ON CONFORMAL MAPPING OF A MULTIPLY-CONNECTED DOMAIN ONTO A CIRCULAR SLIT COVERING SURFACE

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§1. Introduction.

In the present paper we will concern ourselves with conformal mapping of a multiply-connected domain of finite connectivity onto a canonical covering surface whose boundary consists of whole circumferences and circular slits centred at the origin on the basic plane. We will discuss the existence of such a mapping function and its extremality. The purpose of our present investigation is an extension and an improvement of the results obtained in our previous papers [4] and [5].

§2. Preliminaries.

Let B be a multiply-connected domain of finite connectivity on the z-plane. We suppose that each component C_j $(j=1,\cdots,N)$ of its boundary C is a continuum. Let z_0, z_k^0 $(k=1,\cdots,N^0;\ N^0\geqq 0)$ and z_l^∞ $(l=1,\cdots,N^\infty;\ N^\infty\geqq 0)$ be arbitrarily preassigned $N^0+N^\infty+1$ points in B, and positive integers μ_k^0 and μ_l^∞ $(k=1,\cdots,N^0;\ l=1,\cdots,N^\infty)$ be given arbitrarily.[1] Let $\mathfrak{F}$ be the class of analytic functions $w=f(z)$ on B with the following properties:

(a) f has the only zeros z_k^0 $(k=1,\cdots,N^0)$ and the only poles z_l^∞ $(l=1,\cdots,N^\infty)$ with their orders μ_k^0 and μ_l^∞, respectively;[2]

(b) $$w=0,\ \infty\notin\overline{f(B)}-f(B);$$

(c) $$\left|\int_C \lg|f|\,d\arg f\right|<+\infty,$$

where the line integral means $\lim_{n\to\infty}\int_{\partial B_n}\lg|f|\,d\arg f$ with an exhaustion $\{B_n\}$ of B;

(d) $$f(z_0)=1.$$

Let B^* be a subdomain of B whose boundary C^* consists of components C_j^* $(j=1,\cdots,N)$, each being a simple analytic closed curve homotopic to C_j in $B-\sum_{k=1}^{N^0}\{z_k^0\}-\sum_{l=1}^{N^\infty}\{z_l^\infty\}$.[3] We define the *rotation number of the image of* C_j *about* $w=0$ *under* $f\in\mathfrak{F}$ by

$$\nu_j(f)=\frac{1}{2\pi}\int_{C_j^*}d\arg f \qquad (j=1,\cdots,N). \tag{1}$$

Then, it is easily verified by the argument principle that $\nu_j(f)$ $(j=1,\cdots,N)$

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1) Here the case $N^0=0$ or $N^\infty=0$ is permitted.

2) Of course, if $N^0=0$ or $N^\infty=0$, f has no zeros or no poles in B, respectively.

3) Here, in the case $N^0=0$ or $N^\infty=0$, the corresponding summations are taken to be vacuous, and the similar notes should be taken throughout the paper.

are integers not depending on a particular choice of B^*, and satisfy

$$\sum_{j=1}^{N} \nu_j(f) = \sum_{k=1}^{N^0} \mu_k^0 - \sum_{l=1}^{N^\infty} \mu_l^\infty .$$

Conversely, let integers ν_j $(j=1,\cdots,N)$ be given arbitrarily under the condition

$$\sum_{j=1}^{N} \nu_j = \sum_{k=1}^{N^0} \mu_k^0 - \sum_{l=1}^{N^\infty} \mu_l^\infty .$$

Then, there exist functions $f \in \mathfrak{F}$ satisfying $\nu_j(f) = \nu_j$ $(j=1,\cdots,N)$. In fact, it is readily shown that there exists a rational function on the z-plane with the properties, by carrying out, if necessary, a mapping of B onto a domain each boundary component of which separates exterior points of B.

Let $\mathfrak{t}$ be a closed interval $0 \leqq t \leqq 1$. Let the two functions $f_0 \in \mathfrak{F}$ and $f_1 \in \mathfrak{F}$ satisfy the following conditions:

(α) there exists a continuous mapping $w = f(z, t)$ of the topological product $B \times \mathfrak{t}$ into the w-plane such that

$$f(z, 0) = f_0(z), \qquad f(z, 1) = f_1(z);$$

(β) $f(z, t) \in \mathfrak{F}$ for each $t \in \mathfrak{t}$.

Then, we call that f_1 *is homotopic to* f_0 and denote it by $f_0 \sim f_1$. The homotopy relation is obviously an equivalence relation in $\mathfrak{F}$, and thus $\mathfrak{F}$ is divided into classes which are called *homotopy classes*.

LEMMA. *Let* $f_0 \in \mathfrak{F}$, $f_1 \in \mathfrak{F}$. *Then,* $f_0 \sim f_1$ *if and only if* $\nu_j(f_0) = \nu_j(f_1)$ $(j=1,\cdots,N)$.

Proof.[4] Let $f_0 \sim f_1$. Then, f_0 and f_1 satisfy the conditions (α) and (β). We consider

$$\rho_j(t) \equiv \nu_j(f(z, t)) = \frac{1}{2\pi i} \int_{C_j^*} d \arg f(z, t) \qquad (j=1,\cdots,N).$$

Noting to the property (b), we can easily see that each $\rho_j(t)$ is a continuous function in the closed interval $\mathfrak{t}$. However $\rho_j(t)$ takes only integral values. Thus $\rho_j(t) \equiv$ const and especially $\rho_j(0) = \rho_j(1)$. Therefore $\nu_j(f_0) = \nu_j(f_1)$ $(j=1,\cdots,N)$.

Conversely, let $\nu_j(f_0) = \nu_j(f_1)$ $(j=1,\cdots,N)$. We construct a function

$$f(z, t) \equiv \exp\{t(\lg f_1 - \lg f_0) + \lg f_0\}$$

from the both functions f_0 and f_1. Then, it is immediately verified that $f(z, t)$ is a desired mapping which provides for $f_0 \sim f_1$. q. e. d.

§ 3. Theorem.

Let $\mathfrak{H}$ be an arbitrary homotopy class of $\mathfrak{F}$, and let

$$J(f) = \int_C \lg |f| \, d \arg f - 2\pi \sum_{k=1}^{N^0} \mu_k^0 \lg |\mathfrak{f}_k^{0\prime}(0)| - 2\pi \sum_{l=1}^{N^\infty} \mu_l^\infty \lg |\mathfrak{f}_l^{\infty\prime}(0)| \text{[5]}$$

4) Cf. [3].

5) This functional is an extension of one to the present case which Sario introduced in [7].

for $f \in \mathfrak{H}$, where

$$\mathfrak{f}_k^0(\zeta) \equiv f(\zeta^{1/\mu_k^0} + z_k^0) \qquad (k=1,\cdots, N^0),$$

$$\mathfrak{f}_l^\infty(\zeta) \equiv 1/f(\zeta^{1/\mu_l^\infty} + z_l^\infty) \qquad (l=1,\cdots, N^\infty).$$

THEOREM. *There exists a unique element Φ in each homotopy class $\mathfrak{H}$ which minimizes $J(f)$ on $\mathfrak{H}$. Further Φ is the unique element of $\mathfrak{H}$ which maps B onto one of the finitely-sheeted covering surface whose boundary consists of whole circumferences and circular slits centred at the origin on the basic w-plane.*

Proof.[6] We select an arbitrary and fixed element f of $\mathfrak{H}$. Then $\Omega = \lg|f|$ is a potential function on B which is harmonic except for logarithmic singularities with principal parts

$$\mu_k^0 \lg|z - z_k^0|, \qquad -\mu_l^\infty \lg|z - z_l^\infty|$$

at z_k^0, z_l^∞ $(k=1,\cdots, N^0; l=1,\cdots, N^\infty)$, respectively. By (c) and (d), it satisfies

$$\left|\int_C \Omega \frac{\partial \Omega}{\partial n} ds\right| < +\infty, \tag{2}$$

and

$$\Omega(z_0) = 0, \tag{3}$$

respectively, where $\partial/\partial n$ denotes the differentiation along inner normal and ds the line element. And further, by (1), we have

$$\frac{1}{2\pi}\int_{C_j^*} \frac{\partial \Omega}{\partial n} ds = \nu_j(f) \qquad (j=1,\cdots, N). \tag{4}$$

Let $\mathfrak{A}$ be the class of potential functions u which are harmonic on B except for the same logarithmic singularities as Ω at z_k^0, z_l^∞ $(k=1,\cdots, N^0; l=1,\cdots, N^\infty)$, take a constant boundary value on each boundary component of B, and satisfy

$$u(z_0) = 0. \tag{5}$$

Then, it is readily verified by (2) that

$$D_B(\Omega - u) < +\infty \qquad \text{for } u \in \mathfrak{A}.$$

Let $\mathfrak{B}$ be the class of non-constant harmonic functions h on B which have one-valued conjugate harmonic functions and satisfy

$$D_B(h) < +\infty$$

and

$$h(z_0) = 0.$$

(i) *Let h^* be a non-constant harmonic function on B which satisfies the conditions*

$$D_B(h^*) < +\infty, \qquad h^*(z_0) = 0.$$

If there exists a constant c not depending on $u \in \mathfrak{A}$ such that

6) We shall prove the theorem except the case $N^0 = N^\infty = 0$; the exceptional case we can be more easily dealt with in a similar method (cf. [4]).

$$D_B(u, h^*) = c, \tag{6}$$

then we have

$$c = 0 \quad \textit{and} \quad h^* \in \mathfrak{B}.$$

Let

$$-\int_{C_j^*} \frac{\partial h^*}{\partial n} ds = \alpha_j \qquad (j = 1, \cdots, N). \tag{7}$$

Since h^* is harmonic on B, we have

$$\sum_{j=1}^{N} \alpha_j = 0. \tag{8}$$

Let $g(z, z')$ be the Green's function of B with the pole z'. We can take a sufficiently small positive number δ for any given positive number ε such that each component of

$$C^\delta = \{z \mid g(z, z') = \delta\}$$

is a simple analytic closed curve homotopic to C_j $(j = 1, \cdots, N)$, respectively, and

$$|D_{B-B^\delta}(g(z, z'), h^*)| < \varepsilon, \tag{9}$$

where

$$B^\delta = \{z \mid g(z, z') > \delta\}.$$

Using the Green's formula, (7) and (8), and noting that

$$\lim_{r \to 0} \int_{|z-z'|=r} g \frac{\partial h^*}{\partial n} ds = 0,$$

we have

$$D_{B^\delta}(g(z, z'), h^*) = -\int_{C^\delta} g \frac{\partial h^*}{\partial n} ds = -\delta \int_{C^\delta} \frac{\partial h^*}{\partial n} ds = \delta \sum_{j=1}^{N} \alpha_j = 0. \tag{10}$$

By (9) and (10), we have

$$|D_B(g(z, z'), h^*)| \leqq |D_{B^\delta}(g(z, z'), h^*)| + |D_{B-B^\delta}(g(z, z'), h^*)| < \varepsilon.$$

Since ε is an arbitrary positive number, it follows that

$$D_B(g(z, z'), h^*) = 0. \tag{11}$$

Let ω_j $(j = 1, \cdots, N)$ be the harmonic measure of C_j with respect to B, respectively. We can take a sufficiently small positive number δ for any given positive number ε such that

$$C_j^\delta = \{z \mid \omega_j = 1 - \delta\} \quad \text{with} \quad \delta < \frac{\varepsilon}{4|\alpha_j|} \tag{12}$$

is a simple analytic closed curve homotopic to C_j and each component of $C_j^{\delta\prime} = \{z \mid \omega_j = \delta\}$ is a one homotopic component of $C - C_j$, respectively, and

$$|D_{B-B_j^\delta}(\omega_j, h^*)| < \frac{\varepsilon}{2}, \tag{13}$$

where

$$B_j^\delta = \{z \mid \delta < \omega_j < 1-\delta\} \qquad (j=1,\cdots,N).$$

Using the Green's formula, (7) and (8), we have

$$(14) \qquad \begin{aligned} D_{B_j^\delta}(\omega_j, h^*) &= -(1-\delta)\int_{C_j^\delta} \frac{\partial h^*}{\partial n} ds - \delta \int_{C_j^{\delta'}} \frac{\partial h^*}{\partial n} ds \\ &= (1-\delta)\alpha_j + \delta \sum_{\iota \neq j} \alpha_\iota = (1-2\delta)\alpha_j \end{aligned} \qquad (j=1,\cdots,N).$$

By (12), (13) and (14), we have

$$\begin{aligned} |D_B(\omega_j, h^*) - \alpha_j| &\leqq |D_{B_j^\delta}(\omega_j, h^*) - \alpha_j| + |D_{B-B_j^\delta}(\omega_j, h^*)| \\ &< 2\delta|\alpha_j| + \frac{\varepsilon}{2} < \varepsilon \end{aligned} \qquad (j=1,\cdots,N).$$

Since ε is an arbitrary positive number, it follows that

$$(15) \qquad D_B(\omega_j, h^*) = \alpha_j \qquad (j=1,\cdots,N).$$

Now let

$$u_0 \equiv \sum_{k=1}^{N^0} \mu_k^0 g(z, z_k^0) - \sum_{l=1}^{N^\infty} \mu_l^\infty g(z, z_l^\infty) + \gamma,$$

where

$$\gamma = -\sum_{k=1}^{N^0} \mu_k^0 g(z_0, z_k^0) + \sum_{l=1}^{N^\infty} \mu_l^\infty g(z_0, z_l^\infty).$$

Obviously $u_0 \in \mathfrak{A}$. Thus, by the assumption (6), we have

$$D_B(u_0, h^*) = c.$$

On the other hand, by (11), we have

$$(16) \qquad D_B(u_0, h^*) = 0.$$

Thus we obtain $c=0$.

Let

$$u_j \equiv u_0 + \omega_j(z) - \omega_j(z_0) \qquad (j=1,\cdots,N).$$

Obviously $u_j \in \mathfrak{A}$ $(j=1,\cdots,N)$. By (15) and (16), we have

$$D_B(u_j, h^*) = \alpha_j \qquad (j=1,\cdots,N).$$

Thus, by the assumption (6) and $c=0$, we obtain

$$\alpha_j = 0 \qquad (j=1,\cdots,N).$$

This fact means that a conjugate function of h^* is one-valued and thus $h^* \in \mathfrak{B}$.

(ii) *If $\{u^n\}_{n=1}^\infty$ is a sequence of elements of $\mathfrak{A}$ and for any positive number ε there exists an integer n_0 such that*

$$D_B(u^m - u^n) < \varepsilon \qquad \textit{for } m > n_0,\ n > n_0,$$

then $\{u^n\}_{n=1}^\infty$ converges uniformly in the wider sense to an element u of $\mathfrak{A}$ on B.

(iii) *There exists an element U of $\mathfrak{A}$ which minimizes $D_B(\Omega - u)$ among all $u \in \mathfrak{A}$.*

If the functions in question are free of singularity, (ii) and (iii) will be verified by a well known method. In spite of the existence of singularities, this way of proof is valid to the present case; we omit the proof in detail.

(iv) $$\Omega - U \in \mathfrak{B} \quad or \quad \Omega \equiv U.$$

Let

(17) $$d = D_B(\Omega - U) = \min_{u \in \mathfrak{A}} D_B(\Omega - u).$$

For any $u \in \mathfrak{A}$,

$$\frac{U + \lambda u}{1 + \lambda} \in \mathfrak{A}$$

holds for any real λ with $0 < |\lambda| < 1$. Then, by (17), we have

$$D_B\left(\Omega - \frac{U + \lambda u}{1 + \lambda}\right) = D_B\left((\Omega - U) - \frac{\lambda}{1 + \lambda}(u - U)\right) \geqq d.$$

Using (17) again, we get

$$2\frac{\lambda}{1 + \lambda} D_B(\Omega - U, u - U) \leqq \left(\frac{\lambda}{1 + \lambda}\right)^2 D_B(u - U).$$

For any λ with the same sign as $D_B(\Omega - U, u - U)$, we have

$$2\,|D_B(\Omega - U, u - U)| < \left|\frac{\lambda}{1 + \lambda}\right| D_B(u - U).$$

Since $|\lambda|$ can be chosen arbitrarily small for a fixed u, it follows that

$$D_B(\Omega - U, u) = D_B(\Omega - U, U).$$

Since u is an arbitrary element of $\mathfrak{A}$, we obtain (iv) by (i).

Let V be the potential function conjugate to U and its additive constant be determined by the condition

(18) $$\text{a branch of } V(z_0) = 0.$$

We shall show that the analytic function

$$\Phi = \exp(U + iV)$$

is a desired mapping function.

It is obvious that Φ has the zeros z_k^0 $(k = 1, \cdots, N^0)$ and the poles z_l^∞ $(l = 1, \cdots, N^\infty)$ with their orders μ_k^0 and μ_l^∞, respectively. Since $\Omega - U \in \mathfrak{B}$ or $U \equiv \Omega$, we have by (4)

$$\frac{1}{2\pi}\int_{C_j^*} d \arg \Phi = \frac{1}{2\pi}\int_{C_j^*} \frac{\partial U}{\partial n} ds = \frac{1}{2\pi}\int_{C_j^*} \frac{\partial \Omega}{\partial n} ds = \nu_j(f) \quad (j = 1, \cdots, N),$$

and see that Φ is one-valued. By (5) for $u = U$ and (18), we get

$$\Phi(z_0) = 1.$$

Thus, noting the Lemma, we obtain that $\Phi \in \mathfrak{H}$. Further, since U takes a constant boundary value on each boundary component of B, Φ maps B onto a covering surface whose boundary consists of whole circumferences and circular slits centred at the origin on the basic w-plane.

Next we shall show that Φ is the unique extremal function of $\mathfrak{H}$. Let f be an arbitrary element of $\mathfrak{H}$ and let

$$B_r = B - \sum_{k=1}^{N^0} \{ |z - z_k^0| \leqq r \} - \sum_{l=1}^{N^\infty} \{ |z - z_l^\infty| \leqq r \},$$

where r should be chosen suitably sufficiently small. Then, the image curves of $\{|z - z_k^0| = r\}$ $(k = 1, \cdots, N^0)$ and $\{|z - z_l^\infty| = r\}$ $(l = 1, \cdots, N^\infty)$ under f surrounds about $w = 0$ μ_k^0-times $(k = 1, \cdots, N^0)$ and μ_l^∞-times $(l = 1, \cdots, N^\infty)$, respectively, and lies between circumferences

$$|w| = r^{\mu_k^0} |\mathfrak{f}_k^{0\prime}(0)|(1 + \delta(r)) \text{ and } |w| = r^{\mu_k^0} |\mathfrak{f}_k^{0\prime}(0)|(1 - \delta(r)) \qquad (k = 1, \cdots, N^0),$$

and

$$|w| = \frac{1}{r^{\mu_l^\infty} |\mathfrak{f}_l^{\infty\prime}(0)|}(1 + \delta(r)) \text{ and } |w| = \frac{1}{r^{\mu_l^\infty} |\mathfrak{f}_l^{\infty\prime}(0)|}(1 - \delta(r)) \qquad (l = 1, \cdots, N^\infty),$$

respectively, where the positive number $\delta(r)$ does not depend on $f \in \mathfrak{H}$ and

$$\lim_{r \to 0} \delta(r) = 0.$$

Therefore, using the Green's formula, we have

$$\begin{aligned} J(f) &= D_{B_r}(\lg|f|) + \sum_{k=1}^{N^0} \int_{|z - z_k^0| = r} \lg|f|\, d\arg f + \sum_{l=1}^{N^\infty} \int_{|z - z_l^\infty| = r} \lg|f|\, d\arg f \\ &\quad - 2\pi \sum_{k=1}^{N^0} \mu_k^0 \lg|\mathfrak{f}_k^{0\prime}(0)| - 2\pi \sum_{l=1}^{N^\infty} \mu_l^\infty \lg|\mathfrak{f}_l^{\infty\prime}(0)| \\ &= D_{B_r}(\lg|f|) + 2\pi \sum_{k=1}^{N^0} \mu_k^0 \lg|r^{\mu_k^0} \mathfrak{f}_k^{0\prime}(0)| + 2\pi \sum_{l=1}^{N^\infty} \mu_l^\infty \lg|r^{\mu_l^\infty} \mathfrak{f}_l^{\infty\prime}(0)| \\ &\quad - 2\pi \sum_{k=1}^{N^0} \mu_k^0 \lg|\mathfrak{f}_k^{0\prime}(0)| - 2\pi \sum_{l=1}^{N^\infty} \mu_l^\infty \lg|\mathfrak{f}_l^{\infty\prime}(0)| + O(\delta(r)) \\ &= D_{B_r}(\lg|f|) + 2\pi \sum_{k=1}^{N^0} \mu_k^{02} \lg r + 2\pi \sum_{l=1}^{N^\infty} \mu_l^{\infty 2} \lg r + O(\delta(r)). \end{aligned}$$

Thus, we have

$$\begin{aligned} J(f) - J(\Phi) &= D_{B_r}(\lg|f|) - D_{B_r}(\lg|\Phi|) + O(\delta(r)) \\ &= D_{B_r}(\Omega) - D_{B_r}(U) + O(\delta(r)) \\ &= 2 D_{B_r}(U, \Omega - U) + D_{B_r}(\Omega - U) + O(\delta(r)), \end{aligned}$$

which yields, by $r \to 0$,

$$J(f) - J(\Phi) = 2 D_B(U, \Omega - U) + D_B(\Omega - U).$$

Noting that $U \in \mathfrak{A}$, $\Omega - U \in \mathfrak{B}$ and using a similar reasoning as in the proof of (i), we have

$$D_B(U, \Omega - U) = 0$$

and thus

$$J(f) - J(\Phi) = D_B(\Omega - U) \geqq 0.$$

Here, in virtue of the normalizing condition at z_0, we see that the equality in the last inequality holds if and only if $\Omega \equiv U$ or $f \equiv \Phi$.

It remains only to show that Φ is the unique element of $\mathfrak{H}$ which maps B onto one of the covering surfaces whose boundary consists of whole circumferences and circular slits centred at the origin on the basic w-plane. For this purpose, let Φ^* be another element of $\mathfrak{H}$ which gives such a canonical mapping. Then we easily see that Φ^* also must have the same extremality as Φ. Thus we have

$$J(\Phi^*) = J(\Phi)$$

and thus $\Phi^* \equiv \Phi$. q. e. d.

§4. Remarks.

We shall enumerate here the types of the extremal functions Φ for some special homotopy classes $\mathfrak{H}$.

1. The case $N^0 = N^\infty = 0$, $\nu_j \neq 0$ for some j. Φ maps B onto a covering surface of annular type cut along circular slits centred at the origin (cf. Theorem 2 in [4]).

2. The case $N^0 \geqq 1$, $N^\infty = 0$. Φ maps B onto a covering surface of circular type cut along circular slits centred at the origin (cf. Theorem in [5]).

3. The case $\sum_{k=1}^{N^0} \mu_k^0 = \sum_{l=1}^{N^\infty} \mu_l^\infty = P \geqq 1$, $\nu_j = 0$ $(j = 1, \cdots, N)$. Φ maps B onto an exactly P-sheeted covering surface over the entire w-plane cut along circular slits centred at the origin (cf. [1], [6] for the case $P = 1$).

4. The case $N^0 = N^\infty = 0$, $\nu_1 = 1$, $\nu_2 = -1$, $\nu_j = 0$ $(j = 3, \cdots, N)$. Φ maps B onto a schlicht circular slit annulus (cf. [2], [6]).

5. The case $N^0 = 1$, $N^\infty = 0$, $\nu_1 = 1$, $\nu_j = 0$ $(j = 2, \cdots, N)$. Φ maps B onto a schlicht circular slit disk (cf. [2], [6]).

6. The case $N^0 = N^\infty = 0$, $\nu_j = 0$ $(j = 1, \cdots, N)$. Φ must degenerate to $\Phi \equiv 1$.

These are easily verified by the argument principle.

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