ON THE RADICAL OF QUASI-FROBENIUS ALGEBRAS

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Introduction.

Let A be an algebra over a field F. A is called *quasi-Frobenius* if it has a unit element and if every indecomposable direct component of the first regular representation is equivalent to an indecomposable direct component of of the second regular representation. If the two regular representations are equivalent, then A is called a *Frobenius* algebra. Furthermore, A is called *symmetric* if one of the two regular representations can be transformed into the other by a symmetric non-singular matrix.[1] The main purpose of the present work is to study the properties of the radical of these and some other types of algebras.

The first section is preliminary and we make some remarks on simple modules with an algebra A as two-sided operator domain. Then in section 2 we deal mainly with nilpotent (left, right and two-sided) ideals of a quasi-Frobenius algebra. Namely: Let A be a quasi-Frobenius algebra over a field F and let N be its radical. We may assume, essentially without loss of generality, that $M=l(N)=r(N)$ is contained in N^2, where $l(N)$ [$r(N)$] denotes the totality of left [right] annihilators of N (theorem 1). Then we show that a large part of nilpotent ideals of A can be characterized without considering the multiplication by elements of A other than those of N; we show in particular that every nilpotent two-sided ideal of A is such. From this and other results we show in the next section 3 that a quasi-Frobenius algebra is largely determined by its radical. For instance, if two (bound) quasi-Frobenius algebras A and $\tilde{A}$ over F have a same (i.e. isomorphic) radical N, then we have $\bar{A}=A/N \cong \tilde{A}/N=\bar{\tilde{A}}$; we have also $\pi(\kappa)\leftrightarrow\tilde{\pi}(\kappa)$ and $z_{\kappa\lambda}=\tilde{z}_{\tilde{\kappa}\tilde{\lambda}}$ for a unique correspondence $\kappa\leftrightarrow\tilde{\kappa}$ of simple constituents $\bar{A}_\kappa$ and $\bar{\tilde{A}}_{\tilde{\kappa}}$ of $\bar{A}$ and of $\bar{\tilde{A}}$, respectively. Here $M=l_N(N)=r_N(N)$ (annihilators taken in N) is assumed to be contained in N^2 and $z_{\kappa\lambda}$ [$\tilde{z}_{\tilde{\kappa}\tilde{\lambda}}$] denote the (two-sided) Cartan invariants of A [$\tilde{A}$]; (for $\pi(\kappa)$ [$\tilde{\pi}(\tilde{\kappa})$] see section 2). Section 4 is concerned with Frobenius algebras and some supplementary remarks are given on such algebras.

Let now N be a nilpotent algebra over a field F and ρ be its index: $N\supset N^2\supset\cdots\supset N^{\rho-1}\supset N^\rho=0$. In section 5 we discuss a particular class of nilpotent algebras with the property $(l(N):F)=(r(N):F)=(N^{\rho-1}:F)=1$; here, $l(N)$ [$r(N)$] denotes the left [right] annihilators of N. We prove that this property is equivalent to the property that the algebra $F+N$ which is obtained

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1) For the properties of such algebras, see the papers given in the References (above all, see Nakayama [10]).

from N by the adjunction of a unit element is a Frobenius algebra. Several other characterizations of this class of algebras are obtained. Moreover, we show that if an algebra A over F with a unit element has a radical N which itself is a nilpotent algebra of this type, then A is uniquely determined by N up to a semi-simple direct summand, except for the case $N^2=0$. Finally, section 6 is a generalization of previous sections (2, 3 and 5) and aims, not only at quasi-Frobenius algebras, but also at less special class of algebras; we deal there with a certain class of algebras over a field F with a given nilpotent algebra N over F as the radical.

It should be observed that, although we have restricted ourselves to the case of algebras, most of our principal results may be generalized to the case of rings satisfying minimum condition for left and right ideals.

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1. Some remarks on simple modules with an algebra as two-sided operator domain.

Let A be an (associative, finite dimensional) algebra over a field F possessing a unit element; let N be its radical. The residue class algebra $\bar{A}=A/N$ is semisimple and is a direct sum of simple two-sided ideals which themselves are simple algebras:

$$\bar{A}=\bar{A}_1+\bar{A}_2+\cdots+\bar{A}_k;$$

the unit elements $\bar{E}_\kappa$ of each $\bar{A}_\kappa$ $(1\leqq\kappa\leqq k)$ is expressible as a sum of mutually orthogonal idempotent elements $\bar{e}_{\kappa,1}, \bar{e}_{\kappa,2}, \cdots, \bar{e}_{\kappa,f(\kappa)}$ such that left ideals $\bar{A}\bar{e}_{\kappa,i}$ [right ideals $\bar{e}_{\kappa,i}\bar{A}$] are simple, and the unit element $\bar{E}$ of $\bar{A}$ is the sum of these $\bar{e}_{\kappa,i}$ $(1\leqq\kappa\leqq k,\ 1\leqq i\leqq f(\kappa))$. This decomposition of $\bar{E}$ leads to a decomposition of the unit element E of A:

$$E=E_1+E_2+\cdots+E_k,$$
$$E_\kappa=e_{\kappa,1}+e_{\kappa,2}+\cdots+e_{\kappa,f(\kappa)} \qquad (1\leqq\kappa\leqq k),$$

where each $e_{\kappa,i}$ lies in the residue class $\bar{e}_{\kappa,i}\ (\mathrm{mod}\ N)$; A is a direct sum of modules $E_\kappa AE_\lambda$ $(1\leqq\kappa,\ \lambda\leqq k)$. Moreover, there exists for each κ a system of $f(\kappa)^2$ elements $c_{\kappa,ij}$ $(1\leqq i,\ j\leqq f(\kappa))$ such that $c_{\kappa,ii}=e_{\kappa,i}$ and $c_{\kappa,ij}c_{\kappa,hl}=\delta_{jh}c_{\kappa,il}$, δ being the Kronecker's symbol. We call an element a in A is of type[2] (κ,λ) if a lies in $E_\kappa AE_\lambda$, i.e., if a satisfies $E_\kappa aE_\lambda=a$. The idempotent elements $e_{\kappa,i}$ are primitive, and the left ideals $Ae_{\kappa,i}$ [right ideals $e_{\kappa,i}A$] are directly indecomposable; for the sake of brevity we set $e_{\kappa,1}=e_\kappa$ $(1\leqq\kappa\leqq k)$.[3]

Let $\mathfrak{M}$ be a simple (A, A) two-sided module. Then there are two κ, λ such that $E_\kappa\mathfrak{M}E_\lambda=\mathfrak{M}$, $E_{\kappa'}\mathfrak{M}E_{\lambda'}=0$ $(\kappa\neq\kappa'$ or $\lambda\neq\lambda')$; moreover, as we have $N\mathfrak{M}$

2) See Nesbitt [16].

3) For these well known fundamentals of the theory of algebras, see for instance [1], [5].

$=\mathfrak{M}N=0$, we may regard $\mathfrak{M}$ as a simple $(\bar{A}_\kappa, \bar{A}_\lambda)$ module. We shall call such an (A, A) module to be of *type* (κ, λ). Then it is obvious that $\mathfrak{M}$ is decomposed into a direct sum of isomorphic left [right] A-submodules $\mathfrak{M}e_{\lambda,i}$ $(1\leqq i\leqq f(\lambda))$ $[e_{\kappa,j}\mathfrak{M}$ $(1\leqq j\leqq f(\kappa))]$:

$$\mathfrak{M}=\mathfrak{M}e_{\lambda,1}+\mathfrak{M}e_{\lambda,2}+\cdots+\mathfrak{M}e_{\lambda,f(\lambda)}=e_{\kappa,1}\mathfrak{M}+e_{\kappa,2}\mathfrak{M}+\cdots+e_{\kappa,f(\kappa)}\mathfrak{M};$$

$$\mathfrak{M}e_{\lambda,1}\cong\mathfrak{M}e_{\lambda,2}\cong\cdots\cong\mathfrak{M}e_{\lambda,f(\lambda)};\quad e_{\kappa,1}\mathfrak{M}\cong e_{\kappa,2}\mathfrak{M}\cong\cdots\cong e_{\kappa,f(\kappa)}\mathfrak{M}.$$

Furthermore, let e be any primitive idempotent element of A. Then $\bar{e}\in\bar{A}_\mu$ for some μ $(1\leqq\mu\leqq k)$; we have either $\mathfrak{M}e=0$ $(\mu\neq\lambda)$ or $\mathfrak{M}e\cong\mathfrak{M}e_\lambda$ $(\mu=\lambda)$ $[e\mathfrak{M}=0$ $(\mu\neq\kappa)$ or $e\mathfrak{M}\cong e_\kappa\mathfrak{M}$ $(\mu=\kappa)]$. The left [right] modules $\mathfrak{M}e_{\lambda,i}$ $[e_{\kappa,j}\mathfrak{M}]$ are not simple in general. (If the underlying field F is algebraically closed, then these modules are all simple.)

PROPOSITION 1.[4] *Let $\mathfrak{M}$ be a simple (A, A) module of type (κ, λ); let the right A-submodules $e_{\kappa,i}\mathfrak{M}$ $(1\leqq i\leqq f(\kappa))$ of $\mathfrak{M}$ be (all) simple. Let $\mathfrak{l}$ be a simple left A-submodule of $\mathfrak{M}$. Then every A-homomorphism of $\mathfrak{l}$ into $\mathfrak{M}$ is obtained by the right multiplication of an element of A. Conversely, let every A-endomorphism of any simple left A-submodule of $\mathfrak{M}$ be obtained by the right multiplication of an element of A. Then $e_{\kappa,i}\mathfrak{M}$ $(1\leqq i\leqq f(\kappa))$ are (all) simple.*

Proof. Assume that $e_{\kappa,i}\mathfrak{M}$ $(l\leqq i\leqq f(\kappa))$ are simple right A-submodules of $\mathfrak{M}$. Let $\mathfrak{l}_0$ be a simple left submodule of $\mathfrak{M}e_\lambda$. We first prove that any simple left submodule $\mathfrak{l}$ of $\mathfrak{M}$ is obtained from $\mathfrak{l}_0$ by the right multiplication of a regular element of $E_\lambda AE_\lambda$. In fact, let m_0 be any non-zero element in $e_\kappa\mathfrak{l}_0$ and let m by any non-zero element in $e_\kappa\mathfrak{l}$. We now note that $e_\kappa\mathfrak{l}_0$ and $e_\kappa\mathfrak{l}$ are simple left $e_\kappa Ae_\kappa$-modules and that $e_\kappa\mathfrak{M}e_\lambda$ is a simple two-sided $(e_\kappa Ae_\kappa, e_\lambda Ae_\lambda)$ module as well as a simple right $e_\lambda Ae_\lambda$-module. Then consider the expression

$$\begin{aligned} m=mE_\lambda &= me_{\lambda,1}+me_{\lambda,2}+\cdots+me_{\lambda,f(\lambda)} \\ &= me_{\lambda,1}+mc_{\lambda,21}c_{\lambda,12}+\cdots+mc_{\lambda,f(\lambda)1}c_{\lambda,1f(\lambda)}. \end{aligned}$$

At least one $mc_{\lambda,i1}$ does not vanish and we have

$$e_\kappa Ae_\kappa\cdot mc_{\lambda,i1}\cdot e_\lambda Ae_\lambda=e_\kappa\mathfrak{M}e_\lambda=m_0\cdot e_\lambda Ae_\lambda;$$

from this it follows that $mc_{\lambda,i1}=m_0a_ie_\lambda$ for some a_i in $E_\lambda AE_\lambda$ commuting with every $c_{\lambda,jh}$, and hence that $m=m_0(a_1c_{\lambda,11}+a_2c_{\lambda,12}+\cdots+a_{f(\lambda)}c_{\lambda,1f(\lambda)})$; we can therefore choose some regular element a of $E_\lambda AE_\lambda$ such that $m=m_0a$, and we have $\mathfrak{l}=E_\kappa AE_\kappa m=E_\kappa AE_\kappa m_0a=\mathfrak{l}_0a$. We now take the inverse element a' of a in $E_\lambda AE_\lambda$; then $\mathfrak{l}=\mathfrak{l}_0a$ is contained in $\mathfrak{M}a'e_\lambda a$, since $\mathfrak{M}a'=\mathfrak{M}$. After operation of a suitable inner automorphism of A we may therefore assume that $\mathfrak{l}$ is contained in $\mathfrak{M}e_\lambda$. Let now φ be any A-endomorphism of $\mathfrak{l}$; let m be any non-zero element in $e_\kappa\mathfrak{l}$, as before. Then φm is also contained in $e_\kappa\mathfrak{l}$, and hence we have $\varphi m=bm$ for some b in $e_\kappa Ae_\kappa$; but, as $e_\kappa\mathfrak{M}e_\lambda$ is a simple right $e_\lambda Ae_\lambda$-module, $bm=mb'$ for an element b' of $e_\lambda Ae_\lambda$. By this fact the endomorphism φ of $\mathfrak{l}$ is obtained by the right multiplication of the element b' of $e_\lambda Ae_\lambda$.

4) Cf. Ikeda [9].

(Moreover, we see easily that every A-endomorphism of $\mathfrak{l}$ is given by the right multiplication of an element of a division subalgebra of $e_\lambda Ae_\lambda$ (mod $e_\lambda Ne_\lambda$) which is isomorphic to $e_\kappa Ae_\kappa$ (mod $e_\kappa Ne_\kappa$), and conversely; here, the isomorphism of division algebras is given by $\bar{c}m = m\bar{c}'$.) Our first assertion is now immediate from what we have proved above. Assume now conversely that every A-endomorphism of any simple left A-submodule of $\mathfrak{M}$ is given by the right multiplication of an element of A. Let $\mathfrak{l}$ be a simple left A-submodule of $\mathfrak{M}e_\lambda$. Then $e_\kappa\mathfrak{l}$ is a simple left $e_\kappa Ae_\kappa$-module; we take any non-zero element m in $e_\kappa\mathfrak{l}$; as $e_\kappa\mathfrak{M}e_\lambda$ is a simple two-sided $(e_\kappa Ae_\kappa, e_\lambda Ae_\lambda)$ module, we have $e_\kappa Ae_\kappa\cdot m\cdot e_\lambda Ae_\lambda = e_\kappa\mathfrak{M}e_\lambda$. Now consider a mapping φ: $xm\rightarrow xam$, where a is an element of $e_\kappa Ae_\kappa$ and x is any element of A; this mapping is indeed an A-endomorphism of $\mathfrak{l}$, and hence is given by the right multiplication of an element of $e_\lambda Ae_\lambda$. Therefore we have $am=ma'$, where a is in $e_\kappa Ae_\kappa$ and a' is in $e_\lambda Ae_\lambda$; this shows that the module $e_\kappa\mathfrak{M}e_\lambda$ is simple not only as two-sided $(e_\kappa Ae_\kappa, e_\lambda Ae_\lambda)$ module, but also as right $e_\lambda Ae_\lambda$-module, and hence that $e_\kappa\mathfrak{M}$ is a simple right A-module. Our proof is now completed.

The following proposition follows readily from what we have proved:

PROPOSITION 2. *Let $\mathfrak{M}$ be a simple (A, A) module of type (κ, λ): let the left A-submodules $\mathfrak{M}e_{\lambda,i}$ $(1\leqq i\leqq f(\lambda))$ as well as the right A-submodules $e_{\kappa,j}\mathfrak{M}$ $(1\leqq j\leqq f(\kappa))$ of $\mathfrak{M}$ be all simple. Then every simple left [right] A-submodule of $\mathfrak{M}$ is written as $\mathfrak{M}e$ $[e'\mathfrak{M}]$ where e $[e']$ is a primitive idempotent element of type (λ, λ) $[(\kappa, \kappa)]$, and conversely; and, when that is so, $\bar{e}_\kappa\bar{A}\bar{e}_\kappa$ is isomorphic to $\bar{e}_\lambda\bar{A}\bar{e}_\lambda$; moreover, every A-homomorphism of simple left [right] A-submodule of $\mathfrak{M}$ is given by the right [left] multiplication of an element of A.*

PROPOSITION 3. *Let assumptions and notations be as in prop. 2. Let $\mathfrak{M}=\mathfrak{m}_1+\mathfrak{m}_2+\cdots+\mathfrak{m}_{f(\lambda)}$ be any decomposition of $\mathfrak{M}$ into direct sum of simple left A-submodules. Then there exists a system of mutually orthogonal primitive idempotent elements $\tilde{e}_{\lambda,i}$ $(1\leqq i\leqq f(\lambda))$ of $E_\lambda AE_\lambda$ such that $\mathfrak{m}_i=\mathfrak{M}\tilde{e}_{\lambda,i}$. Similarly for right submodules.*

Proof. By prop. 2 we have $\mathfrak{m}_i=\mathfrak{m}e_i'$ $(1\leqq i\leqq f(\lambda))$ for some primitive idempotent element e_i' of $E_\lambda AE_\lambda$. To every e_i' there corresponds a simple left ideal $\bar{A}_\lambda\bar{e}_i'$ of $\bar{A}_\lambda$; and, as is easily be seen, we have $\bar{A}_\lambda=\bar{A}_\lambda\bar{e}_1'+\bar{A}_\lambda\bar{e}_2'+\cdots+\bar{A}_\lambda\bar{e}'_{f(\lambda)}$ (direct sum). Hence there is a system of mutually orthogonal primitive idempotent elements $\tilde{e}_{\lambda,i}$ $(1\leqq i\leqq f(\lambda))$ such that $\bar{A}_\lambda\bar{e}_i'=\bar{A}_\lambda\bar{\tilde{e}}_{\lambda,i}$; our assertion is now evident.

2. Nilpotent ideals of a quasi-Frobenius algebra.

Let A be a quasi-Frobenius algebra over a field F; let N be its radical. Let $A/N=\bar{A}=\bar{A}_1+\bar{A}_2+\cdots+\bar{A}_k$, $f(\kappa)$, $e_{\kappa,i}$, $e_\kappa=e_{\kappa,1}$, $c_{\kappa,ij}$, $E_\kappa=e_{\kappa,1}+e_{\kappa,2}+\cdots+e_{\kappa,f(\kappa)}$ and $E=E_1+E_2+\cdots+E_k$ have the same meaning as in the previous section. The totalities $l(N)$ and $r(N)$ of left annihilators and of right annihilators of N respectively coincide; we denote this by M. Then we have $E_\kappa M = ME_{\pi(\kappa)}$, where $(\pi(1), \pi(2), \cdots, \pi(k))$ is a (unique) permutation of $(1, 2, \cdots, k)$; $E_\kappa M$ $(1 \leqq \kappa \leqq k)$ are non-zero simple two-sided ideals and $M = \sum E_\kappa M = \sum ME_{\pi(\kappa)}$; $e_{\kappa,i}M$ $[Me_{\kappa,i}]$ $(1 \leqq \kappa \leqq k, 1 \leqq i \leqq f(\kappa))$ are simple right [left] ideals. A is uniquely decomposable as the (two-sided) direct sum of a semisimple algebra and an algebra bound to its radical (for short, bound algebra);[5] and, from this point of view, we shall assume in the followings without loss of generality that A is bound to N, i.e. M is contained in N.

Let $\mathfrak{l}$ be a left ideal *of* N (for short, left N-ideal)[6] and let M be contained in $\mathfrak{l}$. Then we obtain the following criterion for $\mathfrak{l}$ to be a left ideal *of* A (for short, left A-ideal):

LEMMA 1. *Let A be a quasi-Frobenius algebra over a field F; let N be its radical. Let $\mathfrak{l}$ be a left N-ideal in N. Then, $\mathfrak{l}$ contains M and is also a left A-ideal if and only if $\mathfrak{l}_N(r_N(\mathfrak{l})) = \mathfrak{l}$.*[7] *A similar assertion holds for right ideals.*

Proof. Let $\mathfrak{l}$ be a left N-ideal in N satisfying $l_N(r_N(\mathfrak{l})) = \mathfrak{l}$. From the definitions we have $r_N(\mathfrak{l}) \subseteqq N$, so that $\mathfrak{l} = l_N(r_N(\mathfrak{l})) \supseteqq l_N(N) = M$; to see that $\mathfrak{l}$ is a left A-ideal we need only to observe $\mathfrak{l} = l_N(r_N(\mathfrak{l})) = N \frown l_A(r_N(\mathfrak{l}))$, where N and $l_A(r_N(\mathfrak{l}))$ are left A-ideals. As to the converse, assume that $\mathfrak{l}$ is a left A-ideal containing M. Then we have $l_A(r_A(\mathfrak{l})) = \mathfrak{l}$ (Nakayama [10], §3); from $\mathfrak{l} \supseteqq M = l_A(N)$ it follows that $r_A(\mathfrak{l})$ is contained in $r_A(M) = r_A(l_A(N)) = N$, so that $r_A(\mathfrak{l}) = r_N(\mathfrak{l})$ and $\mathfrak{l} \subseteqq l_N(r_N(\mathfrak{l})) = l_N(r_A(\mathfrak{l})) \subseteqq (l_A r_A(\mathfrak{l})) = \mathfrak{l}$. This shows $l_N(r_N(\mathfrak{l})) = \mathfrak{l}$.

For each simple two-sided ideal $E_\kappa M$ there is a positive integer h such that $E_\kappa M \subseteqq N^h$ and $E_\kappa M \frown N^{h+1} = 0$. We then say that *$E_\kappa M$ belongs to N^h.*

PROPOSITION 4. *Let $E_\kappa M$ belong to N^h. Then $E_{\pi(\kappa)}M$ also belongs to N^h.*

Proof. We have only to consider the case $\pi(\kappa) \neq \kappa$. Let $E_\kappa M$ belong to N^h and assume that $E_{\pi(\kappa)}M$ were contained in N^{h+1}. Then we can choose $h+1$ elements $x_1, x_2, \cdots, x_{h+1}$ from N such that each x_i is an element of type $(\lambda_{i-1}, \lambda_i)$ where $\lambda_0 = \pi(\kappa)$, $\lambda_{h+1} = \pi^2(\kappa) = \pi(\pi(\kappa))$ and such that $x_1 x_2 \cdots x_{h+1}$ $(\neq 0)$ lies in $E_{\pi(\kappa)}M$; x_1 is of type $(\pi(\kappa), \lambda_1)$ and $x_1 N^h \neq 0$, i.e. x_1 is not contained in $l(N^h)$; as $l(N^h) = r(N^h)$, we have $N^h x_1 \neq 0$. There is therefore an element y of type $(*, \pi(\kappa))$ in N^h with $yx_1 \neq 0$; as y is not in $l(N) = M = r(N)$, we can find an element z in N for which zy $(\neq 0)$ lies in M; but, as y is of type $(*, \pi(\kappa))$, zy is contained in $ME_{\pi(\kappa)}$ as well as in N^{h+1}, i.e. $E_\kappa M = ME_{\pi(\kappa)}$ contains a non-zero element of N^{h+1}. This is a contradiction and hence $E_{\pi(\kappa)}M$ must belong

5) An algebra is said to be *bound to its radical* if the two-sided annihilators of the radical are contained in the radical. See Hall [8].

6) Here N is considered itself to be a nilpotent algebra; so that, when we speak of a left N-ideal, we consider only the left operation of the elements of N.

7) $l_N(S)$ $[r_N(S)]$ denotes the left [right] annihilators of S in N.

to N^m, $m \leqq h$. Now let $\pi^t(\kappa)=\kappa$ and let $E_{\pi^j(\kappa)}M$ belong to N^{m_j} $(2 \leqq j \leqq t)$. From what we have proved above it follows that $h \geqq m \geqq m_2 \geqq \cdots \geqq m_{t-1} \geqq m_t = h$; hence $h=m$ and this completes our proof.

The permutation $(\pi(1), \pi(2), \cdots, \pi(k))$ of $(1, 2, \cdots, k)$ is expressible as an irredundant product of cyclic permutations; let the expression be

$$\text{(a)} \qquad (\kappa_{11}\kappa_{12}\cdots\kappa_{1,r_1})(\kappa_{21}\kappa_{22}\cdots\kappa_{2r_2})\cdots(\kappa_{l1}\kappa_{l2}\cdots a_{lr_l}),$$

where $\pi(\kappa_{11})=\kappa_{12}$, $\pi(\kappa_{12})=\kappa_{13}$ etc. By prop. 4 it follows that, for a factor $(\kappa_{i1}\kappa_{i2}\cdots\kappa_{ir_i})$, the two-sided ideals $E_{\kappa_{ij}}M$ $(1 \leqq j \leqq r_i)$ belong to a same power of of N; in particular, if some $E_{\kappa_{im}}M$ belongs to N^1, then $E_{\kappa_{ij}}M$ $(1 \leqq j \leqq r_i)$ belong to N^1. To every such factor there corresponds a *block* of primitive idempotents $e_{\kappa_{ij},h}$ $(1 \leqq j \leqq r_i, 1 \leqq h \leqq f(\kappa_{ij}))$ and hence a uniquely determined two-sided direct summand of A, which itself is a quasi-Frobenius algebra.[8] (To see this, we have only to observe that an element of type $(\kappa_{ij}, *)$ is either of type $(\kappa_{ij}, \kappa_{ij})$ or of type $(\kappa_{ij}, \kappa_{i,j+1})$.) We now give the following

THEOREM 1. *Let A be a quasi-Frobenius algebra over a field F. Then A is uniquely decomposed into a direct sum A_0+A_1 of two-sided ideals A_0 and A_1. Here, A_0 is itself a generalized uni-serial quasi-Frobenius algebra over F and the square of its radical vanishes; A_1 is itself a quasi-Frobenius algebra with radical $N_1=A_1 \frown N$ and $M_1=l_{A_1}(N_1)=r_{A_1}(N_1)$ is contained in N_1^2.*

Proof. First we note that two-sided direct summand of a quasi-Frobenius algebra is also a quasi-Frobenius algebra. Now in the expression (a) of the permutation $(\pi(1), \pi(2), \cdots, \pi(k))$ of $(1, 2, \cdots, k)$ we assemble all the cyclic factors $(\kappa_{i1}\kappa_{i2}\cdots\kappa_{ir_i})$ belonging to N^1 (i.e. at least one $E_{\kappa_{ij}}M$ belongs to N^1). For each of these factors we obtain a two-sided direct summand of A; let A_0 be the direct sum of thus obtained two-sided ideals. Then the radical $N_0 = A_0 \frown N$ of A_0 satisfies $N_0^2=0$; A_0 is moreover a generalized uni-serial algebra. The rest of our assertions is now immediate from the definitions and from the uniqueness of the decomposition of an algebra into direct sum of indecomposable two-sided ideals.

By virtue of above theorem 1 we can assume, essentially without loss of generality, that $M=l(N)=r(N)$ is contained in N^2. Under this assumption we now consider nilpotent (left, right and two-sided) ideals of A. We have already noted in lemma 1 that a left N-ideal $\mathfrak{l}$ in N containing M is a left A-ideal if and only if $l_N(r_N(\mathfrak{l}))=\mathfrak{l}$. (Similarly for right ideals.) We shall say such a left [right] ideal to be a *closed* left [right] N-ideal.

PROPOSITION 5. *Let A be a quasi-Frobenius algebra over a field F; let N be its radical. Let M be contained in N^2. Then every simple left (A-)ideal*

8) See Artin-Nesbitt-Thrall [1], Ch. 9, Nakayama [14] and Scott [21]. This direct summand is moreover a *generalized uni-serial* algebra and the square of its radical vanishes. For generalized uni-serial algebras see Nakayama [11], [13].

is obtained from a closed left N-ideal by the right multiplication of an element in N. Similarly for right ideals.

Proof. Let $\mathfrak{l}$ be any simple left ideal of A. Then $\mathfrak{l}$ is A-isomorphic to some Ae_κ/Ne_κ; and, when that is so, $\mathfrak{l}$ is contained in the simple two-sided ideal $E_\kappa M = ME_{\pi(\kappa)}$; $\mathfrak{l}$ is therefore A-isomorphic to $Me_{\pi(\kappa)}$. By prop. 3 we may then assume without loss of generality that $\mathfrak{l} = Me_{\pi(\kappa)}$. $Me_{\pi(\kappa)}$ contains an element of the form $xy\ (\neq 0)$, where x and y lie in N. In fact, since $Me_{\pi(\kappa)} \subseteqq N^2$, $Me_{\pi(\kappa)}$ contains a non-zero element of the form $x_1y_1 + x_2y_2 + \cdots + x_sy_s$, where each x_λ and y_λ lie in $N\,(1 \leqq \lambda \leqq s)$; then we can find for some λ an element x of N such that $xy_\lambda e_{\pi(\kappa)}\ (\neq 0)$ is contained in M, hence in $Me_{\pi(\kappa)}$. Now the left ideal $Ax \smile M$ is a closed left N-ideal in N by lemma 1; $Me_{\pi(\kappa)} = Axy = (Ax \smile M)y$ is therefore obtained from a closed left N-ideal $Ax \smile M$ by the right multiplication of an element y of N.

It should be observed that in view of the above prop. 5 we can determine all the left A-ideals contained in M without considering the multiplication by elements of A other than those of N, i.e. *a left N-ideal $\mathfrak{l}$ in M is a left A-ideal if and only if it is expressible as a sum of left N-ideals in M each of which is obtained from a closed left N-ideal by the right multiplication of an element of N.* Similarly for right ideals and hence for two-sided ideals. Such ideals will be called to be *characteristic.*

PROPOSITION 6. *Let A be a quasi-Frobenius algebra over a field F with radical N; let M be contained in N^2. Let $\mathfrak{z}$ be a two-sided N-ideal. Then $\mathfrak{z}$ is also a two-sided A-ideal if and only if $\mathfrak{z} \frown M$ is characteristic and $\mathfrak{z} \smile M$ is a closed two-sided* (*i.e. closed left as well as closed right*) *N-ideal.*

Proof. Suppose that a two-sided N-ideal $\mathfrak{z}$ satisfies our assumptions and put $\bar{\mathfrak{z}} = l(r(\mathfrak{z}))$ (i.e. $l_A(r_A(\mathfrak{z}))$); then $\bar{\mathfrak{z}} \smile M = l(r(\mathfrak{z})) \smile l(N) = l(r(\mathfrak{z}) \frown N) = l(r_N(\mathfrak{z})) = l(r_N(\mathfrak{z} \smile M)) = l(r(\mathfrak{z} \smile M)) = \mathfrak{z} \smile M$.[9] Furthermore, we have $\mathfrak{z} \frown M = \bar{\mathfrak{z}} \frown M$. In fact: Let $\mathfrak{z} \frown M = \mathfrak{z}^{(1)} + \mathfrak{z}^{(2)} + \cdots + \mathfrak{z}^{(t)}$ be the (unique) decomposition of $\mathfrak{z} \frown M$ into the direct sum of simple two-sided A-ideals; we may then set $ME_i = \mathfrak{z}^{(i)}$ for $i = 1, 2, \cdots, t$. From this it follows that $\mathfrak{z} = \mathfrak{z}E_1 + \mathfrak{z}E_2 + \cdots + \mathfrak{z}E_t$ (direct sum), since $\mathfrak{z}E_\lambda \neq 0\ (\lambda > t)$ implies $(\mathfrak{z} \frown M)E_\lambda \neq 0$ which is impossible and since $E_i\ (1 \leqq i \leqq t)$ are mutually orthogonal. We have therefore $r(\mathfrak{z} \frown M) = (\sum_{\kappa=t+1}^{k} E_\kappa A) \smile N = r(\mathfrak{z}) \smile N$; this implies $\bar{\mathfrak{z}} \frown M = l(r(\mathfrak{z})) \frown l(N) = l(r(\mathfrak{z}) \smile N) = l(r(\mathfrak{z} \frown M)) = \mathfrak{z} \frown M$. If, now, we take a suitable two-sided A-ideal $\mathfrak{z}_0$ in M such that $M = \mathfrak{z}_0 + (\mathfrak{z} \frown M)$ (direct sum), we have $\mathfrak{z} \smile M = \bar{\mathfrak{z}} \smile M = \mathfrak{z}_0 + \mathfrak{z} = \mathfrak{z}_0 + \bar{\mathfrak{z}}$ (direct sum); observing that $\mathfrak{z} \subseteqq \bar{\mathfrak{z}}$, this shows $\mathfrak{z} = \bar{\mathfrak{z}} = l(r(\mathfrak{z}))$. Similarly we must have $\mathfrak{z} = r(l(\mathfrak{z}))$. Therefore $\mathfrak{z}$ is a two-sided A-ideal. The converse part of our assertion is trivial.

REMARK. An analogous assertion for a left ideal is not valid. To see this fact, consider an algebra A consisting of all matrices

9) This formula will remain valid if we replace $\mathfrak{z}$ by a left N-ideal $\mathfrak{l}$ such that $\mathfrak{l} \smile M$ is closed.

$$\begin{pmatrix} \alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta_1 & \alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta_3 & 0 & \alpha_2 & 0 & 0 & 0 & 0 & 0 \\ \gamma_1 & -\beta_3 & \beta_4 & \alpha_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta_2 & \alpha_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta_4 & 0 & \alpha_2 & 0 \\ 0 & 0 & 0 & 0 & \gamma_2 & \beta_1 & -\beta_2 & \alpha_1 \end{pmatrix},$$

where α's, β's and γ's are arbitrary elements from a field F. A is in fact a (quasi-)Frobenius algebra possessing the property $M \subseteqq N^2$. Let $\mathfrak{l}$ be a left N-ideal consisting of all matrices

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ c & -b & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b & a & 0 & 0 \end{pmatrix},$$

where a, b and c are in F. It can be easily verified that $\mathfrak{l} \frown M$ and $\mathfrak{l} \smile M$ are both left A-ideals, while $\mathfrak{l}$ itself is not a left A-ideal.

For left and right ideals we have the following

PROPOSITION 7. *Let A be a quasi-Frobenius algebra over a field F with radical N; let M be contained in N^2. Let $\mathfrak{l}$ be a left N-ideal in N. Assume that $\mathfrak{l} \frown M$ is characteristic and that $\mathfrak{l} \smile M$ is a closed left N-ideal. Then there exists a left A-ideal $\mathfrak{l}_0$ in N such that $\mathfrak{l}_0$ is N-isomorphic to $\mathfrak{l}$.*

Proof. By definition, the characteristic left ideal $\mathfrak{l} \frown M$ is decomposed into a direct sum of simple left A-ideals: $\mathfrak{l} \frown M = \mathfrak{l}^{(1)} + \mathfrak{l}^{(2)} + \cdots + \mathfrak{l}^{(t)}$. We can then choose, by prop. 3, a system of mutually orthogonal primitive idempotents $e^{(1)}$, $e^{(2)}, \cdots, e^{(m)}$ $(m = \sum_{\kappa=1}^{k} f(\kappa))$ such that $\sum_{\lambda=1}^{m} e^{(\lambda)} = E$ and $Me^{(i)} = \mathfrak{l}^{(i)}$ for $i = 1, 2, \cdots, t$; we set $E_0 = e^{(1)} + e^{(2)} + \cdots + e^{(t)}$ and $E_1 = e^{(t+1)} + \cdots + e^{(m)}$. Now consider a left N-ideal $\mathfrak{l}_0 = \mathfrak{l}E_0$ and a mapping of $\mathfrak{l}$ onto $\mathfrak{l}_0$ which is given by the right multiplication of E_0, i.e., φ: $x \to xE_0$ $(x \in \mathfrak{l})$. The mapping φ is an N-isomorphism. In fact, suppose that for an x $(\neq 0)$ in $\mathfrak{l}$ we have $xE_0 = 0$; if x is contained in M, i.e. in $\mathfrak{l} \frown M$, then we have $x = xE_0 = 0$, which is a contradiction; if, on the other hand, x is not contained in M, then we can choose an element y in N such that yx $(\neq 0)$ lies in $\mathfrak{l} \frown M$, so that we must have $yxE_0 = yx \neq 0$, which is also a contradiction. Furthermore, we have $\mathfrak{l}_0 = \mathfrak{l}E_0 = (\mathfrak{l} \smile M)E_0$ since $\mathfrak{l} \smile M = \mathfrak{l} + ME_1$ (direct sum); $\mathfrak{l}_0$ is therefore a left A-ideal.

3. Quasi-Frobenius algebras with isomorphic radicals.

Let A be again a quasi-Frobenius algebra with a radical N; as we may assume without essential loss of generality that $M = l(N) = r(N) \subseteqq N^2$, we shall assume throughout this section that a quasi-Frobenius algebra means one which satisfies this condition. Let now $\mathfrak{l}$ be a simple left ideal of A; then by prop. 5 $\mathfrak{l}$ is obtained from a closed left N-ideal $\mathfrak{l}_0$, i.e. $\mathfrak{l} = \mathfrak{l}_0 a$ for some a in N. The mapping f: $x \rightarrow xa$ $(x \in \mathfrak{l}_0)$ is obviously an A-operator homomorphism of $\mathfrak{l}_0$ onto $\mathfrak{l}$; the element a is uniquely determined modulo $r(\mathfrak{l}_0) = r_N(\mathfrak{l}_0)$. We note that $\mathfrak{l}$ can be written as $\mathfrak{l} = Me$ with a primitive idempotent element of A; so that we may assume that $\mathfrak{l} = Me_\lambda$ for some λ and hence that $\mathfrak{l} \cong Ae_\kappa / Ne_\kappa$ where $\kappa = \pi^{-1}(\lambda)$. Now consider an A-endomorphism φ of $\mathfrak{l} = Me_\lambda$. By prop. 2 φ is given by the right multiplication of an element a_φ in $e_\lambda A e_\lambda$[10]: $y \rightarrow ya_\varphi$ $(y \in \mathfrak{l})$. If we combine φ with f, we obtain a homorphism φf: $x \rightarrow xaa_\varphi$ $(x \in \mathfrak{l}_0)$ from $\mathfrak{l}_0$ into $\mathfrak{l}$; put $b_\varphi = aa_\varphi$; then evidently $\mathrm{Ker}(f) = l(a) \frown \mathfrak{l}_0 = l_{\mathfrak{l}_0}(a)$, $\mathrm{Ker}(\varphi f) = l_{\mathfrak{l}_0}(b_\varphi)$; $l_{\mathfrak{l}_0}(a) \subseteqq l_{\mathfrak{l}_0}(b_\varphi)$, where $\mathrm{Ker}(f)$ and $\mathrm{Ker}(\varphi f)$ denote the kernels of these mappings. b_φ is uniquely determined modulo $r(\mathfrak{l}_0) = r_N(\mathfrak{l}_0)$. Conversely take any element c of N such that $\mathfrak{l}_0 c \subseteqq \mathfrak{l}$ and $l_{\mathfrak{l}_0}(a) \subseteqq l_{\mathfrak{l}_0}(c)$. Then the mapping g: $x \rightarrow xc$ $(x \in \mathfrak{l}_0)$ gives an A-homomorphism from $\mathfrak{l}_0$ into $\mathfrak{l}$; as $\mathrm{Ker}(g) = l_{\mathfrak{l}_0}(c) \supseteqq l_{\mathfrak{l}_0}(a) = \mathrm{Ker}(f)$, the correspondence xa $(\in \mathfrak{l}) \leftrightarrow$ class $[x]$ $(\in \mathfrak{l}_0/\mathrm{Ker}(f)) \rightarrow$ class $[x]$ $(\in \mathfrak{l}_0/\mathrm{Ker}(g)) \leftrightarrow xc$ $(\in \mathfrak{l})$ gives an A-endomorphism of $\mathfrak{l}$.

LEMMA 2. *Let A be a quasi-Frobenius algebra over a field F; let N be its radical. Let $\mathfrak{l}$ be a minimal characteristic left N-ideal in M. Then $\mathfrak{l}$ is a simple left A-ideal; moreover, the A-endomorphism ring of $\mathfrak{l}$ is completely determined by the radical N. Similarly for right ideals.*

The first assertion is a direct consequence of prop. 5; the second assertion follows at once from what we have proved above. (Note that M is assumed to be contained in N^2.)

The next proposition follows easily from what we have discussed:

PROPOSITION 8. *Let A be a quasi-Frobenius algebra over a field F; let N be its radical (and let $M \subseteqq N^2$). Let $\mathfrak{l}$ be a minimal characteristic left N-ideal. Then $\mathfrak{l} = \mathfrak{l}_0 a$, where $\mathfrak{l}_0$ is a closed left N-ideal and a is an element in N. When that is so, let $\mathfrak{l}'$ be another minimal characteristic left N-ideal. Then both $\mathfrak{l}$ and $\mathfrak{l}'$ are simple A-ideals; $\mathfrak{l}'$ is A-isomorphic to $\mathfrak{l}$ if and only if there is an element a' in N such that $\mathfrak{l}' = \mathfrak{l}_0 a'$, $\mathfrak{l}_0 \frown l(a') = \mathfrak{l}_0 \frown l(a)$. Moreover, if $M = \mathfrak{l}_1 + \mathfrak{l}_2 + \cdots + \mathfrak{l}_m$ is any decomposition of M into direct sum of minimal characteristic left N-ideals, then $\mathfrak{l}_i$ $(1 \leqq i \leqq m)$ are classified uniquely into classes, each of which consists of such $\mathfrak{l}$'s that every pair $(\mathfrak{l}_j, \mathfrak{l}_h)$ among them satisfies above criterion; further, this classification gives the unique decomposition of M into the direct sum of simple two-sided A-ideals.*

Let A be a quasi-Frobenius algebra over a field F and let N be its radical

10) See also Ikeda [9].

($M \subseteqq N^2$). Consider another (bound) quasi-Frobenius algebra $\tilde{A}$ over F with a radical $\tilde{N}$ which is isomorphic to N; the elements of $\tilde{N}$ may be identified with the elements of N and so we say that A and $\tilde{A}$ have a same radical; then we have from our asssumption $M = l_{\tilde{A}}(N) = r_{\tilde{A}}(N) \subseteqq N^2$. Let $\tilde{E}$, $\tilde{e}_{\kappa,i}$, $\tilde{e}_\kappa = \tilde{e}_{\kappa,1}$, $\tilde{c}_{\kappa,ij}$ ($1 \leqq \kappa \leqq \tilde{k}$, $1 \leqq i, j \leqq \tilde{f}(\kappa)$), $\tilde{E}_\kappa = \sum_{i=1}^{\tilde{f}(\kappa)} \tilde{e}_{\kappa,i}$ and $\tilde{\pi}(\kappa)$ have the same meaning to $\tilde{A}$ as E, $e_{\kappa,i}$ etc. to A. In view of the above prop. 8, we have first of all $\tilde{k} = k$ and we may set $E_\kappa M = \tilde{E}_\kappa M$ ($1 \leqq \kappa \leqq k$); the permutation $\tilde{\pi}(\kappa)$ is given by $\tilde{E}_\kappa M = M\tilde{E}_{\tilde{\pi}(\kappa)}$.

We now note: *Let $\mathfrak{z}_1 \supset \mathfrak{z}_2$ be two-sided N-ideals each of which satisfies the condition of prop.* 6 *and assume that there is no two-sided N-ideal satisfying the condition of prop.* 6 *between $\mathfrak{z}_1$ and $\mathfrak{z}_2$ other than themselves. Then these are two-sided A-ideals as well as two-sided $\tilde{A}$-ideals; if the type of simple (A, A) module $\mathfrak{z}_1/\mathfrak{z}_2$ is (κ, λ), then the type of $\mathfrak{z}_1/\mathfrak{z}_2$, considered as simple $(\tilde{A}, \tilde{A})$ module is $(\kappa, \tilde{\lambda})$, where $\tilde{\lambda} = \tilde{\pi}(\pi^{-1}(\lambda))$.* In fact: The first assertion is immediate by definition and by prop. 6. Now consider the two-sided ideals $\mathfrak{z}_1 \frown M$ and $\mathfrak{z}_2 \frown M$, which are both characteristic; we have obviously $\mathfrak{z}_1 \frown M \supseteqq \mathfrak{z}_2 \frown M$. If $\mathfrak{z}_1 \frown M \neq \mathfrak{z}_2 \frown M$, then there exists a minimal characteristic two-sided N-ideal $\mathfrak{z}_0$ such that $\mathfrak{z}_1 \frown M = (\mathfrak{z}_2 \frown M) + \mathfrak{z}_0$ (direct sum); from this and from our assumptions it follows that $\mathfrak{z}_1 = \mathfrak{z}_2 + \mathfrak{z}_0$ (direct sum). Therefore $\mathfrak{z}_1/\mathfrak{z}_2$ is isomorphic to $\mathfrak{z}_0$ as (A, A) module and, at the same time, as $(\tilde{A}, \tilde{A})$ module; our assertion follows now easily. If, on the other hand, $\mathfrak{z}_1 \frown M = \mathfrak{z}_2 \frown M$, then by prop. 6 we must have $\mathfrak{z}_1 \smile M \supsetneqq \mathfrak{z}_2 \smile M$, and hence $r_N(\mathfrak{z}_1) \subsetneqq r_N(\mathfrak{z}_2)$; we take an element x of $r_N(\mathfrak{z}_2) - r_N(\mathfrak{z}_1)$, i.e. $\mathfrak{z}_2 x = 0$, $\mathfrak{z}_1 x \neq 0$. The mapping $\mathfrak{z}_1 \to \mathfrak{z}_1 x$ is a homomorphism of $\mathfrak{z}_1$ into M; $\mathfrak{z}_1 x$ is direct sum of several minimal characteristic left N-ideals, each of which is isomorphic to Ae_κ/Ne_κ as A-ideal and hence also isomorphic to $\tilde{A}\tilde{e}_\kappa/N\tilde{e}_\kappa$ as $\tilde{A}$-ideal. $\mathfrak{z}_1/\mathfrak{z}_2$ is therefore of type $(\kappa, *)$ as a two-sided $(\tilde{A}, \tilde{A})$ module. Further, a similar consideration shows that $\mathfrak{z}_1/\mathfrak{z}_2$ is of type $(*, \tilde{\lambda})$ as a two-sided $(\tilde{A}, \tilde{A})$ module, where $\tilde{\lambda} = \tilde{\pi}(\pi^{-1}(\lambda))$. This completes the proof.

We consider for some time only the algebra A. Take any λ ($1 \leqq \lambda \leqq k$); then we can find two elements of N, x and y, of type $(*, \lambda)$ and of type $(\lambda, *)$ respectively such that $xy \neq 0$. Among the pairs of elements as this we now take a particular one: Choose as many elements $x_1, x_2, \cdots, x_t$ ($t \geqq 2$) as possible in N such that each x_i ($1 \leqq i \leqq t$) is of type (κ_i, κ_{i+1}), $x_1 x_2 \cdots x_t \neq 0$ and such that for some j ($2 \leqq j \leqq t$) we have $\kappa_j = \lambda$; then put $x = x_1 x_2 \cdots x_{j-1}$, $y = x_j x_{j+1} \cdots x_t$; moreover we may assume that $e_{\kappa_1} x e_\lambda = x$, $e_\lambda y e_{\kappa_{t+1}} = y$. Now we consider two-sided ideals $\mathfrak{z}_1 = AxA$ and $\mathfrak{z}_2 = AyA$ of A; put $\mathfrak{z}_1' = AxN \smile NxA$ and $\mathfrak{z}_2' = AyN \smile NyA$. Then it follows from our definition of x and y that $\mathfrak{z}_1\mathfrak{z}_2' = \mathfrak{z}_1'\mathfrak{z}_2 = \mathfrak{z}_1'\mathfrak{z}_2' = 0$. From this we conclude that we can not choose complete system $\mathfrak{P}_1$ and $\mathfrak{P}_2$ of representatives of residue modules $\mathfrak{z}_1/\mathfrak{z}_1'$ and of $\mathfrak{z}_2/\mathfrak{z}_2'$ repectively, such that every element of $\mathfrak{P}_1$ annihilates every element of $\mathfrak{P}_2$ from the left.

Consider again two algebras A and $\tilde{A}$ as before. By what we have noted, $\mathfrak{z}_1/\mathfrak{z}_1'$ and $\mathfrak{z}_2/\mathfrak{z}_2'$ are $(\tilde{A}, \tilde{A})$ modules of type $(*, \tilde{\lambda})$ and of type $(\lambda, *)$, respectively. (Observe that, although these two-sided modules $\mathfrak{z}_1/\mathfrak{z}_1'$ and $\mathfrak{z}_2/\mathfrak{z}_2'$ are not simple in general, the above arguments can be applied to this case with slight modifications.) Assume now that $\tilde{\lambda}$ were different from λ; then we can choose complete systems $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ of representatives of $\mathfrak{z}_1/\mathfrak{z}_1'$ and of $\mathfrak{z}_2/\mathfrak{z}_2'$ respectively, such that every element of $\mathfrak{Q}_1$ annihilates every element of $\mathfrak{Q}_2$ from the left; this is a contradiction. We must therefore have $\tilde{\lambda}=\lambda$ for each λ, i.e. $\tilde{\pi}(\pi^{-1}(\lambda))=\lambda$, $\tilde{\pi}(\lambda)=\pi(\lambda)$ $(1\leqq\lambda\leqq k)$.

THEOREM 2. *Let A and $\tilde{A}$ be two quasi-Frobenius algebras over a field F with a same radical N (and let $M=l_A(N)=l_{\tilde{A}}(N)\subseteqq N^2$). Then there exists a (unique) 1-1 correspondence between simple constituents of $\bar{A}=A/N$ and of $\bar{\tilde{A}}=\tilde{A}/N$: $\bar{A}_\kappa\leftrightarrow\bar{\tilde{A}}_{\sigma(\kappa)}$ $(1\leqq\kappa\leqq k=\tilde{k})$ (we may, after a suitable reordering, set $\sigma(\kappa)=\kappa$). When that is so, then (α) $\bar{A}_\kappa\cong\bar{\tilde{A}}_\kappa$ $(1\leqq\kappa\leqq k)$, $\bar{A}\cong\bar{\tilde{A}}$; ($\beta$) $\pi(\kappa)=\tilde{\pi}(\kappa)$ $(1\leqq\kappa\leqq k)$; (γ) every composition series of N considered as an (A, A) module is also a composition series of N considered as an $(\tilde{A}, \tilde{A})$ module, and conversely; moreover, the type of every composition factor module considered as (A, A) module is the same as that of the factor module considered as $(\tilde{A}, \tilde{A})$ module; (δ) $z_{\kappa\lambda}=\tilde{z}_{\kappa\lambda}$ where $z_{\kappa\lambda}$, $\tilde{z}_{\kappa\lambda}$ $(1\leqq\kappa,\lambda\leqq k)$ are the (two-sided) Cartan invariants;*[11] *(ε) the two-sided decomposition of A and of $\tilde{A}$ according to blocks induce a same two-sided decomposition of N.*

Proof. The first assertion and (β) follow from what we have shown above. Consider for a κ the minimal characteristic two-sided N-ideal $E_\kappa M=ME_{\pi(\kappa)}=\tilde{E}_\kappa M$ $=M\tilde{E}_{\pi(\kappa)}$; this is decomposed into the direct sum of $f(\kappa)$, as well as of $\tilde{f}(\kappa)$ minimal characteristic right N-ideals; $\tilde{f}(\kappa)$ must therefore be same as $f(\kappa)$. Moreover, we have from lemma 2 that $\bar{e}_\kappa\bar{A}\bar{e}_\kappa\cong\bar{\tilde{e}}_\kappa\bar{\tilde{A}}\bar{\tilde{e}}_\kappa$, which, combined with the well known theorem of Wedderburn-Artin, gives $\bar{A}_\kappa\cong\bar{\tilde{A}}_\kappa$; thus we have proved ($\alpha$). The assertion ($\gamma$) is immediate from what we have discussed; the last two assertions follow readily from the definitions and from (γ).

4. Supplementary remarks on Frobenius algebras.

Let A be a Frobenius algebra over a field F with a radical N, and let A be bound to N. Let $(u_1, u_2, \cdots, u_n)$ be a basis for A which is taken according to $A\supset N\supset M\supset 0$; let the multiplication table of A be $u_iu_j=\sum_{k=1}^n a_{ijk}u_k$, where the coefficients a_{ijk} $(1\leqq i, j, k\leqq n)$ lie in F. Then the parastrophic matrix[12] with $\xi_1, \xi_2, \cdots, \xi_n$ as the values of parameters is of the form

$$P(\xi)=\sum_{k=1}^n P_k\xi_k=\begin{bmatrix} P_{11}(\xi) & P_{12}(\xi) & P_{13}(\xi) \\ P_{21}(\xi) & P_{22}(\xi) & 0 \\ P_{31}(\xi) & 0 & 0 \end{bmatrix}$$

11) If the field F is algebraically closed, then $z_{\kappa\lambda}$ coincide with the ordinary Cartan invariants $c_{\kappa\lambda}$. Cf. Nakayama [14].

12) For the fundamental properties of parastrophic matrices see Frobenius [6], Nakayama [10], Brauer-Nesbitt [4] and Nesbitt [16].

where $P_k=(a_{\lambda\mu k})_{\lambda\mu}$; $P_{13}(\xi)$, $P_{22}(\xi)$ and $P_{31}(\xi)$ are square matrices (observe that $(A/N:F)=(M:F)$). Hence $|P(\xi)|\neq 0$ if and only if $|P_{13}(\xi)|\cdot|P_{22}(\xi)|\cdot|P_{31}(\xi)|\neq 0$, and so we may, without giving any change to $|P(\xi)|$, put $\xi_k=0$ for $k=1, 2, \cdots, t$, where $t=(A/N:F)$. Now we consider ξ_k $(t+1\leqq k\leqq n)$ as indeterminates, while we set $\xi_i=0$ $(1\leqq i\leqq t)$; then we have, from what we noted, that each of $|P_{13}(\xi)|$, $|P_{22}(\xi)|$ and $|P_{31}(\xi)|$ does not identically vanish. Let $\tilde{A}$ be a second Frobenius (bound) algebra over F with the same (i.e. isomorphic) radical N as A; we take a basis $(\tilde{u}_1, \tilde{u}_2, \cdots, \tilde{u}_n)$ for $\tilde{A}$ such that $\tilde{u}_i=u_i$ for $t+1\leqq i\leqq n$. Then, it follows that each one of $|\tilde{P}_{13}(\xi)|$, $|P_{22}(\xi)|$ and $|\tilde{P}_{31}(\xi)|$ does not identically vanish, where $\tilde{P}_{13}(\xi)$ and $\tilde{P}_{31}(\xi)$ have the same meaning to $\tilde{A}$ as $P_{13}(\xi)$ and $P_{31}(\xi)$ (respectively) to A. Therefore we can choose values of $\xi_{t+1}, \xi_{t+2}, \cdots, \xi_n$ from F such that $|P_{13}(\xi)|\cdot|P_{22}(\xi)|\cdot|P_{31}(\xi)|\cdot|\tilde{P}_{13}(\xi)|\cdot|\tilde{P}_{31}(\xi)|\neq 0$;[13] so that we have $|P(\xi)|\neq 0$, $|\tilde{P}(\xi)|\neq 0$. On the other hand, to each non-singular parastrophic matrix of A there corresponds a *non-singular character*[14] of A. From these observations we can easily see the following

PROPOSITION 9. *Let A and $\tilde{A}$ be two Frobenius algebras over a field F with a same radical N. Then for a suitable choice of non-sigular characters $\lambda(x)$ of A and $\tilde{\lambda}(x)$ of $\tilde{A}$, $\lambda(x)=\tilde{\lambda}(x)$ holds for every element x of N; and, when that is so, the (Nakayama's) automorphisms*[15] *φ and $\tilde{\varphi}$ belonging to $\lambda(x)$ and to $\tilde{\lambda}(x)$ respectively satisfy $x^{\tilde{\varphi}}\equiv x^{\varphi}$ $(\mathrm{mod}\, M)$ for every x in N; in particular, they are identical in N^2.*

PROPOSITION 10. *Let A be a Frobenius algebra over a field F; let N be its radical. Let σ be any automorphism of N. Then there exist two non-sigular characters $\lambda(x)$ and $\lambda'(x)$ of A such that $\lambda'(x)=\lambda(x^\sigma)$ for every x in N. Moreover, the automorphisms φ and φ' of A belonging to $\lambda(x)$ and to $\lambda'(x)$, respectively, satisfy $x^{\varphi'}\equiv x^{\sigma\varphi\sigma^{-1}}$ $(\mathrm{mod}\, M)$ for every x in N.*

Proof. Our first assertion is a direct consequence of prop. 9. The second assertion follows from $\lambda'(yx)=\lambda'(x^{\varphi'}y)=\lambda(y^\sigma x^\sigma)=\lambda(x^{\sigma\varphi}y^\sigma)=\lambda(x^{\sigma\varphi\sigma^{-1}\sigma}y^\sigma)=\lambda'(x^{\sigma\varphi\sigma^{-1}}y)$ and from the property of non-singular characters.

5. On a certain class of nilpotent algebras.

Let N be a nilpotent algebra over a field F and let ρ be its index, viz. let $N\supset N^2\supset\cdots\supset N^{\rho-1}\supset N^\rho=0$. We take a basis $(u_1, u_2, \cdots, u_n)$ for N and assume that $u_iu_j=\sum_{k=1}^n a_{ijk}u_k$; the matrix $P(\xi)=\sum_{k=1}^n P_k\xi_k$, where $P_k=(a_{\lambda\mu k})_{\lambda\mu}$ and ξ_k in F, is the parastrophic matrix of N with ξ_k as values of parameters; we have then $\mathrm{rank}\, P(\xi)<n$. We now prove

PROPOSITION 11. *Let N be a nilpotent algebra of index ρ over a field F.*

13) Here the field F must be assumed not to have too few elements.

14) See Azumaya [2], § 2.

15) Nakayama [11], Azumaya [2], Brauer [3], Osima [19], [20].

Then the following four conditions are equivalent each other:

(i) *The algebra $F+N$ which is obtained from N by the adjunction of a unit element is (quasi-)Frobenius;*

(ii) *Every non-zero left ideal $\mathfrak{l}$ [right ideal $\mathfrak{r}$] satisfies $l_N(r_N(\mathfrak{l}))=\mathfrak{l}$ $[r_N(l_N(\mathfrak{r}))=\mathfrak{r}]$;*

(iii) $(r_N(N):F)=(l_N(N):F)=(N^{\rho-1}:F)=1$;[16]

(iv) *The parastrophic matrix of N has the maximal rank $(N:F)-1$.*

Proof. (i)→(ii) and (ii)→(i). First observe that if the algebra $F+N$ is quasi-Frobenius, then it is a Frobenius algebra, and that every left [right] ideal of N is a nilpotent left [right] ideal of $F+N$, and conversely. Our assertions follow then easily by the well known properties of (quasi-)Frobenius algebras. (i)→(iv). Assume shat $F+N$ is a Frobenius algebra. Then, for a basis $(1, u_1, u_2, \cdots, u_n)$ of $F+N$ ($(u_1, u_2, \cdots, u_n)$ being a basis of N), we have a non-singular parastrophic mataix of $F+N$ of the form

$$\bar{P}(\xi)=\sum_{k=1}^{n}\xi_k\bar{P}_k=\begin{bmatrix} \xi_0 & \xi_1 \cdots \xi_n \\ \xi_1 & \\ \vdots & P(\xi) \\ \xi_n & \end{bmatrix},$$

where each ξ_k $(0\leqq k\leqq n)$ lies in F and $P(\xi)$ is the parastrophic matrix of N with ξ_k $(1\leqq k\leqq n)$ as values of parameters. From $|\bar{P}(\xi)|\neq 0$ it follows $\operatorname{rank} P(\xi)=n-1$. (iv)→(iii). Suppose that $(l(N):F)>1$. Then, by the definition of parastrophic matrix, we must have $\operatorname{rank} P(\xi)<n-1$. Therefore (iv) implies $(l(N):F)=1$. Similarly we have $(r(N):F)=1$. (Note that $1\leqq(N^{\rho-1}:F)\leqq(l(N):F)$.) (iii)→(i). Assume that $(l(N):F)=(r(N):F)=(N^{\rho-1}:F)=1$. Then $F+N$ has a unique simple left (also right) ideal $l(N)=r(N)=N^{\rho-1}$; for this ideal we can easily verify $l(r(N^{\rho-1}))=r(l(N^{\rho-1}))=N^{\rho-1}$. Moreover we have $l(r(N))=r(l(N))=N$, $l(r(0))=r(l(0))=0$; $F+N$ is therefore (quasi-)Frobenius.

PROPOSITION 12. *Let A be an algebra with the unit element E over a field F; let N be its radical. Assume that N has the property required in prop. 11. Then A is a (unique) direct sum of a semisimple algebra and a completely primary algebra which is isomorphic to $F+N$, provided that $N^2\neq 0$. If, on the other hand, $N^2=0$, then A is a (unique) direct sum of a semisimple algebra and an algebra which is either isomorphic to $F+N$ or to a matrix algebra consisting of all matrices*

$$\begin{bmatrix} \alpha & 0 \\ \gamma & \beta \end{bmatrix},$$

where α, β and γ are arbitrary elements from F.

Proof. Let E_κ, $e_{\kappa,\iota}$ etc. have the same meaning as in section 1; let ρ be the index of N. By our assumptions we can choose two E_p and E_q among the

16) This characterization was suggested to the author by A. Inatomi.

E_κ such that $E_pN^{\rho-1}E_q = N^{\rho-1}$. Here we note that $f(p)=f(q)=1$ and that $(\bar{e}_p\bar{A}\bar{e}_p : F)=(\bar{e}_q\bar{A}\bar{e}_q : F)=1$, which follow easily from $(N^{\rho-1} : F)=1$. Now take an arbitrary element a of N. Suppose $E_\kappa a \neq 0$ for some E_κ different from E_p; if $E_\kappa a$ is contained in $l(N)=N^{\rho-1}$, then $E_\kappa a = E_pE_\kappa a = 0$, which is impossible; if, on the other hand, $E_\kappa a$ is not contained in $N^{\rho-1}$, then there exists an element x in N such that $E_\kappa ax$ $(\neq 0)$ is contained in $N^{\rho-1}$ and this is also impossible. We must therefore have $E_pN=N$, and similarly $NE_q=N$. Assume now that $N^2 \neq 0$. Then E_p must be the same as E_q, i.e. $E_pNE_p=N$; hence A is a direct sum of a semisimple algebra and E_pAE_p, which is isomorphic to $F+N$. Next assume that $N^2=0$. In this case E_p and E_q may or may not coincide. If $E_p=E_q$, the above consideration is also available; while if $E_p \neq E_q$, then A is a direct sum of a semisimple algebra and $E_pAE_p+E_pAE_q+E_qAE_q$. It is easy to see that the latter algebra is isomorphic to the matrix algebra of our proposition. The uniqueness of the decomposition is immediate. (See Hall [8].)

6. On a class of algebras with isomorphic radicals.

Let N be a nilpotent algebra over a field F satisfying $l_N(N)=r_N(N)\subseteqq N^2$; we write $l_N(N)=r_N(N)=M$ as before. Let A be an algebra with a unit element over F; let its radical be isomorphic to N; we shall, as before, identify the radical of A with N and say that A has N as its radical. Further, we require A to satisfy $l_A(N)=r_A(N)\subseteqq N$, i.e., $l_A(N)=r_A(N)=M\subseteqq N^2$. In this section we consider, for a given N, all algebras as this; throughout the section, by an *algebra* we shall always understand an algebra as above, possessing the given N as its radical. At the outset we introduce the following partial ordering of the class of algebras (with given radical N): *We write $A \succ B$ for two algebras A and B when* (i) *every nilpotent left [right] A-ideal containing M or contained in M is a left [right] B-ideal and* (ii) *every nilpotent two-sided A-ideal is a two-sided B-ideal.* It is easy to see that the relation $\succ$ is in fact a partial ordering. We call A and B *equivalent* if both $A \prec B$ and $B \succ A$; then $\succ$ gives a lattice Λ_N of classes of equivalent algebras.

PROPOSITION 13. *Every Λ_N has a (unique) minimal element. All algebras in the minimal class are isomorphic each other.*

Proof. First of all, we should observe that an algebra $A_0=F+N$ obtained from N by the adjunction of a unit element is minimal; i.e. for any algebra A we have in fact $A \succ A_0$. The first assertion is now evident. The proof for the second assertion is similar to that of prop. 12.

The class of nilpotent algebras discussed in the previous section gives a class of examples of Λ_N which has only one element. Another example of such Λ_N is obtained by taking N as a *free nilpotent algebra*[17] of index $\rho > 1$ over

17) I.e., a nilpotent algebra generated by n bases $u_1, u_2, \cdots, u_n$ over F satisfying only the relations $u_{i_1}u_{i_2}\cdots u_{i_\rho}=0$. This example is due to M. Okuzumi.

an algebraically closed field F.

We now consider the relation between two equivalent algebras A and $\tilde{A}$. As $l(N)=r(N)$, it follows $l(N^{\nu})=r(N^{\nu})$. Most of the discussions made in sections $2-3$ can be applied, under some additional assumptions, to our case with slight modifications. Let E, $e_{\kappa, \iota}$, etc. have the same meaning as before.

(1) For every κ $(1 \leqq \kappa \leqq k)$ there is at least one simple left A-ideal $\mathfrak{l}$ in M such that $E_{\kappa}\mathfrak{l}=\mathfrak{l}$. $\mathfrak{l}$ must be also a simple $\tilde{A}$-ideal; hence $E_{\mu}\mathfrak{l}=\mathfrak{l}$ for some μ. μ is uniquely determined by κ. In fact, let $\mathfrak{l}'$ be a simple left A-ideal (also an $\tilde{A}$-ideal) such that $E_{\kappa}\mathfrak{l}'=\mathfrak{l}'$, $\tilde{E}_{\mu'}\mathfrak{l}'=\mathfrak{l}'$. Assume $\mu' \neq \mu$; if we take any two non-zero element $a \in \mathfrak{l}$ and $a' \in \mathfrak{l}'$, it follows then $\tilde{A}(a+a')=\mathfrak{l}+\mathfrak{l}'$; hence $A(a+a')$ $=E_{\kappa}AE_{\kappa}(a+a')=\mathfrak{l}+\mathfrak{l}'$, which is impossible in general. κ and μ are thus in 1-1 correspondence and so we identity them: $\kappa \leftrightarrow \kappa$ $(1 \leqq \kappa \leqq k=\tilde{k})$. For right ideals a similar 1-1 correspondence can be obtained: $\lambda \leftrightarrow \tilde{\lambda}$ $(1 \leqq \lambda \leqq k)$. We have further that these two correspondences coincide: $\lambda=\tilde{\lambda}$. To see this, we note: *Let z be an element of $N-M$ satisfying $e_{\kappa}ze_{\lambda}=z$. Then the two-sided ideal $\mathfrak{z}=AzA \smile M$ contains a two-sided subideal $\mathfrak{z}'=AzN \smile NzA \smile M$. When that is so, $\mathfrak{z}/\mathfrak{z}'$ is an (A, A) module of type (κ, λ) and at the same time it is an $(\tilde{A}, \tilde{A})$ module of type $(\kappa, \tilde{\lambda})$.* In fact: From assuptions it follows immediately that the factor module $\mathfrak{z}/\mathfrak{z}'$ is an (A, A) module of type (κ, λ). As z is not contained in M, we can take a suitable element a of N such that za lies in M and $\mathfrak{z}'a=0$; then the mapping f: $x \to xa$ $(x \in \mathfrak{z})$ is obviously an A-homomorphism of $\mathfrak{z}$ into M; since $f\mathfrak{z}'=0$, $f\mathfrak{z}$ is a direct sum of several left simple A-ideals and satisfies $E_{\kappa}f\mathfrak{z}=f\mathfrak{z}$. But, the mapping f is at the same time an $\tilde{A}$-homomorphism of $\mathfrak{z}$ into M; moreover, we have $\tilde{E}_{\kappa}f\mathfrak{z}=f\mathfrak{z}$ since $E_{\kappa}f\mathfrak{z}=f\mathfrak{z}$. Therefore, $\mathfrak{z}/\mathfrak{z}'$ must be of type $(\kappa, *)$ as an $(\tilde{A}, \tilde{A})$ module; similarly we see that $\mathfrak{z}/\mathfrak{z}'$ is of type $(*, \tilde{\lambda})$ as an $(\tilde{A}, \tilde{A})$ module. This completes our proof. The proof of $\pi(\kappa)=\tilde{\pi}(\kappa)$ made in section 3 can be now applied to our case and so we have $\tilde{\lambda}=\lambda$.

(2) As to the type of simple two-sided factor module in N, we can say in our case as following: If A has a simple two-sided module $\mathfrak{z}_1/\mathfrak{z}_1'$ of type (κ, λ), then $\tilde{A}$ has at least one $\mathfrak{z}_2/\mathfrak{z}_2'$ of the same type; moreover, when that is so, we can choose a common factor module $\mathfrak{z}_0/\mathfrak{z}_0'$ of type (κ, λ) for both A and $\tilde{A}$. From this we see that A and $\tilde{A}$ have corresponding blocks.

(3) To obtain $f(\kappa)=\tilde{f}(\kappa)$ and $\bar{e}_{\kappa}\bar{A}\bar{e}_{\kappa} \cong \bar{\tilde{e}}_{\kappa}\bar{\tilde{A}}\bar{\tilde{e}}_{\kappa}$, we need some additional assumption; viz., if for every κ there exists a simple two-sided A-ideal $\mathfrak{z}$ of type $(\kappa, *)$ or of type $(*, \kappa)$ such that

(a) $\mathfrak{z}$ contains an element of the form xy where x and y are in N and

(b) $\mathfrak{z}$ satisfies the condition required in prop. 2,

then we have $\tilde{f}(\kappa)=f(\kappa)$, $\bar{e}_{\kappa}\bar{A}\bar{e}_{\kappa} \cong \bar{\tilde{e}}_{\kappa}\bar{\tilde{A}}\bar{\tilde{e}}_{\kappa}$. The proof for this fact is parallel to that of theorem 2. (Note that from the above assumption it follows that the condition (b) is satisfied by $\mathfrak{z}$ even when it is regarded as a two-sided $\tilde{A}$-ideal.) We have thus proved the next theorem:

THEOREM 3. *Let both $A \succ \tilde{A}$ and $\tilde{A} \succ A$. Then there exists a* (*unique*) 1-1 *correspondence between simple constituents of* $\bar{A}$ *and those of* $\bar{\tilde{A}}$: $\bar{A}_\kappa \leftrightarrow \bar{\tilde{A}}_{\sigma(\kappa)}$ (*so we set* $\sigma(\kappa)=\kappa$). *Further, when that is so,* A *and* $\tilde{A}$ *have corresponding blocks. If moreover for every* κ *there exists a simple two-sided* A*-ideal of type* $(\kappa, *)$ *or of type* $(*, \kappa)$ *satisfying the above conditions* (a) *and* (b), *then* $\bar{A}_\kappa \cong \bar{\tilde{A}}_\kappa$ $(1 \leqq \kappa \leqq k = \tilde{k})$ *and* $\bar{A} \cong \bar{\tilde{A}}$.

As to the radical N of a quasi-Frobenius algebra A ($M \subseteqq N^2$ being assumed), we have the following

PROPOSITION 14. *Let* A *be a quasi-Frobenius algebra over a field* F *with a radical* N (*and let* $M \subseteqq N^2$). *Then* Λ_N *has a* (*unique*) *maximal class* Γ *and* A *is an element of* Γ; *further, when that is so,* Γ *consists of all quasi-Frobenius algebras with radical* N. *If* A *is moreover a Frobenius* [*weakly symmetric*] *algebra, then every algebra in* Γ *is also a Frobenius* [*weakly symmetric*] *algebra.*

Our assertions follow from prop. 5, from prop. 6, from the well known properties of quasi-Frobenius algebras and from the definitions.

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