ON NORMAL GENERAL CONNECTIONS

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In a previous paper [7], the author showed that for a space $\mathfrak{X}$ with a regular general connection Γ which is denoted as

$$\Gamma = \partial u_i \otimes (P^i_j d^2 u^j + \Gamma^i_{jh} du^j \otimes du^h)$$

in terms of local coordinates $u^1, \cdots, u^n$ of $\mathfrak{X}$ and

$$P = \lambda(\Gamma) = \partial u_i \otimes P^i_j du^j$$

is an isomorphism of the tangent bundle $T(\mathfrak{X})$ of $\mathfrak{X}$, its covariant differential operator D can be written as product of its basic covariant differential operator $\bar{D}$ and the homomorphism φ of the tangent tensor bundle of $\mathfrak{X}$ naturally derived from P.[1] $\bar{D}$ operates on contravariant tensors and covariant tensors as covariant differential operators defined by the contravariant part $'\Gamma$ and the covariant part $''\Gamma$ of Γ respectively, which are both classical affine connections, that is

$$\lambda('\Gamma) = \lambda(''\Gamma) = I.$$

Therefore, the formulas with regard to $\bar{D}$ are simple and analogous to the classical ones. These results were obtained chiefly by making use of the regularity of the tensor field P.

In the present paper, the author will show that these concepts can be generalized in a sense for normal general connections[2] which are not necessarily regular but include the regular ones.

§ 1. Normal tensor fields of type (1, 1).

Let $\mathfrak{X}$ be a differentiable manifold[3] of dimension n. A tensor field P of type (1, 1) on $\mathfrak{X}$ is called *normal*, if the homomorphism defined by P on the tangent bundle $T(\mathfrak{X})$ of $\mathfrak{X}$ is an isomorphism on the image $P(T_x(\mathfrak{X}))$ at each point $x \in \mathfrak{X}$ and $\dim P(T_x(\mathfrak{X})) = m$ is constant.

Let a normal tensor field P of type (1, 1) on $\mathfrak{X}$ be given. Then the union

$$P(\mathfrak{X}) = \bigcup_{x \in \mathfrak{X}} P(T_x(\mathfrak{X})) \tag{1.1}$$

is naturally regarded as a subbundle of $T(\mathfrak{X})$ whose fibre

$$P_x(\mathfrak{X}) = P(T_x(\mathfrak{X}))$$

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1) See [7], § 3.

2) See [8], § 3.

3) In the present paper, we deal with only manifolds, mappings with suitable differentiabilities for our purpose.

is an m-dimensional vector space. Since P is an isomorphism of $P(\mathfrak{X})$,

$$N_x(\mathfrak{X}) = \text{kernel of } P \mid T_x(\mathfrak{X})$$

is of dimension $n-m$ and

$$T_x(\mathfrak{X}) = P_x(\mathfrak{X}) \oplus N_x(\mathfrak{X}).$$

The union

$$N(\mathfrak{X}) = \bigcup_{x \in \mathfrak{X}} N_x(\mathfrak{X}) \tag{1.2}$$

is also regarded as a subbundle of $T(\mathfrak{X})$ and

$$T(\mathfrak{X}) = P(\mathfrak{X}) \oplus N(\mathfrak{X}) \tag{1.3}$$

as vector bundles over $\mathfrak{X}$.

Let us denote the projections of $T(\mathfrak{X})$ onto $P(\mathfrak{X})$ and $N(\mathfrak{X})$ according to the decomposition (1.3) of $T(\mathfrak{X})$ respectively by

$$A: \quad T(\mathfrak{X}) \to P(\mathfrak{X}), \qquad A \mid P(\mathfrak{X}) = 1, \tag{1.4}$$

$$N: \quad T(\mathfrak{X}) \to N(\mathfrak{X}), \qquad N \mid N(\mathfrak{X}) = 1. \tag{1.5}$$

A and N are also regarded as tensor fields of type (1.1) on $\mathfrak{X}$.

If we take a field of frame $\{V_\lambda\}$ of $\mathfrak{X}$ defined on a neighborhood, such that

$$\{V_1, \cdots, V_m\} \quad \text{is a field of frames of } P(\mathfrak{X})$$

and

$$\{V_{m+1}, \cdots, V_n\} \quad \text{is a field of frames of } N(\mathfrak{X}),$$

then we have easily

$$\begin{cases} P(V_\alpha) = W_\alpha^\beta V_\beta, & P(V_A) = 0, \quad |W_\alpha^\beta| \neq 0, \\ A(V_\alpha) = V_\alpha, & A(V_A) = 0, \\ N(V_\alpha) = 0, & N(V_A) = V_A. \end{cases} \tag{1.6}$$
[4]

Let us denote the homomorphisms of the cotangent bundle $T^*(\mathfrak{X})$ of $\mathfrak{X}$, which are the dual mappings of P, A, N at each point x of $\mathfrak{X}$, by the same notations P, A, N respectively. Then, for the field of the dual frames $\{U^\lambda\}$ of $\{V_\lambda\}$, we have

$$\begin{cases} P(U^\alpha) = W_\beta^\alpha U^\beta, & P(U^A) = 0, \\ A(U^\alpha) = U^\alpha, & A(U^A) = 0, \\ N(U^\alpha) = 0, & N(U^A) = U^A. \end{cases} \tag{1.7}$$

Lastly we define a tensor field Q of type (1.1) by

$$Q = \begin{cases} P^{-1} & \text{on } P_x(\mathfrak{X}), \\ 0 & \text{on } N_x(\mathfrak{X}), \end{cases} \tag{1.8}$$

then we have

$$PQ = QP = A, \tag{1.9}$$

4) The indices run as follows:

$$\lambda, \mu, \nu, \cdots, i, j, h, \cdots = 1, 2, \cdots, n;$$
$$\alpha, \beta, \gamma, \cdots = 1, 2, \cdots, m;$$
$$A, B, C, \cdots = m+1, \cdots, n.$$

$$\left\{\begin{array}{l} AP=PA=P, \qquad AQ=QA=Q, \\ NP=PN=NQ=QN=AN=NA=0. \end{array}\right. \tag{1.10}$$

In the following, we denote the homomorphisms, which are extended onto any tensor product bundle

$$T(\mathfrak{X})^{\otimes(p,q)}=T(\mathfrak{X})^{\otimes p}\otimes T^*(\mathfrak{X})^{\otimes q}, \qquad p,\ q=0,\ 1,\ 2,\ \cdots \tag{1.11}$$

from P, Q, A, N, making use of tensor products of the homomorphisms respectively, by the same symbols. We say that any tensor field $V\in\Psi(T(\mathfrak{X})^{\otimes(p,q)})$ of $\mathfrak{X}$ invariant under A or N belongs to $P(\mathfrak{X})$ or $N(\mathfrak{X})$ respectively and it may be denoted as

$$V\in\Psi(P(\mathfrak{X})^{\otimes(p,q)}) \quad \text{or} \quad \Psi(N(\mathfrak{X})^{\otimes(p,q)}),$$

because it can be written only in terms of V_α, U^β or V_A, U^B.

§ 2. General connections.

Let $\mathfrak{M}_n^2$ be the semi-group whose any element is written as a set of real numbers $(a^i_j,\ a^i_{jh})$ and its multiplication is given by the formulas: For any elements $\alpha,\ \beta\in\mathfrak{M}_n^2$, the components of $\alpha\beta$ are

$$\begin{aligned} &a^i_j(\alpha\beta)=a^i_k(\alpha)\,a^k_j(\beta), \\ &a^i_{jh}(\alpha\beta)=a^i_k(\alpha)\,a^k_{jh}(\beta)+a^i_{kl}(\alpha)\,a^k_i(\beta)\,a^l_h(\beta), \end{aligned} \tag{2.1}$$

and $\mathfrak{L}_n^2$ be the subgroup of $\mathfrak{M}_n^2$ such that $|a^i_j(\alpha)|\neq 0$. Let σ: $\mathfrak{M}_n^2\rightarrow M_n^1=\mathrm{End}(R^n)$ be the natural homomorphism which maps $(a^i_j,\ a^i_{jh})$ to (a^i_j). M_n^1 is regarded as a sub-semi-group of $\mathfrak{M}_n^2$, identifying (a^i_j) with $(a^i_j,\ 0)$.

A general connection Γ of $\mathfrak{X}$ is by definition a cross-section of the tensor product bundle $T(\mathfrak{X})\otimes\mathfrak{D}^2(\mathfrak{X})$[5] *over $\mathfrak{X}$ which is written as*

$$\Gamma=\partial u_i\otimes(P^i_j d^2u^j+\Gamma^i_{jh}du^j\otimes du^h) \tag{2.2}$$

in terms of local coordinates u^i of $\mathfrak{X}$. Let the coordinates u^i be defined on a neighborhood U, then we have a mapping f_U: $U\rightarrow\mathfrak{M}_n^2$ by

$$a^i_j\cdot f_U=P^i_j, \qquad a^i_{jh}\cdot f_U=\Gamma^i_{jh}. \tag{2.3}$$

For any two coordinate neighborhoods $(U,\ u^i)$, $(V,\ v^i)$, $U\cup V\neq\phi$, we have

$$(\sigma\cdot g_{VU})f_U=f_V\,g_{VU}, \tag{2.4}$$

where g_{VU}: $U\cap V\rightarrow\mathfrak{L}_n^2$ is the coordinate transformation of the vector bundles $\mathfrak{T}^2(\mathfrak{X})$[5] and $\mathfrak{D}^2(\mathfrak{X})$ over $\mathfrak{X}$ given by

$$a^i_j\cdot g_{VU}=\frac{\partial v^i}{\partial u^j}, \qquad a^i_{jh}\cdot g_{VU}=\frac{\partial^2 v^i}{\partial u^h\partial u^j}. \tag{2.5}$$

The system $\{f_U\}$ satisfying (2.4) characterizes Γ. Since we have from (2.4) the equation

$$(\sigma\cdot g_{VU})(\sigma\cdot f_U)=(\sigma\cdot f_V)(\sigma\cdot g_{VU}), \tag{2.6}$$

5) See [6], § 1.

P^i_j are the components of a tangent tensor field of type (1, 1) of $\mathfrak{X}$ which we denote by

$$\lambda(\Gamma)=\partial u_i \otimes P^i_j du^j = P. \tag{2.7}$$

For Γ, we define a bundle homomorphism $\varphi=\varphi_\Gamma$ which maps any tensor product bundle composed of the tangent bundles and the cotangent bundles of order 1 or 2 of $\mathfrak{X}$ into the one replaced $\mathfrak{T}^2(\mathfrak{X})$ and $\mathfrak{D}^2(\mathfrak{X})$ by $T(\mathfrak{X})$ and $T^*(\mathfrak{X}) \otimes T^*(\mathfrak{X})$ respectively and is given by

$$\begin{gathered} \varphi(\partial u_j)=P^i_j \partial u_i, \qquad \varphi(\partial^2 u_{jh})=\Gamma^i_{jh}\partial u_i, \\ \varphi(d^2u^i)=-\Lambda^i_{jh}du^j\otimes du^h, \\ \varphi(du^i)=du^i, \\ \varphi(du^{i_1}\otimes\cdots\otimes du^{i_q}\otimes du^h)=P^{i_1}_{j_1}\cdots P^{i_q}_{j_q}du^{j_1}\otimes\cdots\otimes du^{j_q}\otimes du^h, \qquad q\geqq 1, \end{gathered} \tag{2.8}$$

where

$$\Lambda^i_{jh}=\Gamma^i_{jh}-\frac{\partial P^i_j}{\partial u^h}. \tag{2.9}$$

Making use of φ, we define the covariant differential operator $D=D_\Gamma$ of the general connection Γ by

$$D=\varphi\cdot d.^{6)} \tag{2.10}$$

Now, let $\tilde{\mathfrak{L}}^2_n$ be the semi-group whose any element is written as a set of real numbers (a^i_j, a^i_{jh}, p^i_j) such that $|a^i_j|\neq 0$ and its multiplication is given by the formulas: For any elements α, $\beta\in\tilde{\mathfrak{L}}^2_n$, the components of $\alpha\beta$ are

$$\begin{cases} a^i_j(\alpha\beta)=a^i_k(\alpha)a^k_j(\beta), \\ a^i_{jh}(\alpha\beta)=a^i_k(\alpha)a^k_{jh}(\beta)+a^i_{kl}(\alpha)p^k_j(\beta)a^l_h(\beta), \\ p^i_j(\alpha\beta)=p^i_k(\alpha)p^k_j(\beta). \end{cases} \tag{2.11}$$

Let us denote the natural homomorphism of $\tilde{\mathfrak{L}}^2_n$ onto $L^1_n=\mathrm{GL}(n, R)\subset M^1_n$ which maps (a^i_j, a^i_{jh}, p^i_j) to (a^i_j) by the same symbol σ. $\mathfrak{L}^2_n$ is regarded as a subgroup of $\tilde{\mathfrak{L}}^n_2$, identifying (a^i_j, a^i_{jh}) with (a^i_j, a^i_{jh}, a^i_j).

For each coordinate neighborhood (U, u^i), we define a mapping $\tilde{f}_U$: $U\to\tilde{\mathfrak{L}}^2_n$ by

$$a^i_j\cdot\tilde{f}_U=\delta^i_j, \qquad a^i_{jh}\cdot\tilde{f}_U=\Lambda^i_{jh}, \qquad p^i_j\cdot\tilde{f}_U=-P^i_j. \tag{2.12}$$

Then, for any two coordinate neighborhoods (U, u^i), (V, v^i), $U\cap V\neq\phi$, we have

$$g_{VU}\tilde{f}_U=\tilde{f}_V(\sigma\cdot g_{VU}),^{7)} \tag{2.13}$$

which is equivalent to (2.4).

Therefore, *that a general connection Γ of $\mathfrak{X}$ is given is equivalent to that for each coordinate neighborhood U of $\mathfrak{X}$ a mapping f_U: $U\to\mathfrak{M}^2_n$ (or $\tilde{f}_U$: $U\to\tilde{\mathfrak{L}}^2_n$) is given and the system $\{f_U\}$ (or $\{\tilde{f}_U\}$) satisfies* (2.4) (*or* (2.13)).

Lastly, we show that Γ can be written as

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6) See [7], § 1.

7) See (2.28) of [7].

(2.14) $$\Gamma=\partial u_i\otimes\{d(P^i_j du^j)+\Lambda^i_{jh}du^j\otimes du^h\}.$$

§ 3. Normal general connections and their contravariant parts and covariant parts.

A general connection Γ is called *normal* if $\lambda(\Gamma)=P$ is normal.

Let Γ be a normal general connection of $\mathfrak{X}$ and let us make use of the consideration in § 1 for $P=\lambda(\Gamma)$.

Let q_U: $U\to\mathfrak{M}^2_n$ be a mapping defined by

(3.1) $$a^i_j\cdot q_U=Q^i_j,\qquad a^i_{jh}\cdot q_U=0.$$

Since Q^i_j are the components of the tensor field Q, we have

$$(\sigma\cdot g_{VU})q_U=q_V(\sigma\cdot g_{VU})$$

for any coordinate neighborhoods U, V, $U\cap V\neq\phi$. By means of (2.4), we get easily

$$(\sigma\cdot g_{VU})(q_U f_U)=(q_V f_V)g_{VU},$$

hence the system $\{f'_U=q_U f_U\}$ defines a general connection $'\Gamma$. Since we have

(3.2) $$a^i_j\cdot f'_U=Q^i_k P^k_j=A^i_j,\qquad a^i_{jh}\cdot f'_U=Q^i_k\Gamma^k_{jh}={}'\Gamma^i_{jh},$$

Γ' is locally written as

(3.3) $$\begin{aligned}'\Gamma&=\partial u_i\otimes(A^i_j d^2u^j+{}'\Gamma^i_{jh}du^j\otimes du^h)\\&=\partial u_k Q^k_i\otimes(P^i_j d^2u^j+\Gamma^i_{jh}du^j\otimes du^h).\end{aligned}$$

We call $'\Gamma$ *the contravariant part of* Γ. $'\Gamma$ is clearly normal and $A=\lambda('\Gamma)$ is the projection of $T(\mathfrak{X})$ onto $P(\mathfrak{X})$.

Next, let $\tilde q_U$: $U\to\tilde{\mathfrak{L}}^2_n$ be a mapping defined by

$$a^i_j\cdot\tilde q_U=\delta^i_j,\qquad a^i_{jh}\cdot\tilde q_U=0,\qquad p^i_j\cdot\tilde q_U=Q^i_j.$$

Then, we have

$$(\sigma\cdot g_{VU})\tilde q_U=\tilde q_V(\sigma\cdot g_{VU}),$$

here we consider as $L^1_n\subset\mathfrak{L}^2_n\subset\tilde{\mathfrak{L}}^2_n$. By means of (2.13), we get easily

$$g_{VU}(\tilde f_U\tilde q_U)=(\tilde f_V\tilde q_V)(\sigma\cdot g_{VU}),$$

hence the system $\{\tilde f''_U=\tilde f_U\tilde q_U\}$ defines a general connection $''\Gamma$. Since we have

(3.4) $$a^i_j\cdot\tilde f''_U=\delta^i_j,\qquad a^i_{jh}\cdot\tilde f''_U=\Lambda^i_{kh}Q^k_j={}''\Lambda^i_{jh},\qquad p^i_j\cdot f''_U=-A^i_j,$$

the connection $''\Gamma$ can be locally written as

(3.5) $$\begin{aligned}''\Gamma&=\partial u_i\otimes(A^i_j d^2u^j+{}''\Gamma^i_{jh}du^j\otimes du^h)\\&=\partial u_i\otimes\{d(A^i_j du^j)+\Lambda^i_{kh}Q^k_j du^j\otimes du^h\}\end{aligned}$$

by means of (2.14), hence we have

(3.6) $$''\Gamma=\partial u_i\otimes\{P^i_j d(Q^j_k du^k)+\Gamma^i_{jh}(Q^j_k du^k)\otimes du^h\}$$

and

$$''\Gamma^i_{jh} = \Gamma^i_{kh} Q^k_j + P^i_k \frac{\partial Q^k_j}{\partial u^h}. \tag{3.7}$$

We call $''\Gamma$ *the covariant part of* Γ. $''\Gamma$ is also a normal general connection and $A = \lambda(''\Gamma)$.

Here, for any tensor field M of type (1, 1) on $\mathfrak{X}$, we define a bundle homomorphism ι_M of tensor product bundles of order 1 of $\mathfrak{X}$ as follows:

$$\begin{aligned} &\iota_M = (M \mid T(\mathfrak{X}))^{\otimes p} \quad \text{on} \quad T(\mathfrak{X})^{\otimes p}, \\ &\iota_M = (M \mid T(\mathfrak{X}))^{\otimes p} \otimes (M \mid T^*(\mathfrak{X}))^{\otimes (q-1)} \otimes 1 \quad \text{on} \quad T(\mathfrak{X})^{\otimes (p,q)}, \\ &\qquad p \geqq 0, \ q \geqq 1, \end{aligned} \tag{3.8}$$

where $M \mid T(\mathfrak{X})$ and $M \mid T^*(\mathfrak{X})$ are the homomorphisms induced from M on $T(\mathfrak{X})$ and $T^*(\mathfrak{X})$.

Now, we put

$$\varphi' = \varphi_{'\Gamma} \quad \text{and} \quad \varphi'' = \varphi_{''\Gamma},$$

which are defined for $'\Gamma$ and $''\Gamma$ analogously to (2.8), that is

$$\left\{ \begin{aligned} &\varphi'(\partial u_j) = \varphi''(\partial u_j) = A^i_j \partial u_i, \\ &\varphi'(\partial^2 u_{jh}) = {'\Gamma^i_{jh}} \partial u_i, \quad \varphi''(\partial^2 u_{jh}) = {''\Gamma^i_{jh}} \partial u_i, \\ &\varphi'(d^2 u^i) = -{'\Lambda^i_{jh}} du^j \otimes du^h, \quad \varphi''(d^2 u^i) = -{''\Lambda^i_{jh}} du^j \otimes du^h, \\ &\varphi'(du^i) = \varphi''(du^i) = du^i, \\ &\varphi'(du^{i_1} \otimes \cdots \otimes du^{i_q} \otimes du^h) = \varphi''(du^{i_1} \otimes \cdots \otimes du^{i_q} \otimes du^h) \\ &\qquad = A^{i_1}_{j_1} \cdots A^{i_q}_{j_q} du^{j_1} \otimes \cdots \otimes du^{j_q} \otimes du^h, \quad q \geqq 1. \end{aligned} \right. \tag{3.9}$$

Clearly, we have

$$\varphi' = \varphi'' = \iota_A \quad \text{on} \quad T(\mathfrak{X})^{\otimes (p,q)}; \quad p, q = 0, 1, 2, \cdots. \tag{3.10}$$

THEOREM 3.1.[8] *For a normal general connection* Γ, *we define a bundle homomorphism* $\bar{\mu}$ *by*

$$\bar{\mu} = \bar{\mu}_\Gamma = \left\{ \begin{array}{ll} \varphi' & \textit{on tangent bundles of order 1 or 2,} \\ \varphi'' & \textit{on cotangent bundles of order 1 or 2,} \end{array} \right. \tag{3.11}$$

then it holds good

$$\iota_A \cdot \varphi = \bar{\varphi} \cdot \bar{\mu}, \tag{3.12}$$

where $\bar{\varphi}$ *is the restriction of* $\varphi = \varphi_\Gamma$ *on tensor product bundles* $T(\mathfrak{X})^{\otimes (p,q)}$ *of order* 1 *and* $\bar{\varphi} = \iota_P$.

Proof. By means of (2.8), (3.8), (3.2), (3.4), (1.9) and (1.10), we get

$$\begin{aligned} \iota_A \varphi(\partial u_j) &= \iota_A(P^i_j \partial u_i) = P^i_j A^h_i \partial u_h = A^i_j P^h_i \partial u_h = \bar{\varphi}\bar{\mu}(\partial u_j), \\ \iota_A \varphi(\partial^2 u_{jh}) &= \iota_A(\Gamma^i_{jh} \partial u_i) = \Gamma^i_{jh} A^k_i \partial u_k = {'\Gamma^i_{jh}} P^k_i \partial u_k \\ &= \bar{\varphi}\varphi'(\partial^2 u_{jh}) = \bar{\varphi}\bar{\mu}(\partial^2 u_{jh}), \\ \iota_A \varphi(d^2 u^i) &= \iota_A(-\Lambda^i_{jh} du^j \otimes du^h) = -\Lambda^i_{jh} A^j_k du^k \otimes du^h \\ &= -{''\Lambda^i_{jh}} P^j_k du^k \otimes du^h = \bar{\varphi}\varphi''(d^2 u^i) = \bar{\varphi}\bar{\mu}(d^2 u^i), \\ \iota_A \varphi(du^i) &= du^i = \bar{\varphi}\bar{\mu}(du^i), \end{aligned}$$

8) See Theorem 3.1 of [7].

$$\begin{aligned}\iota_A\varphi(du^{i_1}\otimes\cdots\otimes du^{i_q}\otimes du^h) &= \iota_A(P^{i_1}_{j_1}\cdots P^{i_q}_{j_q}du^{j_1}\otimes\cdots\otimes du^{j_q}\otimes du^h)\\ &= P^{i_1}_{j_1}\cdots P^{i_q}_{j_q}A^{j_1}_{k_1}\cdots A^{j_q}_{k_q}du^{k_1}\otimes\cdots\otimes du^{k_q}\otimes du^h\\ &= A^{i_1}_{j_1}\cdots A^{j_q}_{i_q}P^{j_1}_{k_1}\cdots P^{j_q}_{k_q}du^{k_1}\otimes\cdots\otimes du^{k_q}\otimes du^h\\ &= \bar{\varphi}\bar{\mu}(du^{i_1}\otimes\cdots\otimes du^{i_q}\otimes du^h),\end{aligned}$$

hence it must be

$$\iota_A\cdot\varphi=\iota_P\cdot\bar{\mu}.$$

We call $\bar{\mu}=\bar{u}_\Gamma$ *the basic homomorphism* of the normal general connection Γ. Putting

$$\bar{D}=\bar{D}_\Gamma=\bar{\mu}\cdot d, \tag{3.13}$$

we call this *the basic covariant differential operator of* Γ. By means of (2.10) and (3.13), we get easily the following

THEOREM 3.2. *For the covariant differentiation and the basic covariant differentiation of a normal general connection* Γ, *it holds good*

$$\iota_A\cdot D=\iota_P\cdot\bar{D}. \tag{3.14}$$

§ 4. Basic covariant differentiations.

For any tensor field $V\in\Psi(T(\mathfrak{X})^{\otimes(p,q)})$ with local components $V^{i_1\cdots i_p}_{j_1\cdots j_q}$, its basic covariant differential

$$\bar{D}V=\partial u_{i_1}\otimes\cdots\otimes\partial u_{i_p}\otimes du^{j_1}\otimes\cdots\otimes du^{j_q}\otimes\bar{D}V^{i_1\cdots i_p}_{j_1\cdots j_q}$$

is given by the formulas:

$$\bar{D}V^{i_1\cdots i_p}_{j_1\cdots j_q}=V^{i_1\cdots i_p}_{j_1\cdots j_q|h}du^h, \tag{4.1}$$

$$\begin{aligned}V^{i_1\cdots i_p}_{j_1\cdots j_q|h} &= A^{i_1}_{k_1}\cdots A^{i_p}_{k_q}\frac{\partial V^{k_1\cdots k_p}_{h_1\cdots h_q}}{\partial u^h}A^{h_1}_{j_1}\cdots A^{h_q}_{j_q}\\ &+\sum_{s=1}^{p}A^{i_1}_{k_1}\cdots A^{i_{s-1}}_{k_{s-1}}{}'\Gamma^{i_s}_{k_s h}A^{i_{s+1}}_{k_{s+1}}\cdots A^{i_p}_{k_p}V^{k_1\cdots k_p}_{h_1\cdots h_q}A^{h_1}_{j_1}\cdots A^{h_q}_{j_q}\\ &-\sum_{t=1}^{q}A^{i_1}_{k_1}\cdots A^{i_p}_{k_p}V^{k_1\cdots k_p}_{h_1\cdots h_p}A^{h_1}_{j_1}\cdots A^{h_{t-1}}_{j_{t-1}}{}''\Lambda^{h_t}_{j_t h}A^{h_{t+1}}_{j_{t+1}}\cdots A^{h_q}_{j_q},\end{aligned} \tag{4.2}$$

which are obtained from (3.9), (3.11) and (3.13).[9]

Now, from (1.10), (3.2) and (3.4), we get

$$A^i_k{}'\Gamma^k_{jh}={}'\Gamma^i_{jh},\qquad {}''\Lambda^i_{kh}A^k_j={}''\Lambda^i_{jh}, \tag{4.3}$$

hence we have from (3.9)

$$\iota_A\cdot\bar{\mu}=\bar{\mu}. \tag{4.4}$$

THEOREM 4.1. *For the basic covariant differentiation of a normal general connection* Γ, *it holds good*

$$\iota_A\cdot\bar{D}=\bar{D} \tag{4.5}$$

and for any tensor field $V\in\Psi(T(\mathfrak{X})^{\otimes(p,q)})$ *we have*

9) See (7.4) of [6] and (2.15) of [7].

$$V_{|h}A^h_i \otimes du^i \in \Psi(P(\mathfrak{X})^{\otimes(p,q+1)}),$$

where $\bar{D}V = V_{|h} \otimes du^h$.

Proof. (4.5) follows immediately from (4.4) and the definition of $\bar{D}$. With regard to the second part, we have

$$\begin{aligned} V_{|h}A^h_i \otimes du^i &= (1 \otimes A)\bar{D}V \\ &= (1 \otimes A)\iota_A\bar{D}V = (1 \otimes A)(A \otimes 1)\bar{D}V \\ &= (A \otimes A)\bar{D}V = A\bar{D}V \in \Psi(P(\mathfrak{X})^{\otimes(p,q+1)}), \end{aligned}$$

where we use the notation A according to the convention stated in §1.

Now, we say that a tensor field V of $\mathfrak{X}$ is *basic* or *normal* if $AV = V$ or $NV = V$ respectively. We will show that if V is basic, the formula (4.2) becomes very simple as the classical one.

At first, (4.2) can be easily rewritten as

$$\begin{aligned} V^{i_1\cdots i_p}_{j_1\cdots j_q|h} = &\frac{\partial}{\partial u^h}(A^{i_1}_{k_1}\cdots A^{i_1}_{k_p}V^{k_1\cdots k_p}_{h_1\cdots h_q}A^{h_1}_{j_1}\cdots A^{h_q}_{j_q}) \\ &+\sum_{s=1}^{p} A^{i_1}_{k_1}\cdots A^{i_{s-1}}_{k_{s-1}}{}'\Lambda^{i_s}_{k_s h}A^{i_{s+1}}_{k_{s+1}}\cdots A^{i_p}_{k_p}V^{k_1\cdots k_p}_{h_1\cdots h_q}A^{h_1}_{j_1}\cdots A^{h_q}_{j_q} \\ &-\sum_{t=1}^{q} A^{i_1}_{k_1}\cdots A^{i_p}_{k_p}V^{k_1\cdots k_p}_{h_1\cdots h_q}A^{h_1}_{j_1}\cdots A^{h_{t-1}}_{j_{t-1}}{}''\Gamma^{h_t}_{j_t h}A^{h_{t+1}}_{j_{t+1}}\cdots A^{h_q}_{j_q}. \end{aligned} \tag{4.2'}$$

Now, let $V \in \Psi(P(\mathfrak{X})^{\otimes(p,q)})$ with local components $V^{i_1\cdots i_p}_{j_1\cdots j_q}$, then we have

$$A^{i_1}_{k_1}\cdots A^{i_p}_{k_p}V^{k_1\cdots k_p}_{h_1\cdots h_q}A^{h_1}_{j_1}\cdots A^{h_q}_{j_q} = V^{i_1\cdots i_p}_{j_1\cdots j_q}. \tag{4.6}$$

Since A is a projection, it follows that

$$A^{i_s}_k V^{i_1\cdots k\cdots i_p}_{j_1\cdots\cdots\cdots j_q} = V^{i\cdots\cdots\cdots\cdots i_q}_{j_1\cdots k\cdots j_q}A^k_{j_t} = V^{i_1\cdots i_p}_{j_1\cdots j_q}, \quad s = 1, \cdots, p; \quad t = 1, \cdots, q. \tag{4.7}$$

Clearly the conditions (4.6) and (4.7) are equivalent to each other. Putting these relations into (4.2′), we obtain the following

THEOREM 4.2. *Let Γ be a normal general connection. For any tensor field V of type (p, q) with local components $V^{i_1\cdots i_p}_{j_1\cdots j_q}$ invariant under A the components of its basic covariant differential $\bar{D}V$ are given by the formula:*

$$V^{i_1\cdots i_p}_{j_1\cdots j_q|h} = \frac{\partial V^{i_1\cdots i_p}_{j_1\cdots j_q}}{\partial u^h} + \sum_{s=1}^{p} {}'\Lambda^{i_s}_{k h}V^{i_1\cdots k\cdots i_p}_{j_1\cdots\cdots\cdots j_q} - \sum_{t=1}^{q} {}''\Gamma^k_{j_t h}V^{i_1\cdots\cdots\cdots i_p}_{j_1\cdots k\cdots j_q}, \tag{4.8}$$

where

$$\begin{cases} {}'\Lambda^i_{jh} = Q^i_k\Gamma^k_{jh} - \dfrac{\partial A^i_j}{\partial u^h}, \\ {}''\Gamma^i_{jh} = \Gamma^j_{kh}Q^k_j + P^i_k\dfrac{\partial Q^k_j}{\partial u^h}. \end{cases} \tag{4.9}$$

The formula (4.8) is a natural extension of (3.7) of [7], since ${}'\Lambda^i_{jh} = {}'\Gamma^i_{jh}$, when Γ is regualar.

Analogously, a tensor field V of (p, q) with local components $V^{i_1\cdots i_p}_{j_1\cdots j_q}$ is a tensor field of $N(\mathfrak{X})$, if and only if

$$N^{i_1}_{k_1}\cdots N^{i_p}_{k_p}V^{k_1\cdots k_p}_{h_1\cdots h_q}N^{h_1}_{j_1}\cdots N^{h_q}_{j_q}=V^{i_1\cdots i_p}_{j_1\cdots j_q} \tag{4.10}$$

or

$$N^{i_s}_{k}V^{i_1\cdots k\cdots i_p}_{j_1\cdots\cdots j_q}=V^{i_1\cdots\cdots i_p}_{j_1\cdots k\cdots j_q}N^{k}_{j_t}=V^{i_1\cdots i_p}_{j_1\cdots j_q}, \quad s=1, \cdots, p; \quad t=1, \cdots, q. \tag{4.11}$$

Hence, for such tensor field $V\in\Psi(N(\mathfrak{X})^{\otimes(p,q)})$, we have

$$A^{i_s}_{k}V^{i_1\cdots k\cdots i_p}_{j_1\cdots\cdots j_q}=V^{i_1\cdots\cdots i_p}_{j_1\cdots k\cdots j_q}A^{k}_{j_t}=0 \tag{4.12}$$

and so we get from (4.2′) the formulas:

$$V^{i_1\cdots i_p}_{j_1\cdots j_q|h}=0, \qquad \text{when} \quad p+q\geqq 2, \tag{4.13}$$

$$\begin{cases} V^{i}{}_{|h}='A^{i}_{jh}V^{j}, \\ V_{j|h}=-''\Gamma^{i}_{jh}V_{i}. \end{cases} \tag{4.14}$$

§ 5. Normal convariant differentiations.

Making use of the tensor N in place of Q, we shall define a covariant differentiation.

For each coordinate neighborhood (U, u^i), let n_U: $U\to\mathfrak{M}^2_n$ and $\tilde{n}_U\to\tilde{\mathfrak{L}}^2_n$ be the mappings defined by

$$a^{i}_{j}\cdot n_U=N^{i}_{j}, \qquad a^{i}_{jh}\cdot n_U=0 \tag{5.1}$$

and

$$a^{i}_{j}\cdot\tilde{n}_U=\delta^{i}_{j}, \qquad a^{i}_{jh}\cdot\tilde{n}_U=0, \qquad p^{i}_{j}\cdot\tilde{n}_U=N^{i}_{j}, \tag{5.2}$$

then the systems $\{n_U f_U\}$ and $\{\tilde{f}_U\tilde{n}_U\}$ define two general connections $'\Gamma_n$ and $''\Gamma_n$ of $\mathfrak{X}$ respectively as the systems $\{f'_U=q_U f_U\}$ and $\{\tilde{f}''_U=\tilde{f}_U\tilde{q}_U\}$ in § 3. Since we have

$$(N^{i}_{j},\ 0)(P^{i}_{j},\ \Gamma^{i}_{jh})=(0,\ N^{i}_{k}\Gamma^{k}_{jh}),$$
$$(\delta^{i}_{j},\ A^{i}_{jh},\ -P^{i}_{j})(\delta^{i}_{j},\ 0,\ N^{i}_{j})=(\delta^{i}_{j},\ A^{i}_{kh}N^{k}_{j},\ 0),$$

$'\Gamma_n$ and $''\Gamma_n$ are tensor fields of type (1, 2) on $\mathfrak{X}$ with local components as

$$\begin{cases} 'N^{i}_{jh}=N^{i}_{k}\Gamma^{k}_{jh}, \\ ''N^{i}_{jh}=A^{i}_{kh}N^{k}_{j}=\left(\Gamma^{i}_{kh}-\dfrac{\partial P^{i}_{k}}{\partial u^h}\right)N^{k}_{j} \end{cases} \tag{5.3}$$

respectively.

Now, let φ_n' and φ_n'' be the bundle homomorphisms for the general connections $'\Gamma_n$ and $''\Gamma_n$ defined as $\varphi=\varphi_\Gamma$ for Γ. Then we have clearly

$$\begin{cases} \iota_N\varphi(\partial u_j)=P^{i}_{j}N^{k}_{i}\partial u_k=0=\varphi_n'(\partial u_j), \\ \iota_N\varphi(\partial^2 u_{jh})=\Gamma^{i}_{jh}N^{k}_{i}\partial u_k='N^{i}_{jh}\partial u_i=\varphi_n'(\partial^2 u_{jh}), \\ \iota_N\varphi(d^2u^i)=-A^{i}_{jh}N^{j}_{k}du^k\otimes du^h=-''N^{i}_{jh}du^j\otimes du^h=\varphi_n''(d^2u^i), \\ \iota_N\varphi(du^i)=du^i=\varphi_n''(du^i), \\ \iota_N\varphi(du^{i_1}\otimes\cdots\otimes du^{i_q}\otimes du^h)=\varphi_n''(du^{i_1}\otimes\cdots\otimes du^{i_q}\otimes du^h)=0, \quad q\geqq 1. \end{cases} \tag{5.4}$$

Putting

$$\bar{D}_n = \iota_N \cdot D, \tag{5.5}$$

we call this *the normal covariant differential operator of* Γ. From (5.4), we see that $\bar{D}_n$ *is identical with the covariant differential operators of* $'\Gamma_n$ *or* $''\Gamma_n$ *for contravariant or covariant tensor fields respectively.*

THEOREM 5.1. *For the normal covariant differentiation of* Γ, *it holds good*

$$\iota_N \cdot \bar{D}_n = \bar{D}_n$$

and for any tensor field $V \in \Psi(T(\mathfrak{X})^{\otimes(p,q)})$ *with local components* $V^{i_1\cdots i_p}_{j_1\cdots j_q}$ *we have*

$$\begin{cases} \bar{D}_n V^{i_1\cdots i_p}_{j_1\cdots j_q} = 0, \quad \text{when} \quad p+q \geqq 2, \\ \bar{D}_n V^i = {'N^i_{jh}} V^j du^h, \\ \bar{D}_n V_j = -{''N^i_{jh}} V_i du^h. \end{cases} \tag{5.7}$$

The proof is evident.

Lastly, since we have from (4.9)

$$'A^i_{kh} N^k_j = \left(Q^i_l \Gamma^l_{kh} - \frac{\partial A^i_k}{\partial u^h}\right) N^k_j = Q^i_l \left(\Gamma^l_{kh} - \frac{\partial P^l_k}{\partial u^h}\right) N^k_j = Q^i_l A^l_{kh} N^k_j = Q^i_l {''N^l_{jh}}$$

and

$$N^i_k {''\Gamma^k_{jh}} = N^i_k \left(\Gamma^k_{lh} Q^l_j + P^k_l \frac{\partial Q^l_j}{\partial u^h}\right) = {'N^i_{kh}} Q^k_j,$$

the *formula* (4.14) *can be rewritten as*

$$\begin{cases} V^i{}_{|h} = Q^i_l {''N^l_{kh}} V^k, \\ V_{j|h} = -{'N^k_{lh}} V_k Q^l_j \end{cases} \tag{5.8}$$

where $V^k \partial u_k$ *and* $V_k du^k$ *are vector fields of* $N(\mathfrak{X})$.

§ 6. Some general connections derived from a normal general connection.

From a normal general connection Γ, we obtained the four normal general connections $'\Gamma$, $''\Gamma$, $'\Gamma_n$, $''\Gamma_n$, which are given by (3.2), (3.3), (3.5), (3.7), (5.3), that is

$$\Gamma: \quad (P^i_j, \Gamma^i_{jh}),$$

$$\begin{cases} '\Gamma: & (A^i_j, Q^i_k \Gamma^k_{jh}), \\ ''\Gamma: & \left(A^i_j, \Gamma^i_{kh} Q^k_j + P^i_k \dfrac{\partial Q^k_j}{\partial u^h}\right), \\ '\Gamma_n: & (0, N^i_k \Gamma^k_{jh}) = (0, {'N^i_{jh}}), \\ ''\Gamma_n: & \left(0, \left(\Gamma^i_{kh} - \dfrac{\partial P^i_k}{\partial u^h}\right) N^k_j\right) = (0, {''N^i_{jh}}) \end{cases} \tag{6.1}$$

with respect to local coordinates u^i.

Let us calculate the components of the normal general connections which are derived from the four general connections by the same manner.

Since $\lambda('\Gamma)=A$, with regard to $'\Gamma$, we have

$$'('\Gamma):\quad (A^i_j,\ A^{i\prime}_k\Gamma^k_{jh})=(A^i_j,\ Q^i_k\Gamma^k_{jh}),$$

hence

$$'('\Gamma)='\Gamma. \tag{6.2}$$

$$\Gamma^{\cdot}\equiv''('\Gamma):\quad \left(A^i_j,\ '\Gamma^i_{kh}A^k_j+A^i_k\frac{\partial A^k_j}{\partial u^h}\right)=\left(A^i_j,\ Q^i_l\Gamma^l_{kh}A^k_j+A^i_k\frac{\partial A^k_j}{\partial u^h}\right), \tag{6.3}$$

$$'('\Gamma)_n:\quad (0,\ N^{i\prime}_k\Gamma^k_{jh})=(0,\ 0), \tag{6.4}$$

$$''('\Gamma)_n:\quad \left(0,\ \left('\Gamma^i_{kh}-\frac{\partial A^i_k}{\partial u^h}\right)N^k_j\right),$$

and

$$\left('\Gamma^i_{kh}-\frac{\partial A^i_k}{\partial u^h}\right)N^k_j=Q^i_l\Gamma^l_{kh}N^k_j-\frac{\partial(Q^i_lP^l_k)}{\partial u^h}N^k_j=Q^i_l\left(\Gamma^l_{kh}-\frac{\partial P^l_k}{\partial u^h}\right)N^k_j=Q^{i\prime\prime}_lN^l_{jh},$$

that is

$$\Gamma^{\cdot}_n\equiv''('\Gamma)_n:\quad (0,\ Q^i_k{''}N^k_{jh}). \tag{6.5}$$

Next, since $\lambda(''\Gamma)=A$, with regard to $''\Gamma$, we have

$$\Gamma^{\cdot\cdot}\equiv'(''\Gamma):\quad (A^i_j,\ A^{i\prime\prime}_k\Gamma^k_{jh})=\left(A^i_j,\ A^i_l\Gamma^l_{kh}Q^k_j+P^i_k\frac{\partial Q^k_j}{\partial u^h}\right). \tag{6.6}$$

$$''(''\Gamma):\quad \left(A^i_j,\ ''\Gamma^i_{kh}A^k_j+A^i_k\frac{\partial A^k_j}{\partial u^h}\right),$$

and

$$''\Gamma^i_{kh}A^k_j+A^i_k\frac{\partial A^k_j}{\partial u^h}=\left(\Gamma^i_{lh}Q^l_k+P^i_l\frac{\partial Q^l_k}{\partial u^h}\right)A^k_j+A^i_k\frac{\partial A^k_j}{\partial u^h}$$

$$=\Gamma^i_{kh}Q^k_j+P^i_l\frac{\partial Q^l_k}{\partial u^h}A^k_j+P^i_lQ^l_k\frac{\partial A^k_j}{\partial u^h}=\Gamma^i_{kh}Q^k_j+P^i_k\frac{\partial Q^k_j}{\partial u^h}$$

hence

$$''(''\Gamma)=''\Gamma. \tag{6.7}$$

$$\Gamma^{\cdot\cdot}_n\equiv'(''\Gamma)_n:\quad (0,\ N^i_k{''}\Gamma^k_{jh})=(0,\ 'N^i_{kh}Q^k_j). \tag{6.8}$$

$$''(''\Gamma)_n:\quad \left(0,\ \left(''\Gamma^i_{kh}-\frac{\partial A^i_k}{\partial u^h}\right)N^k_j\right),$$

and

$$\left(''\Gamma^i_{kh}-\frac{\partial A^i_k}{\partial u^h}\right)N^k_j=\left(\Gamma^i_{lh}Q^l_k+P^i_l\frac{\partial Q^l_k}{\partial u^h}-\frac{\partial A^i_k}{\partial u^h}\right)N^k_j=-\frac{\partial P^i_l}{\partial u^h}Q^l_kN^k_j=0,$$

that is

$$''(''\Gamma)_n:\quad (0,\ 0). \tag{6.9}$$

Since $\lambda('\Gamma_n)=\lambda(''\Gamma_n)=0$, we have easily

$$\begin{cases} '('\Gamma_n):\quad (0,\ 0),\\ ''('\Gamma_n):\quad (0,\ 0),\\ '('\Gamma_n)_n=''('\Gamma_n)_n='\Gamma_n \end{cases} \tag{6.10}$$

and

$$(6.11)\qquad \begin{cases} '(''\Gamma_n)\colon & (0,\ 0), \\ ''(''\Gamma_n)\colon & (0,\ 0), \\ '(''\Gamma_n)_n = ''(''\Gamma_n)_n = ''\Gamma_n. \end{cases}$$

Furthermore, with regard to the normal general connections

$$\Gamma^{\cdot} = ''('\Gamma) \quad \text{and} \quad \Gamma^{\cdot\cdot} = '(''\Gamma),$$

we have from (6.1), (6.3), (6.6) the relations:

$$'(\Gamma^{\cdot}) = '(''('\Gamma))\colon \quad \left(A^i_j,\ A^i_l\,'\Gamma^l_{kh}A^k_j + A^i_k\frac{\partial A^k_j}{\partial u^h}\right)$$

and

$$A^i_l\,'\Gamma^l_{kh}A^k_j + A^i_k\frac{\partial A^k_j}{\partial u^h} = A^i_l(Q^l_t\Gamma^t_{kh})A^k_j + A^i_k\frac{\partial A^k_j}{\partial u^h}$$

$$= Q^i_l\Gamma^l_{kh}A^k_j + A^i_k\frac{\partial A^k_j}{\partial u^h} = \Gamma^{\cdot}{}^i_{jh};$$

$$''(\Gamma^{\cdot\cdot}) = ''('(''\Gamma))\colon \quad \left(A^i_j,\ A^i_l\,''\Gamma^l_{kh}A^k_j + A^i_k\frac{\partial A^k_j}{\partial u^h}\right)$$

and

$$A^i_l\,''\Gamma^l_{kh}A^k_j + A^i_k\frac{\partial A^k_j}{\partial u^h} = A^i_l\left(\Gamma^l_{th}Q^t_k + P^l_t\frac{\partial Q^t_k}{\partial u^h}\right)A^k_j + A^i_k\frac{\partial A^k_j}{\partial u^h}$$

$$= A^i_l\Gamma^l_{kh}Q^k_j + P^i_l\frac{\partial Q^l_k}{\partial u^h}A^k_j + A^i_k\frac{\partial A^k_j}{\partial u^h}$$

$$= A^i_l\Gamma^l_{kh}Q^k_j + P^i_k\frac{\partial Q^k_j}{\partial u^h} = \Gamma^{\cdot\cdot}{}^i_{jh}.$$

THEOREM 6.1. *For a normal general connection Γ, the normal general connections $\Gamma^{\cdot} = ''('\Gamma)$ and $\Gamma^{\cdot\cdot} = '(''\Gamma)$ satisfy the following conditions*:

$$(6.12)\qquad \begin{cases} '(\Gamma^{\cdot}) = ''(\Gamma^{\cdot}) = \Gamma^{\cdot}, \\ '(\Gamma^{\cdot\cdot}) = ''(\Gamma^{\cdot\cdot}) = \Gamma^{\cdot\cdot} \end{cases}$$

and

$$(6.13)\qquad '(\Gamma^{\cdot})_n = ''(\Gamma^{\cdot})_n = '(\Gamma^{\cdot\cdot})_n = ''(\Gamma^{\cdot\cdot})_n = 0.$$ [10)]

Proof. (6.12) is evident from (6.2), (6.7) and the above relations for $\Gamma^{\cdot}$ and $\Gamma^{\cdot\cdot}$. Regarding to (6.13), we have

$$'(\Gamma^{\cdot})_n\colon \quad (0,\ N^i_k\Gamma^{\cdot}{}^k_{jh}) = (0,\ 0),$$

$$''(\Gamma^{\cdot})_n\colon \quad \left(0,\ \left(\Gamma^{\cdot}{}^i_{kh} - \frac{\partial A^i_k}{\partial u^h}\right)N^k_j\right)$$

and

$$\left(\Gamma^{\cdot}{}^i_{kh} - \frac{\partial A^i_k}{\partial u^h}\right)N^k_j = \left(A^i_l\frac{\partial A^l_k}{\partial u^h} - \frac{\partial A^i_k}{\partial u^h}\right)N^k_j = -A^i_l A^l_k\frac{\partial N^k_j}{\partial u^h} + A^i_k\frac{\partial N^k_j}{\partial u^h} = 0;$$

$$'(\Gamma^{\cdot\cdot})_n\colon \quad (0,\ N^i_k\Gamma^{\cdot\cdot}{}^k_{jh}) = (0,\ 0),$$

10) 0 denotes the trivial general connection whose components all vanish.

$$''(\Gamma^{\cdot\cdot})_n: \quad \left(0, \left(\Gamma^{\cdot\cdot i}_{\ \ kh} - \frac{\partial A^i_k}{\partial u^h}\right) N^k_j\right)$$

and

$$\left(\Gamma^{\cdot\cdot i}_{\ \ kh} - \frac{\partial A^i_k}{\partial u^h}\right) N^k_j = \left(P^i_l \frac{\partial Q^l_k}{\partial u^h} - \frac{\partial A^i_k}{\partial u^h}\right) N^k_j = - P^i_l Q^l_k \frac{\partial N^k_j}{\partial u^h} + A^i_k \frac{\partial N^k_j}{\partial u^h} = 0.$$

COROLLARY 6.2. *For the normal general connections $\Gamma^{\cdot}$ and $\Gamma^{\cdot\cdot}$, their covariant differentiations and their basic covariant differentiations are identical with each other respectively.*

THEOREM 6.3. *For a normal general connection Γ, we have the formulas:*

$$('\Gamma)^{\cdot} = ('\Gamma)^{\cdot\cdot} = \Gamma^{\cdot}, \quad (''\Gamma)^{\cdot} = (''\Gamma)^{\cdot\cdot} = \Gamma^{\cdot\cdot}. \tag{6.14}$$

Proof. By means of (6.2), (6.7) and (6.12), we get

$$('\Gamma)^{\cdot} = ''('('\Gamma)) = ''('\Gamma) = \Gamma^{\cdot},$$
$$('\Gamma)^{\cdot\cdot} = '(''('\Gamma)) = '(\Gamma^{\cdot}) = \Gamma^{\cdot},$$
$$(''\Gamma)^{\cdot} = ''('(''(\Gamma)) = ''(\Gamma^{\cdot\cdot}) = \Gamma^{\cdot\cdot},$$
$$(''\Gamma)^{\cdot\cdot} = '(''(''\Gamma)) = '(''\Gamma) = \Gamma^{\cdot\cdot}.$$

Theorem 6.1 shows that out of the normal general connections naturally derived from a normal general connection Γ, $\Gamma^{\cdot}$ and $\Gamma^{\cdot\cdot}$ are the most convenient ones and we may consider them as belonging to $P(\mathfrak{X})$.

Furthermore, we get easily from (6.5) and (6.8) the relations:

$$'(\Gamma^{\cdot}_n) = ''(\Gamma^{\cdot}_n) = 0, \quad '(\Gamma^{\cdot}_n)_n = ''(\Gamma^{\cdot}_n)_n = \Gamma^{\cdot}_n \tag{6.15}$$

and

$$'(\Gamma^{\cdot\cdot}_n) = ''(\Gamma^{\cdot\cdot}_n) = 0, \quad '(\Gamma^{\cdot\cdot}_n) = ''(\Gamma^{\cdot\cdot}_n)_n = \Gamma^{\cdot\cdot}_n. \tag{6.16}$$

Lastly, we show the results with respect to the general connections derived from a normal general connection Γ in a diagram. If we regard this diagram as the genealogical tree of the descendants of a normal general connection Γ, it shows that

(i) all the descendants are normal general connections,

(ii) their normal parts and $\Gamma^{\cdot}$ and $\Gamma^{\cdot\cdot}$ out of their basic parts are generically fixed,

(iii) $'\Gamma$ and $''\Gamma$ are not exterminable,

and

(iv) the genealogical tree is composed of at most the ten general connections: Γ, $'\Gamma$, $''\Gamma$, $\Gamma^{\cdot}$, $\Gamma^{\cdot\cdot}$, $'\Gamma_n$, $''\Gamma_n$, $\Gamma^{\cdot}_n$, $\Gamma^{\cdot\cdot}_n$, 0.

- Γ
  - $'\Gamma$
    - $'\Gamma \longrightarrow$
    - $''('\Gamma) = \Gamma^{\cdot}$
      - $\Gamma^{\cdot} \longrightarrow$
      - $\Gamma^{\cdot} \longrightarrow$
      - 0
      - 0
    - 0
    - $''('\Gamma)_n = \Gamma^{\cdot}_n$
      - 0
      - 0
      - $\Gamma^{\cdot}_n \longrightarrow$
      - $\Gamma^{\cdot}_n \longrightarrow$
  - $''\Gamma$
    - $'(''\Gamma) = \Gamma^{\cdot\cdot}$
      - $\Gamma^{\cdot\cdot} \longrightarrow$
      - $\Gamma^{\cdot\cdot} \longrightarrow$
      - 0
      - 0
    - $''\Gamma \longrightarrow$
    - $'(''\Gamma)_n = \Gamma_n^{\cdot\cdot}$
      - 0
      - 0
      - $\Gamma^{\cdot\cdot}_n \longrightarrow$
      - $\Gamma^{\cdot\cdot}_n \longrightarrow$
    - 0
  - $'\Gamma_n$
    - 0
    - 0
    - $'\Gamma_n \longrightarrow$
    - $'\Gamma_n \longrightarrow$
  - $''\Gamma_n$
    - 0
    - 0
    - $''\Gamma_n \longrightarrow$
    - $''\Gamma_n \longrightarrow$

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