is an arbitrary simple left subideal of Ae_κ and isomorphic to $Ae_{\pi(\kappa)}/Ne_{\pi(\kappa)}$ and x's are suitable units of $e_\kappa Ae_\kappa$, and the same for $e_{\pi(\lambda)}A$.* (iii) *$f(\kappa) = 1$ if the largest completely reducible right subideal of $e_{\pi(\kappa)}A$ is not simple, and the same for $f(\pi(\kappa))$ and for Ae_κ.*

Proof. A has a unit element E by what we have mentioned above. By lemma 4 we have $r(N) = l(N)$, and we denote this by M. There is a permutation π of $(1, 2, \cdots, k)$ such that $E_{\pi(\kappa)}M = ME_\kappa$ $(1 \leqq \kappa \leqq k)$; each ME_κ is a simple two-sided ideal of A. All of our assertions are now immediate consequences of lemma 2.

COROLLARY. *Let A be a primary ring satisfying* (*) *for simple left ideals as well as for simple right ideals. Then A is either a quasi-Frobenius ring or a completely primary ring.*

The following theorem is a direct consequence of lemma 3.

THEOREM 2. *Let A be an algebra over a field F satisfying* (*) *for simple left ideals as well as for simple right ideals. Then besides* (i), (ii) *and* (iii) (*in theorem* 1), *A has the property*: (iv) *For each κ either Ae_κ has a unique simple left subideal, or $e_{\pi(\kappa)}A$ has a unique simple right subideal.*

REMARK. If A is an algebra over an algebraically closed field and if A satisfies (*) for simple left ideals, then by lemma 1 A satisfies also (a) for simple left ideals. Therefore by Ikeda [3], prop. 1 A is a quasi-Frobenius algebra whenever A has a left unit element.[2]

THEOREM 3. *Let A be a ring which has the properties* (i), (ii), (iii) *and* (iv). *Then A satisfies* (*) *for simple left ideals as well as for simple right ideals.*

2) Y. Kawada [4], theorem 3.

Proof. Since the largest completely reducible subideal $r(N)e_{\kappa,\iota}$ of $Ae_{\kappa,\iota}$ is a direct sum of simple left subideals isomorphic to $Ae_{\pi(\kappa)}/Ne_{\pi(\kappa)}$, we have $r(N)e_{\kappa,\iota}=E_{\pi(\kappa)}r(N)e_{\kappa,\iota}$ and hence $r(N)E_{\kappa}=E_{\pi(\kappa)}r(N)E_{\kappa}$ $(1\leqq\kappa\leqq k)$. From this it follows that $r(N)E_{\kappa}=E_{\pi(\kappa)}r(N)$ and that $r(N)E_{\kappa}$ is a two-sided ideal $(1\leqq\kappa\leqq k)$. Similarly, we have that $l(N)E_{\kappa}=E_{\pi(\kappa)}l(N)$ is a two-sided ideal $(1\leqq\kappa\leqq k)$. Furthermore, we can see in the same way as the proof of Ikeda [2], theorem 2 that $r(N)E_{\kappa}$ and $l(N)E_{\kappa}$ are both simple two-sided ideals; therefore we must have $r(N)=l(N)$, and we shall denote this by M. Now let $\mathfrak{l}$ be a simple left ideal which is isomorphic to $Ae_{\pi(\kappa)}/Ne_{\pi(\kappa)}$ and let $\mathfrak{l}'$ by any left ideal which is isomorphic to $\mathfrak{l}$. Both $\mathfrak{l}$ and $\mathfrak{l}'$ are contained in the simple two-sided ideal $E_{\pi(\kappa)}M=ME_{\kappa}$. If $e_{\pi(\kappa)}M=e_{\pi(\kappa)}ME_{\kappa}$ is a simple right subideal of ME_{κ}, then by lemma 1 $\mathfrak{l}'$ is written as $\mathfrak{l}a$ by a regular element a of $E_{\kappa}A_{\kappa}E_{\kappa}$; the element a can be taken to be a regular element of A. If, on the other hand, $e_{\pi(\kappa)}M$ is not simple, then by (iii) and (iv) it follows that $f(\kappa)=1$ and Me_{κ} is a simple left ideal, i. e. $Me_{\kappa}=ME_{\kappa}$ is itself a simple left subideal. Therefore we have $\mathfrak{l}'=ME_{\kappa}=\mathfrak{l}=\mathfrak{l}\cdot E$. Thus A satisfies (*) for simple left ideals. Similarly, we see that A satisfies (*) for simple right iderls.

REMARK. In theorem 3, the assumption (iv) can not be omitted. For example, let A be an algebra of order 9 over the field R of rational numbers with a basis $(1, \omega, \omega^2, m, \omega m, \omega^2 m, m\omega, \omega m\omega, \omega^2 m\omega)$; let the multiplication table be

	1	ω	ω^2	m	ωm	$\omega^2 m$	$m\omega$	$\omega m\omega$	$\omega^2 m\omega$
1	1	ω	ω^2	m	ωm	$\omega^2 m$	$m\omega$	$\omega m\omega$	$\omega^2 m\omega$
ω	ω	ω^2	3	ωm	$\omega^2 m$	$3m$	$\omega m\omega$	$\omega^2 m\omega$	$3m\omega$
ω^2	ω^2	3	3ω	$\omega^2 m$	$3m$	$3\omega m$	$\omega^2 m\omega$	$3m\omega$	$3\omega m\omega$
m	m	$m\omega$	$-(\omega^2 m+\omega m\omega)$	0	0	0	0	0	0
ωm	ωm	$\omega m\omega$	$-(3m+\omega^2 m\omega)$	0	0	0	0	0	0
$\omega^2 m$	$\omega^2 m$	$\omega^2 m\omega$	$-(3\omega m+3m\omega)$	0	0	0	0	0	0
$m\omega$	$m\omega$	$-(\omega^2 m+\omega m\omega)$	$3m$	0	0	0	0	0	0
$\omega m\omega$	$\omega m\omega$	$-(3m+\omega^2 m\omega)$	$3\omega m$	0	0	0	0	0	0
$\omega^2 m\omega$	$\omega^2 m\omega$	$-(3\omega m+3m\omega)$	$3\omega^2 m$	0	0	0	0	0	0

We can easily see that A satisfies (i), (ii) and (iii), and that A does not satisfy (iv); furthermore A does not satisfy (*) for simple left ideals or for simple right ideals.

4. A remark on the duality relation of two-sided ideals in a ring.

Let A be a ring with radical N satisfying the duality relations (γ_1) $l(r(\mathfrak{z}))=\mathfrak{z}$ and (γ_2) $r(l(\mathfrak{z}))=\mathfrak{z}$ for every simple two-sided ideal $\mathfrak{z}$, for $\mathfrak{z}=0$ and for $\mathfrak{z}=N$. Then we can verify, in a similar way as the proof of Nakayama [5], theorem 7, that A has a unit element, $r(N)=l(N)$ (we denote this by M) and that $E_{\pi(\kappa)}M=ME_{\kappa}$ $(1\leqq\kappa\leqq k)$ are simple two-sided ideals. We now give

THEOREM 4. *Let A be a bound ring*[3] *with radical N. Let (γ_1) and (γ_2) be valid for every simple two-sided ideal $\mathfrak{z}$, for $\mathfrak{z}=0$ and for every two-sided ideal $\mathfrak{z}$ which contains $M(=r(N)=l(N))$. Then (γ_1) and (γ_2) are satisfied by every two-sided ideal $\mathfrak{z}$ of A.*

Proof. We have $r(N)=l(N)=M\subseteqq N$ by what we have cited and by the definition of a bound ring. $M=\sum E_{\pi(\kappa)}M=\sum ME_\kappa$ is the unique decomposition of M into direct sum of simple two-sided ideals. Let $\mathfrak{z}$ be an arbitrary two-sided ideal. Then we can prove (γ_1) and (γ_2) for $\mathfrak{z}$ in an analogous manner as the proof of prop. 6 of [1].

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3) A bound ring is a ring in which every two-sided annihilator of the radical is contained in the radical.