A THEOREM OF RENEWAL TYPE

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1. Introduction.

Let

$$X_1, X_2, X_3, \cdots \tag{1.1}$$

be a sequence of independent random variables with an identical distribution. Set

$$S_n = \sum_{k=1}^{n} X_k. \tag{1.2}$$

Moreover if X_k, $k=1, 2, \cdots$, are non-negative, $N(t)$ is defined to be the biggest n for which $S_n \leqq t$, and $H(t)=EN(t)$, then one obtains

$$H(t+h)-H(t) \to \frac{h}{\mu} \quad \text{as} \quad t \to \infty, \tag{1.3}$$

where $h>0$ is a constant and $\mu = EX_1 > 0$ to be supposed. The fact (1.3) is now a classical renewal theorem due to Blackwell [1, 2] and was proved also by Doob [4], Kesten and Runnenburg [6]. Also see Smith [8].

(1.3) is also true even if X_n is not non-negative provided that μ, $h>0$ and X_n is non-lattice. This was first proved by Chung and Pollard [3] under some restrictions and later generally proved by Maruyama [7].

Since

$$H(t+h)-H(t) = \sum_{n=1}^{\infty} P(t < S_n \leqq t+h),$$

(1.3) is equivalent to

$$\lim_{t\to\infty} \sum_{n=1}^{\infty} P(t < S_n \leqq t+h) = \frac{h}{\mu}. \tag{1.4}$$

Now in the present paper we shall consider the relation

$$\lim_{t\to\infty} \sum_{n=1}^{\infty} a_n P(t < S_n \leqq t+h) = \frac{ha}{\mu}. \tag{1.5}$$

It is easily seen that if

$$\lim_{n\to\infty} a_n = a \tag{1.6}$$

then (1.5) is valid. We want to generalize this relation assuming instead of (1.6) that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} a_k = a. \tag{1.7}$$

In this connection we have to mention Smith's results [8] in which he

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considered the density of S_n in place of $P(t < S_n \leqq t+h)$ and showed under some conditions on the distributions of X_n that

$$\lim_{t\to\infty} \sum_{n=1}^{\infty} a_n h_n(t) = \frac{a}{\mu}, \tag{1.8}$$

assuming that

$$\frac{1}{p}\sum_{k=n}^{n+p} a_k \to a \tag{1.9}$$

as $p\to\infty$, uniformly with respect to n, where X_n is not necessarily non-negative.

We would like to notify that when we deal with (1.5), under (1.7) instead of (1.9) we shall find that the situation will be quite different. For instance if (1.7) is assumed, (1.5) does not necessarily hold.

2. The theorem and a lemma.

We shall state the theorem. Let $\{X_n\}$ be a sequence of independent random variables with identical distributions which are not necessarily non-negative.

THEOREM. *Suppose that a sequence of real numbers* $\{a_n\}$ *satisfies*

$$\frac{1}{n}\sum_{k=1}^{n} a_k = a + o\left(\frac{1}{\sqrt{n}}\right), \tag{2.1}$$

X_k *has a probability density with the finite third moment and the probability density of the sum* $S_n = \sum_1^n X_k$ *belongs for some* n *to* L_r *for some* $1 < r \leqq 2$. *Then the following relation should be valid*:

$$\lim_{t\to\infty} \sum_{n=1}^{\infty} a_n P(t < S_n \leqq t+h) = \frac{ha}{\mu}, \tag{2.2}$$

provided that $EX_n = \mu > 0$.

If (2.1) *is replaced by*

$$\frac{1}{n}\sum_{k=1}^{n} a_n = a + o\left(\frac{1}{n^{\alpha}}\right), \quad \alpha \leqq \frac{1}{2}, \tag{2.3}$$

then (2.2) *does not necessarily hold.*

We shall show the last result in §5 below after we shall have completed the proof of theorem.

To prove the theorem we shall require the following lemmas which we shall prove in the later sections.

LEMMA 1. *Under the conditions of the theorem, each of*

$$\sum_{n\mu > t+\sqrt{t}} \sqrt{n}\,|P(t < S_n \leqq t+h) - P(t < S_{n-1} \leqq t+h)|, \tag{2.4}$$

$$\sum_{n\mu < t-\sqrt{t}} \sqrt{n}\,|P(t < S_n \leqq t+h) - P(t < S_{n-1} \leqq t+h)| \tag{2.5}$$

and

(2.6)
$$\sum_{t-\sqrt{t}<n\mu<t+\sqrt{t}} \sqrt{n}\,|P(t<S_n\leqq t+h)-P(t<S_{n-1}\leqq t+h)|$$

are bounded over $0<t<\infty$.

If this lemma is supposed to be proved, the theorem easily follows making use of the theorem of Chung, Pollard and Maruyama quoted in §1. In fact, setting

(2.7)
$$\frac{1}{n}\sum_1^n a_k = A_n,$$

we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} a_n P(t<S_n\leqq t+h)\\
&=\sum_{n=1}^{\infty}(nA_n-(n-1)A_{n-1})P(t<S_n\leqq t+h) \qquad (A_0=0)\\
&=\sum_{n=1}^{\infty} nA_n\{P(t<S_n\leqq t+h)-P(t<S_{n+1}\leqq t+h)\},
\end{aligned}$$

because $nA_n\cdot P(t<S_n\leqq t+h)$ converges to zero as $n\to\infty$ (since $P(t<S_n\leqq t+h)$ diminishes exponentially).

Hence writing $P(t<S_n\leqq t+h)=\tau_n(t)$, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} a_n P(t<S_n\leqq t+h)\\
&=\sum_{n=1}^{\infty} n(A_n-a)(\tau_n(t)-\tau_{n+1}(t))+a\sum_{n=1}^{\infty} n(\tau_n(t)-\tau_{n+1}(t))\\
&=\sum_{n=1}^{\infty} n(A_n-a)\{\tau_n(t)-\tau_{n+1}(t)\}+a\sum_{n=1}^{\infty}\tau_n(t),
\end{aligned}$$

the last member of which converges to ah/μ. So it suffices to show that the first term converges to zero. We divide that into three parts as

$$\sum_{n\mu<t-\sqrt{t}}+\sum_{t-\sqrt{t}<n\mu<t+\sqrt{t}}+\sum_{n\mu>t+\sqrt{t}}=L_1+L_2+L_3.$$

Lemma shows that

$$\begin{aligned}
L_2&=\sum_{t-\sqrt{t}<n\mu<t+\sqrt{t}} O\left(\frac{1}{\sqrt{n}}\right) n\,|\tau_n(t)-\tau_{n+1}(t)|=o(1),\\
L_1&=\sum_{n\mu<t-\sqrt{t}} o\left(\frac{1}{\sqrt{n}}\right)\cdot n\,|\tau_n(t)-\tau_{n+1}(t)|\\
&=o\left(\sum_{n\mu<t-\sqrt{t}}\sqrt{n}\,|\tau_n(t)-\tau_{n+1}(t)|\right)=o(1)\cdot o(1),
\end{aligned}$$

and

$$L_3=\sum_{n\mu>t+\sqrt{t}} o\left(\frac{1}{\sqrt{n}}\right) n\,|\tau_n(t)-\tau_{n+1}(t)|=o(1),$$

which proves the theorem.

3. Proof of (2.4), (2.5) of the lemma.

To prove the lemma we use the following elementary facts. Let $\varphi(u)$ be a characteristic function of a random variable which has a probability density with mean 0 and the finite third moment. We then have

$$\varphi^{(k)}(u)=o(1), \qquad \text{for} \quad k=1, 2, 3, \tag{3.1}$$

$$|\varphi(u)|\leqq 1-\frac{\sigma^2u^2}{4}\leqq e^{-\sigma^2u^2/4} \qquad \text{for} \quad |u|<\varepsilon \tag{3.2}$$

for some small $\varepsilon>0$, σ^2 being the variance of the variable,

$$|\varphi(u)|<e^{-c} \qquad \text{for} \quad |u|>\varepsilon \text{ and some } c>0, \tag{3.3}$$

$$\varphi(u)=o(1) \qquad \text{as} \quad |u|\to\infty, \tag{3.4}$$

and

$$\varphi'(u)=-\sigma^2u+o(u) \qquad \text{for small} \quad u. \tag{3.5}$$

Moreover if the density of $S_n=\sum_1^n X_k$, $\{X_k\}$ being a sequence of independent random variables with identical distributions, for some n belongs to L_r $(1<r\leqq 2)$ for some r, then

$$\int_{-\infty}^{\infty}|\varphi(u)|^n du<\infty \tag{3.6}$$

for large n, $\varphi(u)$ being the characteristic function of X_k. See for instance Gnedenko and Kolmogorov [5].

Now we proceed to prove the lemma. Applying the well known inversion formula, we can express (2.4) as

$$\begin{aligned} K(t)&=\frac{1}{2\pi}\sum_{n\mu>t+\sqrt{t}}\sqrt{n}\left|\int_x^{x+h}dy\int_{-\infty}^{\infty}f^n(u)e^{-iuy}du-\int_x^{x+h}dy\int_{-\infty}^{\infty}f^{n+1}(u)e^{-iuy}du\right| \\ &=\frac{1}{2\pi}\sum_{n\mu>t+\sqrt{t}}\sqrt{n}\left|\int_x^{x+h}dy\int_{-\infty}^{\infty}f^n(u)(1-f(u))e^{-iuy}du\right| \\ &=\frac{1}{2\pi}\sum_{n\mu>t+\sqrt{t}}\sqrt{n}\left|\int_{x-n\mu}^{x+h-n\mu}dy\int_{-\infty}^{\infty}\varphi^n(u)(1-f(u))e^{-iuy}du\right|, \end{aligned} \tag{3.7}$$

where $\varphi(u)$ is the characteristic function of $X_k-\mu$.

We now apply the integration by parts three times in the inner integral which gives

$$K_1(y)=\int_{-\infty}^{\infty}\varphi^n(u)(1-f(u))e^{-iuy}du=\frac{1}{i^3y^3}\int_{-\infty}^{\infty}e^{-iuy}\frac{d^3}{du^3}\{\varphi^n(u)g(u)\}du, \tag{3.8}$$

where we have written

$$1-f(u)=g(u). \tag{3.9}$$

The integrated terms are of the form

$$\left[\left(-\frac{1}{iy}\right)^k e^{-iuy}\frac{d^{k-1}}{du^{k-1}}\{\varphi^{n-1}(u)g(u)\}\right]_{u=-\infty}^{\infty}, \qquad k=1, 2, 3,$$

which vanish because of (3.1) and (3.4). Also we used

(3.10) $$g^{(k)}(u)=o(1)\qquad k=0,1,2,3,$$

(3.8) then turns out to

$$\frac{1}{i^3y^3}\sum_{k=0}^{3}\binom{3}{k}\int_{-\infty}^{\infty}e^{-iyu}\frac{d^{3-k}}{du^{3-k}}\varphi^n(u)\cdot\frac{d^k}{du^k}g(u)\,du$$
$$=\frac{1}{i^3y^3}n(n-1)(n-2)\int_{-\infty}^{\infty}e^{-iyu}\varphi^{n-3}(u)\{\varphi'(u)\}^3g(u)\,du$$
$$+\frac{3}{i^3y^3}n(n-1)\int_{-\infty}^{\infty}e^{-iyu}\varphi^{n-2}(u)\varphi'(u)\varphi''(u)\,g(u)\,du$$
$$+\frac{3}{i^3y^3}n\int_{-\infty}^{\infty}e^{-iyu}\varphi^{n-1}(u)\varphi'''(u)\,g(u)\,du$$
$$+\frac{3}{i^3y^3}n(n-1)\int_{-\infty}^{\infty}e^{-iyu}\varphi^{n-2}(n)\{\varphi'(u)\}^2g'(u)\,du$$
$$+\frac{3}{i^3y^3}n\int_{-\infty}^{\infty}e^{-iyu}\varphi^{n-1}(u)\varphi''(u)\,g'(u)\,du$$
$$+\frac{3}{i^3y^3}n\int_{-\infty}^{\infty}e^{-iyu}\varphi^{n-1}(u)\varphi'(u)\,g''(u)\,du$$
$$+\frac{1}{i^3y^3}n\int_{-\infty}^{\infty}\varphi^n(u)g'''(u)\,du$$
$$=\sum_{k=1}^{7}J_k,$$

say. We shall estimate each of J_k. First we shall consider J_1. Let ε be a positive number so small that (3.2) is true.

(3.11) $$J_1=\frac{1}{i^3y^3}\left(\int_{|u|\leqq\varepsilon}+\int_{|u|>\varepsilon}\right)=J_{11}+J_{12},$$

say. By (3.1), (3.2) and

$$g(u)=o(u)\qquad\text{for small }\ |u|,$$

(3.12) $$|J_{11}|\leqq C\frac{n^3}{|y|^3}\int_{|u|\leqq\varepsilon}e^{-(n-3)\sigma^2u^2/4}|u|^4du,$$

where σ^2 is the variance of X_k and C is a constant independent of n. Hereafter we shall use the generic notation C to express a constant independent of n, y, t which may differ on each occurrence.

The above expression comes to

(3.13) $$|J_{11}|\leqq C\frac{n^{1/2}}{|y|^3}\int_{|z|\leqq\varepsilon\sqrt{n}}e^{-z^2/8}|z|^4dz\leqq\frac{cn^{1/2}}{|y|^3}.$$

As to J_{12} we see, by (3.1), (3.3) and (3.10),

$$|J_{12}|\leqq\frac{cn^3}{|y|^3}\int_{|u|>\varepsilon}|\varphi(u)|^{n-3}du$$
$$\leqq\frac{cn^3}{|y|^3}\int_{|u|>\varepsilon}e^{-\{(n-3)c-\alpha\}}|\varphi(u)|^{\alpha}du,$$

taking n large enough and letting α be such that (3.6) holds with α for n. We then obtain

$$|J_{12}| \leqq \frac{cn^3}{|y|^3} e^{-nc} \int_{-\infty}^{\infty} |\varphi(u)|^{\alpha} du = \frac{cn^3}{|y|^3} e^{-nc} \tag{3.14}$$

(3.13) and (3.14) give

$$|J_1| \leqq \frac{cn^{1/2}}{|y|^3} + \frac{cn^3}{|y|^3} e^{-nc} = \frac{cn^{1/2}}{|y|^3} + \frac{cn^c e^{-nc}}{|y|^3}. \tag{3.15}$$

Similar estimates will be obtained for other J's except J_3. As to J_3 a better estimate will be valid:

$$|J_3| \leqq \frac{c}{|y|^3} + \frac{cn}{|y|^3} e^{-nc}.$$

After all we have

$$|K_1(y)| \leqq \frac{cn^{1/2}}{|y|^3} + \frac{cn^c e^{-nc}}{|y|^3}. \tag{3.16}$$

Inserting this into (3.7) we obtain

$$\begin{aligned} K(t) &\leqq C \sum_{n\mu > t+\sqrt{t}} (n + n^c e^{-nc}) \left| \int_{t-n\mu}^{t+h-n\mu} \frac{dy}{y^3} \right| \\ &= C \sum_{n\mu > t+\sqrt{t}} \frac{nh}{(n\mu - (t+h))^3} + C \sum_{n\mu > t+\sqrt{t}} n^c e^{-nc} \frac{1}{(n\mu - (t+h))^3} \\ &\leqq C \int_{t+\sqrt{t}}^{\infty} \frac{x}{(x-(t+h))^3} dx + \frac{c}{t^{3/2}} \sum_{n\mu > t} n^c e^{-nc} \\ &\leqq C \frac{1}{t} \int_{(\sqrt{t}-h)/(t+h)}^{\infty} \frac{z}{(1-z)^3} dz + o(1) \\ &= o(1) + o(1) = o(1). \end{aligned}$$

We hence complete the proof of (2.4).

We may prove (2.5) quite similarly. In fact,

$$\begin{aligned} L_1(x) &= \sum_{n\mu < t-\sqrt{t}} \sqrt{n}\, |\tau_n(t) - \tau_{n+1}(t)| \\ &= \sum_{n\mu < t-\sqrt{t}} \sqrt{n} \left| \int_{t-n\mu}^{t+h-n\mu} dy \int_{-\infty}^{\infty} \varphi^n(u)(1-f(u)) e^{-iuy} du \right| \\ &\leqq C \sum_{n\mu < t-\sqrt{t}} \sqrt{n} \left| \int_{t-n\mu}^{t+h-n\mu} \frac{n^{1/2}}{|y|^3} + \frac{cn^c e^{-nc}}{|y|^3} dy \right|. \end{aligned}$$

We here used the estimate (3.16) again. Hence

$$\begin{aligned} L_1(x) &\leqq C \sum_{n\mu < t-\sqrt{t}} (n + n^c e^{-nc}) \int_{t-n\mu}^{t+h-n\mu} \frac{dy}{y^3} \\ &\leqq C \sum_{n\mu < t-\sqrt{t}} \frac{nh}{(t-n\mu)^3} + C \sum_{n\mu < t-\sqrt{t}} n^c e^{-nc} \frac{1}{(t-n\mu)^3} \\ &\leqq C \int_0^{t-\sqrt{t}} \frac{x}{(t-x)^3} dx + \frac{1}{t^{3/2}} \sum_1^{\infty} n^c e^{-nc} \end{aligned}$$

$$= \frac{1}{t}\int_0^{t-1/\sqrt{t}} \frac{z}{(1-z)^3}\,dz + o(1)$$
$$= o(1) + o(1) = o(1).$$

4. Proof of (2.6) of the lemma.

(2.6) can be written as in the proof of (2.4) or (2.5) as

$$
\begin{aligned}
M(t) &= \sum_{t-\sqrt{t}<n\mu<t+\sqrt{t}} \sqrt{n}\left|\int_{x-n\mu}^{x+h-n\mu} dy \int_{-\infty}^{\infty} \varphi^n(u)(1-f(u))e^{-\imath uy}\,du\right| \\
&\leqq c\sqrt{t} \sum_{t-\sqrt{t}<n\mu<t+\sqrt{t}} \int_{-\infty}^{\infty} |\varphi(u)|^n\,|1-f(u)|\,du \\
&\leqq c\sqrt{t}\sum\int_{|u|\leqq\varepsilon} + c\sqrt{t}\sum\int_{|u|\geqq\varepsilon} \\
&= M_1(t) + M_2(t),
\end{aligned}
\tag{4.1}
$$

say. The same argument as in the estimate of J_{12} in the proof of (2.4), leads us to, with same notations,

$$
\begin{aligned}
M_2(t) &\leqq c\sqrt{t} \sum_{t-\sqrt{t}<n\mu<t+\sqrt{t}} e^{-nc}\int_{-\infty}^{\infty} |\varphi(u)|^{\alpha}\,du \\
&= ct \sum_{t-\sqrt{t}<n\mu} e^{-nc} = o(\sqrt{t}\,e^{-tc}) = o(1).
\end{aligned}
\tag{4.2}
$$

We further divide the integral in $M_1(t)$ as

$$\int_{|u|<1/t} + \int_{\varepsilon>|u|>1/t}$$

the former of which is

$$\int_{|u|<1/t} |\varphi(u)|^n\,|1-f(u)|\,du \leqq \int_{|u|<1/t} du = o\left(\frac{1}{t}\right). \tag{4.3}$$

The second integral does not exceed

$$\int_{\varepsilon>|u|>1/t} |1-f(u)|\,\frac{|\varphi(u)|^{t-\sqrt{t}}}{1-|\varphi(u)|}(1-|\varphi(u)|^{2t^{1/2}})\,du$$

which in turn, by (3.2) and the fact that

$$|1-f(u)| \leqq c\,|u|, \qquad \varphi(u) = 1 - \frac{\sigma^2 u^2}{2} + o(u^2),$$

is not greater than

$$
\begin{aligned}
& c\int_{\varepsilon>|u|>1/t} |u|\,\frac{e^{-\sigma^2u^2t/2}}{\sigma^2u^2/2}\left(1-\left(1-\frac{u^2\sigma^2}{2}\right)^{2t^{1/2}}\right)du \\
&\leqq c\int_{\varepsilon>|u|>1/t} |u|^{-1}e^{-cu^2t}u^2t^{1/2}\,du \\
&\leqq c\frac{1}{\sqrt{t}}\int_{-\varepsilon\sqrt{t}}^{\varepsilon\sqrt{t}} e^{-cv}v\,dv \leqq \frac{c}{\sqrt{t}}.
\end{aligned}
\tag{4.4}
$$

(4.3) and (4.4) give $M_2(t) = o(1)$ which with (4.2) proves (2.6) of the lemma.

5. A negative result.

We shall show that if the condition (1.5) is replaced by the weaker one

$$A_n = a + o\left(\frac{1}{n^\alpha}\right), \qquad \alpha \leqq \frac{1}{2} \tag{5.1}$$

then the theorem ceases to be true.

Let X_i depend on the normal law with mean 1 and variance 1. We consider the sequence

$$a_n = (-1)^n n^p. \tag{5.2}$$

If $p < 1$, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} a_k = 0,$$

so that $a = 0$ in the theorem. We also have

$$a_n P(x < S_n \leqq x + h) = (-1)^n n^p \int_x^{x+h} \frac{1}{\sqrt{2\pi n}} e^{-(y-n)^2/2n} dy.$$

We verify that the sequence

$$n^p \frac{1}{\sqrt{2\pi n}} e^{-(y-n)^2/2n}$$

is non-decreasing if $n < \sqrt{y^2 + \beta^2} - \beta$ and is non-increasing if $n > \sqrt{y^2+\beta^2} - \beta$, where $\beta = p - 1/2$.

Take h so small that

$$\sqrt{x^2 + \beta^2} + 1 > \sqrt{(x+h)^2 + \beta^2}$$

and to be x and N so that

$$N < \sqrt{x^2+\beta^2} - \beta < \sqrt{(x+h)^2 + \beta^2} - \beta < N + 1 \tag{5.3}$$

and N is even.

Putting

$$u_n = n^p \int_x^{x+h} \frac{1}{\sqrt{2\pi n}} e^{-(y-n)^2/2n} dy,$$

we can write

$$\begin{aligned} &\sum_{n=1}^{\infty} a_n P(x < S_n \leqq x+h) \\ &= \{(-u_1 + u_2) + \cdots (-u_{N-1} + u_N)\} - u_{N+1} + \{(u_{N+2} - u_{N+3} + \cdots\} \\ &= S_1 - u_{N+1} + S_2. \end{aligned} \tag{5.4}$$

Obviously $S_1 > 0$, $S_2 > 0$.

Now

$$u_{N+1} \leqq N^p \int_x^{x+h} \frac{1}{\sqrt{2\pi N}} e^{-(x-N)^2/2N} dy \leqq \frac{h}{\sqrt{2\pi}} N^{p-1/2}, \tag{5.5}$$

while for $n < N$

$$u_n - u_{n-1} = \int_x^{x+h}\left(n^p \frac{1}{\sqrt{2\pi n}} e^{-(y-n)^2/2n} - (n-1)^p \frac{1}{\sqrt{2\pi(n-1)}} e^{-(y-(n-1))^2/2(n-1)}\right) dy$$
$$= \frac{1}{\sqrt{2\pi}} \int_x^{x+h} \frac{1}{2} e^{-(y-n_1)^2/2n_1} n_1^{p-5/2}[(2p-1)n_1 + (y^2 - n_1^2)]\, dy,$$

where $n \geqq n_1 \geqq n-1$. Hence

$$u_n - u_{n-1} \geqq \frac{1}{2\sqrt{2\pi}} \int_x^{x+h} e^{-(y-n)^2/2n} n^{p-5/2}((2p-1)(n-1) + (y^2 - n^2))\, dy$$
$$\geqq \frac{2p-1}{2\sqrt{2\pi}} x^{p-3/2} \int_x^{x+h} e^{-(y-n)^2/2x}\, dy$$
$$+ \frac{1}{2\sqrt{2\pi}} (2x - A\sqrt{x}) x^{p-5/2} \int_x^{x+h} e^{-(y-n)^2/2x} (y-n)\, dy,$$

if $n > x - A\sqrt{x}$. Hence

$$\sum_{N \geqq n} (u_n - u_{n-1}) \geqq \sum_{N \geqq n > x - A\sqrt{x}} (u_n - u_{n-1})$$
$$\geqq \frac{2p-1}{2\sqrt{2\pi}} x^{p-3/2} \int_x^{x+h} \sum_{N \geqq n > x - A\sqrt{x}} e^{-(y-n)^2/2x}\, dy$$
$$+ \frac{1}{2\sqrt{2\pi}} (2x - A\sqrt{x}) x^{p-5/2} \int_x^{x+h} \sum_{N \geqq n > x - A\sqrt{2}} e^{-(y-n)^2/2x} (y-n)\, dy$$
$$\geqq \frac{2p-1}{2\sqrt{2\pi}} x^{p-3/2} \int_x^{x+h} dy \int_{x - A\sqrt{x}}^{N-1} e^{-(y-z)^2/2x}\, dz$$
$$+ \frac{1}{2\sqrt{2\pi}} (2x - A\sqrt{x}) x^{p-5/2} \int_x^{x+h} dy \int_{x - A\sqrt{x}}^{N-1} e^{-(y-z)^2/2x} (y-z)\, dz$$
$$\geqq \left(p - \frac{1}{2}\right) \frac{h}{\sqrt{2\pi}} x^{p-1/2} \int_{1/\sqrt{x}}^{A} e^{-s^2/2}\, ds + \frac{x - A/2 \cdot \sqrt{x}}{\sqrt{2\pi}} x^{p-3/2} \int_{1/\sqrt{x}}^{A} s e^{-s^2/2}\, ds.$$

Hence we have

$$S_1 \geqq K(N),$$

where

$$K(N) \sim \left(p - \frac{1}{2}\right) \frac{h}{\sqrt{2\pi}} N^{p-1/2} \int_{1/\sqrt{N}}^{A} e^{-s^2/2}\, ds + \frac{N^{p-3/2}}{\sqrt{2\pi}} \int_{1/\sqrt{N}}^{A} s e^{-s^2/2}\, ds.$$

We similarly have omitting details, that for some small ε_1, ε_2 and ε_3 and for an arbitrary B,

$$S_2 \geqq \sum_{x + B\sqrt{x} \geqq n \geqq N+2} (u_n - u_{n+1})$$
$$\geqq -\left(p - \frac{1}{2}\right) \frac{h}{\sqrt{2\pi}} (x - \varepsilon_1)^{p-1/2} \int_0^B e^{-s^2/2}\, ds + \frac{(x - \varepsilon_2)^{p-1/2}}{\sqrt{2\pi}} \int_{\varepsilon_3}^{B} s e^{-s^2/2}\, ds$$
$$\sim -\left(p - \frac{1}{2}\right) \frac{h}{\sqrt{2\pi}} N^{p-1/2} \int_0^B e^{-s^2/2}\, ds + \frac{N^{p-1/2}}{\sqrt{2\pi}} \int_{\varepsilon_3}^{B} s e^{-s^2/2}\, ds.$$

Taking A, B and ε_3 appropriately and also such that

$$\left(\int_{1/\sqrt{N}}^{A} + \int_{\varepsilon_3}^{B}\right) s e^{-s^2/2}\, ds > 1 + c, \qquad c > 0,$$

we finally obtain

$$\sum_{n=1}^{\infty} a_n P(x < S_n \leqq x + h) \geqq o(N^{p-1/2}) + \frac{ch}{\sqrt{2\pi}} N^{p-1/2}$$

which proves our assertion.

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