ON THE GROWTH ON MINIMAL POSITIVE HARMONIC FUNCTIONS IN A PLANE REGION

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Under the same title Kjellberg [1] offered an important and suggestive result:

In any planar domain D, let $v_1, \cdots, v_n$ *be* n $(\geqq 2)$ *non-proportional minimal positive harmonic functions, tending to zero in a vicinity of every finite boundary point. Let* ρ_ν *be the order of* v_ν *defined by*

$$\rho_\nu = \overline{\lim_{r\to\infty}} \frac{\log M_\nu(r)}{\log r}, \qquad M_\nu(r) = \max_{|z|=r} v_\nu(z).$$

Then it holds that

$$\sum_1^n \frac{1}{\rho_\nu} \leqq 2.$$

Here n may be ∞.

In the present paper we shall give a perfect criterion for a point to be regular for the Dirichlet problem in terms of the growth of a certain functional. Our result may be considered so as to fill up the gap in case of $n=1$ which is excluded in the above theorem. We need some preparations on positive harmonic functions.

Let D be a planar domain bounded by an infinite number of analytic Jordan curves $\{C_j\}$ whose only one clustering point is the point at infinity. Let $P(D)$ be the class of positive harmonic functions in D with the vanishing boundary value at any finite boundary point. Let $G(D)$ and $K(D)$ be two subclasses of $P(D)$ such that $u \in G(D)$ is equivalent to

$$0 < \int_{\Sigma_1^\infty C_\nu} \frac{\partial}{\partial n} u(z)\, ds < \infty$$

and $u \in K(D)$ is equivalent to

$$\int_{\Sigma_1^\infty C_\nu} \frac{\partial}{\partial n} u(z)\, ds = \infty.$$

All classes $P(D)$, $G(D)$ and $K(D)$ are evidently positively linear spaces. Martin [3] proved that any minimal positive harmonic function $m_j(z)$ can be obtained as the limit function

$$m_i(z) = \lim_{n\to\infty} \frac{g(z, p_{in})}{g(z_0, p_{in})}$$

along a suitable non-compact sequence (p_{in}), where $g(z, p)$ is the Green func-

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tion of D with singularity at p. And further, any element $u \in P(D)$ can be written as a positively linear combination of these minimals:

$$u(z)=\int_{\varDelta} m(z)\,d\mu,$$

where the integral is taken over the set $\varDelta$ of Martin minimal point with a suitable non-negative Radon measure μ. In our previous paper [4] we proved that any minimal u in $G(D)$ is obtained as the limit function

$$\lim_{n\to\infty} g(z,\ p_n)$$

along a suitable non-compact sequence (p_n) and, if the above limit function exists and does not reduce to the constant zero, then the function belongs to the class $G(D)$. Therefore we can say that the irregularity of the point at infinity is equivalent to the fact $P(D)\equiv G(D)$ and the regularity of the point ∞ to the fact $P(D)\equiv K(D)$. Let D_0 be the domain $r_0<|z|<\infty$. We may assume, with no loss of generality, that $D_0 \supset D$. Then $P(D_0)\equiv G(D_0)$ holds and further there is a one-to-one positively linear mapping S from $G(D)$ into $P(D_0)$ which preserves the singularity and the minimality and, if $G(D)$ is of dimension one, S reduces then to an onto mapping and vice versa [2], [4]. The dimension of a linear space means here the maximal cardinal number of linearly independent vectors.

By its construction $Su \geqq u$ for any $u\in G(D_0)$. Since Su has the expression $N\log(|z|/r_0)$ with a positive constant N, we can say that

$$\varlimsup_{z\to\infty}\frac{u(z)}{\log|z|}\leqq\varlimsup_{z\to\infty}\frac{Su(z)}{\log|z|}=N<\infty.$$

If u has the growth $\varlimsup_{z\to\infty} u(z)/\log|z|=+\infty$ and belongs to the class $G(D)$, then Su has the growth $\varlimsup_{z\to\infty} Su(z)/\log|z|=+\infty$, which is a contradiction. If there holds $\varlimsup_{z\to\infty} u(z)/\log|z|<N<\infty$ for a function $u\in P(D)$ and if $u(z)\in K(D)$, then $Su(z)\equiv\infty$. On the other hand, $Su(z)\leqq(N+\varepsilon)(\log|z|-\log r_0)$ holds by its construction, which is absurd. Therefore $u(z)$ must belong to the class $G(D)$.

Let $\psi(r)$ be the functional defined by

$$\int_{\{|z|=r\}\frown D} u(r,\ \theta)\,d\theta, \qquad u(r,\ \theta)\equiv u(z)\in P(D).$$

By Green's formula, we get a relation

$$-\int_{\{|z|=r\}\frown D}\frac{\partial}{\partial n}u(r,\ \theta)\,ds=\int_{\Sigma_1^\infty C_j\frown\{|z|<r\}}\frac{\partial}{\partial n}u(r,\ \theta)\,ds.$$

The left hand side is equal to $r\psi'(r)$, since $u(r,\ \theta)=0$ on each C_r. Therefore we have

$$r\psi'(r)=t(r), \qquad t(r)=\int_{\Sigma_1^\infty C_j\frown\{|z|<r\}}\frac{\partial}{\partial n}u(r,\ \theta)\,ds.$$

$t(r)$ is a non-decreasing continuous function of r positive for $r>r_1$. For any member u of $P(D)$ we have

$$0 < c \leqq \overline{\lim_{R\to\infty}} \frac{\psi(R)}{\log R} \leqq \overline{\lim_{r\to\infty}} \frac{2\pi M(r)}{\log r}, \qquad M(r) = \max_{|z|=r} u(z).$$

When $t(r)$ is bounded, then we have

$$\overline{\lim_{R\leftarrow\infty}} \frac{\psi(R)}{\log R} \leqq N < \infty, \qquad N = \lim_{R\to\infty} t(R).$$

This is the case when the point at infinity is an irregular point, since $t(r)$, then, is bounded.

THEOREM 1. *If the point at infinity is an irregular point of D for the Dirichlet problem, then $G(D)$ is of dimension one, $K(D)$ is empty and $u \in G(D)$ has the growth*

$$0 < c \leqq \overline{\lim_{z\to\infty}} \frac{u(z)}{\log|z|} \leqq N < \infty.$$

If there hold the above inequalities for a function $u(z) \in P(D)$, then $u(z) \in G(D)$ and $z=\infty$ is an irregular point.

The above theorem gives a characterization of the regularity and the irregularity of a point.

COROLLARY 1. *If $z=\infty$ is a regular point, then there exists at least a member $u(z)$ of $P(D)$ satisfying*

$$\overline{\lim_{z\to\infty}} \frac{u(z)}{\log|z|} = +\infty,$$

and vice versa.

We shall give another perfect criterion for the regularity by making use of the functional $\psi(r)$.

COROLLARY 2. *The point at infinity is a regular point for the Dirichlet problem if and only if there exists at least one minimal positive harmonic function satisfying the following condition*

$$\lim_{r\to\infty} \frac{\psi(r)}{\log r} = +\infty.$$

Proof. Let $z=\infty$ be a regular point, then we have $\lim_{r\to\infty} t(r) = +\infty$ and hence

$$\psi(r) - \psi(r_1) = \int_{r_1}^{r} \frac{t(x)}{x}\, dx$$

satisfies

$$\lim_{r\to\infty} \frac{\psi(r)}{\log r} = +\infty.$$

It there holds the above equality for a minimal positive harmonic function $u(z)$, then we have

$$\lim_{r\to\infty}\frac{M(r)}{\log r}=+\infty, \qquad M(r)=\max_{|z|=r}u(z),$$

since there holds

$$\psi(r)\leqq 2\pi M(r).$$

Hence we can say by theorem 1 that $z=\infty$ is a regular point.

Let γ_n be a sufficiently smooth curve lying in D and separating the origin from the point at infinity, which tends to the point at infinity for $n\to\infty$. Let G_n be the finite domain which is the intersection of the finite domain bounded by the curve γ_n and the domain D.

COROLLARY 3. *If $z=\infty$ is a regular point, then there holds*

$$\lim_{n\to\infty}\frac{\omega_n(z)}{D_n(\omega_n)}=0,$$

where $\omega_n(z)$ is the harmonic measure $\omega(z,\gamma_n,G_n)$ and $D_n(\omega_n)$ the Dirichlet integral extended over the domain G_n.

Proof. Let G_n' be the domain bounded by γ_n and C_1, and Ω_n be the harmonic measure $\omega(z,\gamma_n,G_n')$. Then there holds the inequality

$$\Omega_m(z)\geqq\omega_m(z), \qquad z\in G_m'.$$

This implies that there holds the inequality

$$-\frac{\partial}{\partial n}\omega_m(z)\geqq-\frac{\partial}{\partial n}\Omega_m(z)$$

on γ_m, where $\partial/\partial n$ is the inner normal derivative.

Thus there holds

$$D_m(\omega_m(z))=-\int_{\gamma_m}\frac{\partial}{\partial n}\omega_m(z)\,ds\geqq-\int_{\gamma_m}\frac{\partial}{\partial n}\Omega_m(z)\,ds=D_{G_m'}(\Omega_m).$$

This implies that

$$\frac{\omega_m(z)}{D_m(\omega_m)}\leqq\frac{\Omega_m(z)}{D_{G_m'}(\Omega_m)}.$$

On the other hand, it is well known that the right hand side tends to the Green function $g_{B_1}(z,\infty)$ of an infinite domain B_1 bounded by a single curve C_1. Therefore the left hand side, by taking a suitable subsequence if necessary, tends to either a non-trivial function $u\in P(D)$ or a trival function zero. If $u\in P(D)$, then u has the growth not greater than that of $g_{B_1}(z,\infty)$. On the other hand, $g_{B_1}(z,\infty)$ satisfies

$$0<c\leqq\overline{\lim_{z\to\infty}}\frac{g_{B_1}(z,\infty)}{\log|z|}\leqq N<\infty.$$

Therefore by theorem 1 we can say that $z=\infty$ is an irregular point.

Finally, we state a remark. Let $f(z)$ be such an integral function that the point at infinity is an irregular point for a domain D on which $|f(z)|>1$ holds. Then $f(z)$ reduces to a polynomial. Indeed, $\log|f(z)|$ is a positive harmonic function on D vanishing identically on every finite boundary point. By theorem 1 $\log|f(z)|/\log|z| \leqq N < \infty$ for any $|z|>r$. This shows that $f(z)$ is a polynomial.

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