REMARKS CONCERNING TWO QUASI-FROBENIUS RINGS WITH ISOMORPHIC RADICALS

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The purpose of this short note is to make some supplementary remarks to the author's previous work [2] and refine theorem 2 of [2]. Let A and $\tilde{A}$ be two quasi-Frobenius rings and let the radical $\tilde{N}$ of $\tilde{A}$ be isomorphic to the radical N of A; we shall identify $\tilde{N}$ with N and say that A and $\tilde{A}$ have the same radical N. Let

$$A=\sum_{\kappa=1}^{k}\sum_{\imath=1}^{f(\kappa)}Ae_{\kappa,\imath}$$

be a decomposition of A into direct sum of indecomposable left ideals; the elemeats $e_{\kappa,\imath}$ $(1\leqq\kappa\leqq k,\ 1\leqq i\leqq f(\kappa))$ are mutually orthogonal primitive idempotents of A such that $Ae_{\kappa,\imath}\cong Ae_{\lambda,\jmath}$ if and only if $\kappa=\lambda$. We put $e_{\kappa,1}=e_{\kappa}$, $\sum_{\imath}e_{\kappa,\imath}=E_{\kappa}$; $E=\sum_{\kappa}E_{\kappa}$ is the unit element of A. Further, let $\tilde{e}_{\kappa,\imath}$, $\tilde{E}_{\kappa}$, etc. have the same meaning to $\tilde{A}$ as $e_{\kappa,\imath}$, E_{κ}, etc. to A. For a subset S of A we denote the left [right] annihilators of S by $l_A(S)$ $[r_A(S)]$; the notations $l_{\tilde{A}}(*)$, $l_N(*)$ etc. may be defined similarly.

Remembering theorem 1 of [2], we shall assume in this note that both A and $\tilde{A}$ are bound rings and that $M=l_N(N)=r_N(N)$ is contained in N^2. Then by theorem 2 of [2] $\bar{A}=A/N$ is isomorphic to $\bar{\tilde{A}}=\tilde{A}/N$; moreover, there is a (unique) 1-1 correspondence between the simple constituents of $\bar{A}$ and those of $\bar{\tilde{A}}$. So that we may assume, after a suitable reordering, that $\bar{A}_{\kappa}=\bar{A}\bar{E}_{\kappa}$ corresponds to $\bar{\tilde{A}}_{\kappa}=\bar{\tilde{A}}\bar{\tilde{E}}_{\kappa}$ in this correspondence $(1\leqq\kappa\leqq k)$.

PROPOSITION 1. *Let A and $\tilde{A}$ be as above. Let $\mathfrak{l}\supset\mathfrak{l}'$ be two left A-ideals contained in N and let the factor module $\mathfrak{l}/\mathfrak{l}'$ be simple and isomorphic to Ae_{κ}/Ne_{κ}. Assume moreover that $\mathfrak{l}$ and $\mathfrak{l}'$ are left $\tilde{A}$-ideals. Then $\mathfrak{l}/\mathfrak{l}'$ is also a simple $\tilde{A}$-module and is isomorphic to $\tilde{A}\tilde{e}_{\kappa}/N\tilde{e}_{\kappa}$. Similarly for right ideals.*

Proof. First we assume that $\mathfrak{l}\smile M=\mathfrak{l}'\smile M$. Then we must have $\mathfrak{l}\frown M\supsetneqq\mathfrak{l}'\frown M$, and there exists a minimal left A-ideal $\mathfrak{l}_0$ in M such that $\mathfrak{l}\frown M=\mathfrak{l}'\frown M+\mathfrak{l}_0$; from this it follows that $\mathfrak{l}=\mathfrak{l}'+\mathfrak{l}_0$ and the assumption $\mathfrak{l}/\mathfrak{l}'\cong Ae_{\kappa}/Ne_{\kappa}$ shows that $\mathfrak{l}_0\cong Ae_{\kappa}/Ne_{\kappa}$. As $\mathfrak{l}_0$ is also a left $\tilde{A}$-ideal, we have that $\mathfrak{l}/\mathfrak{l}'\cong\mathfrak{l}_0\cong\tilde{A}\tilde{e}_{\kappa}/N\tilde{e}_{\kappa}$ is a simple $\tilde{A}$-module. Now assume that $\mathfrak{l}\smile M\supsetneqq\mathfrak{l}'\smile M$. In this case, $\mathfrak{l}\frown M$ must coincide with $\mathfrak{l}'\frown M$. In fact, if $\mathfrak{l}\frown M\supsetneqq\mathfrak{l}'\frown M$, we have for a suitable left A-ideal $\mathfrak{l}^*$ in M $\mathfrak{l}\frown M=\mathfrak{l}'\frown M+\mathfrak{l}^*$, which implies $\mathfrak{l}=\mathfrak{l}'+\mathfrak{l}^*$ since $\mathfrak{l}/\mathfrak{l}'$ is a simple A-module. This contradicts the assumption $\mathfrak{l}\smile M\supsetneqq\mathfrak{l}'\smile M$. Now, note that $\mathfrak{l}\smile M/\mathfrak{l}'\smile M=\mathfrak{l}\smile(\mathfrak{l}'\smile M)/\mathfrak{l}'\smile M\cong\mathfrak{l}/\mathfrak{l}'$ (as A-modules

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and at the same time as $\tilde{A}$-modules); as $\mathfrak{l}/\mathfrak{l}' \cong Ae_\kappa/Ne_\kappa$ (as an A-module), $\mathfrak{l}^\smile M/\mathfrak{l}'^\smile M$ is a simple A-module and hence is a simple $\tilde{A}$-module (by lemma 1 of [2]). Therefore $\mathfrak{l}/\mathfrak{l}'$ is a simple left $\tilde{A}$-module. In order to show that $\mathfrak{l}/\mathfrak{l}'$ is isomorphic to $\tilde{A}\tilde{e}_\kappa/N\tilde{e}_\kappa$ as an $\tilde{A}$-module, we take an element $a\ (\neq 0)$ in $r_N(\mathfrak{l}') - r_N(\mathfrak{l})$;[1] the mapping $x \to xa\ (x \in l)$ is an A-homomorphism of $\mathfrak{l}$ into M, and by our assumptions we have $\mathfrak{l}a \cong Ae_\kappa/Ne_\kappa$. This mapping, however, is also an $\tilde{A}$-homomorphism of $\mathfrak{l}$; then, since $\mathfrak{l}a \cong \tilde{A}\tilde{e}_\kappa/N\tilde{e}_\kappa$ (as an $\tilde{A}$-module) we have $\mathfrak{l}/\mathfrak{l}' \cong \tilde{A}\tilde{e}_\kappa/N\tilde{e}_\kappa$.

Let the index of N be ρ: $N \supset N^2 \supset \cdots \supset N^{\rho-1} \supset N^\rho = 0$. For the ring A we denote by $t^\tau_{\kappa\lambda}$ the number of simple submodules of the completely reducible module $N^\tau e_\lambda/N^{\tau+1}e_\lambda$ which are isomorphic to the simple module Ae_κ/Ne_κ ($1 \leqq \kappa, \lambda \leqq k$, $0 \leqq \tau \leqq \rho-1$; $N^0 = A$). The same number for the ring $\tilde{A}$ will be denoted by $\tilde{t}^\tau_{\kappa\lambda}$. Then the numbers

$$c_{\kappa\lambda} = \sum_{\tau=0}^{\rho-1} t^\tau_{\kappa\lambda} \quad \text{and} \quad \tilde{c}_{\kappa\lambda} = \sum_{\tau=0}^{\rho-1} \tilde{t}^\tau_{\kappa\lambda}$$

are the left Cartan invariants of A and $\tilde{A}$, respectively.[2]

PROPOSITION 2. *Let A and $\tilde{A}$ be as above. Then there exists an N-isomorphism between two left N-ideals Ne_κ and $N\tilde{e}_\kappa$; and, by this isomorphism every left A-subideal of Ne_κ is mapped onto a left $\tilde{A}$-subideal of $N\tilde{e}_\kappa$ and conversely. Moreover, let $\mathfrak{l} \supset \mathfrak{l}'$ be two $\tilde{A}$-subideals of Ne_κ such that $\mathfrak{l}/\mathfrak{l}' \cong Ae_\lambda/Ne_\lambda$; let $\tilde{\mathfrak{l}}$, $\tilde{\mathfrak{l}}'$ be the corresponding left $\tilde{A}$-subideals of $N\tilde{e}_\kappa$ (by this isomorphism). Then $\tilde{\mathfrak{l}}/\tilde{\mathfrak{l}}' \cong \tilde{A}\tilde{e}_\lambda/N\tilde{e}_\lambda$. Similarly for right ideals.*

Proof. At the outset we observe that the left A-ideal Ne_κ has the unique minimal subideal Me_κ; similarly, $N\tilde{e}_\kappa$ has the unique minimal subideal $M\tilde{e}_\kappa$. We may assume, without loss of generality, that Me_κ coincides with $M\tilde{e}_\kappa$. Now consider a mapping φ: $x \to x\tilde{e}_\kappa\ (x \in Ne_\kappa)$ and a mapping $\tilde{\varphi}$: $y \to ye_\kappa\ (y \in N\tilde{e}_\kappa)$; by the proof of prop. 7 of [2] φ is an N-isomorphism of Ne_κ into $N\tilde{e}_\kappa$ and maps every A-subideal of Ne_κ onto an $\tilde{A}$-subideal of $N\tilde{e}_\kappa$; similar fact is valid for $\tilde{\varphi}$. Further, φ and $\tilde{\varphi}$ are onto mappings; in fact, the composed mapping $\tilde{\varphi}\varphi$: $x \to (x\tilde{e}_\kappa)e_\kappa$ is an N-isomorphism of Ne_κ into itself and maps every A-subideal of Ne_κ onto an A-subideal. Therefore, considering the composition length of Ne_κ, both φ and $\tilde{\varphi}$ must be onto mappings. We have thus proved our first assertion. Let now $\mathfrak{l} \supset \mathfrak{l}'$ be two A-subideals of Ne_κ such that $\mathfrak{l}/\mathfrak{l}' \cong Ae_\lambda/Ne_\lambda$. Then $\varphi(\mathfrak{l})$ and $\varphi(\mathfrak{l}')$ are $\tilde{A}$-subideals of $N\tilde{e}_\kappa$ and $\varphi(\mathfrak{l})/\varphi(\mathfrak{l}')$ is a simple $\tilde{A}$-module. Assume that $\varphi(\mathfrak{l})/\varphi(\mathfrak{l}')$ is isomorphic to $\tilde{A}\tilde{e}_\mu/N\tilde{e}_\mu$, say. As $M\tilde{e}_\kappa$ is the unique minimal $\tilde{A}$-subideal of $\varphi(\mathfrak{l})$ (and of $\varphi(\mathfrak{l}')$), there exists an element $\tilde{a}$ of N such that $\varphi(\mathfrak{l})\tilde{a} \neq 0$, $\varphi(\mathfrak{l}')\tilde{a} = 0$; here, $\varphi(\mathfrak{l})\tilde{a}$ is obviously a simple left $\tilde{A}$-ideal and $\varphi(\mathfrak{l})\tilde{a} \cong \tilde{A}\tilde{e}_\mu/N\tilde{e}_\mu$. But, $\varphi(\mathfrak{l})\tilde{a} = \mathfrak{l}\cdot\tilde{e}_\kappa\tilde{a}$ ($\tilde{e}_\kappa\tilde{a}$ is an element of N) and $\varphi(\mathfrak{l}')\tilde{a} = \mathfrak{l}'\cdot\tilde{e}_\kappa\tilde{a} = 0$ show that $\mathfrak{l}/\mathfrak{l}'$ is isomorphic to $\varphi(\mathfrak{l})\tilde{a}$ ($\cong Ae_\mu/Ne_\mu$) as an A-modoule; we have hence $\mu = \lambda$.

1) As $\mathfrak{l}^\smile M \supsetneqq \mathfrak{l}'^\smile M$, we must have $r_N(\mathfrak{l}') \supsetneqq r_N(\mathfrak{l})$.

2) For these notions see Artin-Nesbitt-Thrall [1], Ch. 9.

The following theorem is now immediate.

THEOREM. *Let A and $\tilde{A}$ be as above. Then we have* $t^{\tau}_{\kappa\lambda} = \tilde{t}^{\tau}_{\kappa\lambda}$ $(1 \leqq \kappa, \lambda \leqq k, 0 \leqq \tau \leqq \rho - 1)$; *in particular, the left Cartan invariants* $c_{\kappa\lambda}$ *of A coincide with those,* $\tilde{c}_{\kappa\lambda}$, *of* $\tilde{A}$. *The same fact is also valid for the right-hand side invariants.*

REFRENCES

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