A SUPPLEMENT TO "ON PFLUGER'S SUFFICIENT CONDITION FOR A SET TO BE OF CLASS $N_{\mathfrak{B}}$"

By Mitsuru Ozawa

1. In the present note we shall supplement several remarks to our previous paper [4], in which we established the following theorem:

If the condition (A) and

$$\limsup_{n\to\infty}\left(\alpha\sum_{j=1}^{n}\log\mu_j-\left(1-\frac{\alpha}{2}\right)\log\nu(n)\right)=+\infty \tag{1}$$

hold for an α $(0<\alpha\leqq 2)$ and for a suitable exhaustion $\{D_n\}$ of D, then the α-dimensional measure $m_\alpha(E)$ is equal to zero.

The condition (A) means that there is a positive number δ such that for any n and k there holds an inequality

$$\delta\cdot l_{2n}^{(k)^2}\leqq A_{2n}^{(k)} \qquad \text{or} \qquad \delta\cdot d_{2n}^{(k)^2}\leqq A_{2n}^{(k)}.$$

In the present note we make use of the same notations as those in the previous paper. It is plausible to conjecture that the condition (A) can be excluded. Although we cannot yet settle this problem completely up to now, our results will offer some informations for Painlevé problem.

2. Let R be a ring domain bounded by two rectifiable Jordan curves C_1, the outer boundary, and C_2, the inner boundary. Further we set that A_i is the area of the domain T_i bounded by C_i. Let l' be the minimum length with respect to the Euclidean metric in the closed curve family in R separating C_1 and C_2.

Lemma 1. *The modulus* $\operatorname{mod} R$ *of* R *satisfies an inequality*

$$\frac{1}{2\pi}\operatorname{mod} R\leqq\frac{A_1-A_2}{l'^2}.$$

Proof. It is well known that the extremal length $\lambda(\Gamma)$ of the family of all closed curves γ separating C_1 and C_2 in R is equal to $2\pi/\operatorname{mod} R$. On the other hand, we have

$$\lambda(\Gamma)\equiv\sup_{\rho}\frac{\left(\inf_{\gamma}\int_{\gamma}\rho ds\right)^2}{\iint_R\rho^2dxdy}\geqq\frac{l'^2}{A_1-A_2}.$$

This shows the desired result.

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LEMMA 2. (Golusin [1].) *Under the same assumption as in lemma* 1, *we have*

$$2 \bmod R \leqq \log A_1 - \log A_2.$$

LEMMA 3. *If the least length l' satisfies an inequality*

$$al' \geqq 2\sqrt{\pi}(A_1 - A_2)^{1/2}$$

for a constant a $(0 < a < 1)$, then there holds an inequality

$$\exp \bmod R \leqq 1 + \frac{1}{1-a} \, \frac{4\sqrt{\pi}(A_1 - A_2)^{1/2}}{l'}.$$

Proof. Let $\log(1/r)$ be the modulus of R, then there exists a regular function $f(z)$ which maps conformally an annulus $r < |z| < 1$ onto R. Let $f(z)$ have the local expansion $f(z) = \sum_{n=1}^{\infty} c_n (z - z_0)^n$ at $z_0 = (1 + r)e^{i\theta}/2$, then we have

$$\begin{aligned} |f'(z_0)|^2 = |c_1|^2 &\leqq \sum_{n=1}^{\infty} n |c_n|^2 \left(\frac{1-r}{2}\right)^{2n-2} \\ &= \frac{4\pi^{-1}}{(1-r)^2} \iint\limits_{|z-z_0|<(1-r)/2} |f'(z)|^2 dxdy \\ &\leqq \frac{4\pi^{-1}}{(1-r)^2} \iint\limits_{r<|z|<1} |f'(z)|^2 dxdy \\ &= (A_1 - A_2) \frac{4\pi^{-1}}{(1-r)^2}. \end{aligned}$$

Thus we have

$$l' \leqq \int_0^{2\pi} |f'(z_0)| \frac{1+r}{2} d\theta \leqq \sqrt{A_1 - A_2} \frac{1+r}{1-r} \cdot 2\pi \frac{1}{\sqrt{\pi}},$$

which implies the desired result under our assumption.

THEOREM 1. *If there holds*

$$\limsup_{n\to\infty} \left(\frac{\alpha}{2} \log\log \mu_n + \alpha \sum_{j=1}^{n-1} \log \mu_j - \left(1 - \frac{\alpha}{2}\right) \log \nu(n) \right) = +\infty \tag{2}$$

for an α $(0 < \alpha \leqq 2)$ and for a suitable exhaustion $\{D_n\}$, then the α-dimensional measure of E is equal to zero.

Proof. Let $l_n^{(k)\prime}$ be the minimum Euclidean length in the closed curve family in $R_n^{(k)}$ separating $C_{1n}^{(k)}$ and $C_{2n}^{(k)}$. By lemma 1 we have

$$\left(\frac{1}{2\pi} \log \mu_n\right)^{\alpha/2} \leqq \frac{\sum_{k=1}^{\nu(n)} (A_{1n}^{(k)})^{\alpha/2}}{\sum_{k=1}^{\nu(n)} (l_n^{(k)\prime})^\alpha}$$

which implies

$$\left(\frac{1}{2\pi}\right)^{\alpha/2} (\log \mu_n)^{\alpha/2} \sum_{k=1}^{\nu(n)} (l_n^{(k)\prime})^\alpha \leqq \left(\sum_{k=1}^{\nu(n)} A_{1n}^{(k)}\right)^{\alpha/2} \nu(n)^{1-\alpha/2}$$

by Hölder inequality. By lemma 2 we can apply the same reasoning as in [4], and we have finally

$$\left(\frac{1}{2\pi}\right)^{\alpha/2}(\log\mu_n)^{\alpha/2}\prod_{j=1}^{n-1}\mu_j^{\alpha}\sum_{k=1}^{\nu(n)}(l_n^{(k)'})^{\alpha}\leqq M\cdot\nu(n)^{1-\alpha/2}.$$

This implies the desired result.

COROLLARY 1. *If there exists an infinite number of indices n_p such that μ_{n_p} satisfies an inequality*

$$\text{(B)}\qquad 1+\varepsilon\leqq\mu_{n_p}$$

for any positive number ε and for any n_p, then the existence of an exhaustion satisfying the condition

$$(3)\qquad \lim_{p\to\infty}\left(\alpha\sum_{j=1}^{n_p}\log\mu_j-\left(1-\frac{\alpha}{2}\right)\log\nu(n_p)\right)=+\infty$$

implies $m_\alpha(E)=0$.

Proof. If μ_{n_p} is bounded from above, then the condition (3) is equivalent to the condition (2). If μ_{n_p} is not bounded, then $D=E^c$ belongs to the class O_G and hence E is of logarithmic capacity zero. This implies that $m_\alpha(E)=0$ for any α with $0<\alpha\leqq 2$.

COROLLARY 2. *If the order of $\log\nu(n)$ is not less than n and if (3) holds for an exhaustion and for an α $(0<\alpha\leqq 2)$, then $m_\alpha(E)=0$.*

Proof. By the previous corollary 1, we may assume that μ_n tends to 1. Then the order of $\sum_{n=1}^{m}\log\mu_n$ is less than n, which contradicts our assumptions.

Corollary 2 is due to a remark given by K. Hayashi.

THEOREM 2. *If there exists an infinite number of indices n_p for which an inequality*

$$a_{n_p}l_{n_p}^{(k)'}\geqq 2\sqrt{\pi}\,(A_{1n_p}^{(k)}-A_{2n_p}^{(k)})^{1/2}$$

remains valid for any k $(1\leqq k\leqq\nu(n_p))$ and for a constant a_{n_p} $(0<a_{n_p}<1)$ independent of k, then the existence of an exhaustion $\{D_n\}$ satisfying the condition

$$(*)\qquad \lim_{p\to\infty}\left(\alpha\log(\mu_{n_p}-1)+\alpha\log(1-a_{n_p})+\alpha\sum_{j=1}^{n_p-1}\log\mu_j-\left(1-\frac{\alpha}{2}\right)\log\nu(n_p)\right)=+\infty$$

implies $m_\alpha(E)=0$. If there exists a constant a such that $a_{n_p}\leqq a<1$ for any n_p, then we can exclude the term involving a_{n_p} in (*).

Proof. Lemma 3 is used instead of lemma 1.

3. Further in [4] we established the following result: Let μ_n' be the minimum of $\mu_n^{(k)'}$ $(1\leqq k\leqq\nu(n))$, where $\mu_n^{(k)'^2}$ is the ratio of two areas $A_{1n}^{(k)}$ and and $A_{2n}^{(k)}$. If

$$(4)\qquad \limsup_{n\to\infty}\left(\alpha\sum_{j=1}^{n}\log\mu_j'-\left(1-\frac{\alpha}{2}\right)\log\nu(n)\right)=+\infty$$

holds for an exhaustion $\{D_n\}$ and the condition (A) remains valid, then $m_\alpha(E) = 0$.

Let t_n be the maximum of $t_n^{(l)}$, which is defined by an inequality

$$t_n^{(l)} A_{2,n-1}^{(l)} \geqq \sum A_{1n}^{(k)},$$

where the summation is taken over all the $C_{1n}^{(k)}$ contained in the domain bounded by $C_{2,n-1}^{(l)}$. Evidently we may put $t_n \leqq 1$. Then we have

THEOREM 3. *If the condition* (A) *and*

$$(5) \qquad \limsup_{n\to\infty}\left(\alpha \sum_{j=1}^{n} \log \mu_j' - \left(1 - \frac{\alpha}{2}\right) \log \nu(n) - \frac{\alpha}{2} \sum_{j=2}^{n} \log t_j\right) = +\infty$$

hold for an α $(0 < \alpha \leqq 2)$, *then*

$$\liminf_{n\to\infty} \sum_{k=1}^{\nu(n)} (d_{2n}^{(k)})^\alpha = 0$$

and hence $m_\alpha(E) = 0$.

Evidently (4) implies (5). Next we enter into an inverse discourse. We assume that all the $A_{1n}^{(k)}$ are equal to A_n. This assumption is denoted by (C) and is really satisfied by a symmetric product set of a general linear Cantor set. Let s_n be the minimum of $s_n^{(l)}$ defined by $A_{2,n-1}^{(l)} s_n^{(l)} \leqq \sum A_{1n}^{(k)}$, where the summation is taken over all the k such that $C_{1n}^{(k)}$ is contained in the domain bounded by $C_{2,n-1}^{(l)}$. By the isoperimetric inequality we have $A_{2n}^{(k)} \leqq \pi d_{2n}^{(k)2}/4$, where $d_{2n}^{(k)}$ is the diameter of $C_{2n}^{(k)}$. Let $\hat{\mu}_j$ be the maximum of all the $\mu_j^{(k)\prime}$.

THEOREM 4. *If*

$$\liminf_{n\to\infty} \sum_{k=1}^{\nu(n)} (d_{2n}^{(k)})^\alpha = 0$$

holds under our assumption (C) *for an exhaustion* $\{D_n\}$, *then we have*

$$\limsup_{n\to\infty}\left(\alpha \sum_{j=1}^{n} \log \hat{\mu}_j - \left(1 - \frac{\alpha}{2}\right) \log \nu(n) - \frac{\alpha}{2} \sum_{j=2}^{n} \log s_j\right) = +\infty.$$

This theorem 4 implies the following fact.

COROLLARY 2. *Let E be a two-dimensional closed set constructed as a symmetric product set of a general linear Cantor set in Ohtsuka's sense* [3], *then the condition* (4) *is a perfect condition in order that*

$$\liminf_{n\to\infty} \sum_{k=1}^{\nu(n)} (d_{2n}^{(k)})^\alpha = 0.$$

In fact, in the above case the condition (A) is satisfied, since all the $C_{2n}^{(k)}$ are squares, and s_n and t_n are equal to 1. The condition (4) is quite similar to a perfect condition for a symmetric product set of a general linear Cantor set to be of α-capacity zero, which is due to Ohtsuka [3].

4. We show that the condition (A) cannot be excluded in theorem 3 for

$\alpha < 2$. Let I_x be the unit interval $[0, 1]$ on the x-axis and I_{1x}^i, $i = 1, 2$, two subintervals of I_x such that they are disjoint and have the same length $(1/2)^\beta$ and the remaing set $I_x - I_{1x}^1 \cup I_{1x}^2$ is an interval, where β is determined later. In each $I_{n-1,x}^j$, we consider two subintervals I_{nx}^k, $k = 1, 2$, such that they are disjoint and have the same length $1/2^{\beta n}$ and the remaining set in an interval. We continue this process indefinitely and then have a general linear Cantor set $C_{\beta x}$. Similarly we construct I_{ny}^k and a general linear Cantor set $C_{\gamma y}$ on the y-axis. Let E be the product set $C_{\beta x} \times C_{\gamma y}$. Then we have

$$\mu_n' = \mu_n^{(k)\prime} = 2^{\beta + \gamma - 2}, \qquad \nu(n) = 2^n \times 2^n, \qquad t_n = 1$$

and

$$\left(\frac{1}{2^{\beta n}}\right)^\alpha \nu(n) = 2^{(2 - \beta\alpha) n}.$$

If $\gamma > \beta$, then the diameter of $I_{nx}^k \times I_{ny}^l$ is equal to $(1 + 2^{(\beta - \gamma) n})^{1/2} / 2^{\beta n}$ and the α-dimensional measure of the nth step is equal to

$$\left(\frac{1}{2^{\beta n}}\right)^\alpha (1 + 2^{n(\beta - \gamma)})^{\alpha/2} \cdot 2^{2n}$$

except a constant positive factor and this tends to 1 if $\beta\alpha = 2$ as n tends to ∞, that is,

$$\lim_{n \to \infty} \sum_{k=1}^{\nu(n)} (d_{2n}^{(k)})^\alpha > 0. \tag{6}$$

In this case, we have

$$\alpha \sum_{j=1}^{n} \log \mu_j' - \left(1 - \frac{\alpha}{2}\right) \log \nu(n) = \frac{2n}{\beta} (\gamma - 1) \log 2.$$

Therefore, if $\beta = 2/\alpha$ and $\gamma > \beta > 1$, then (4) remains valid, while (6) also holds. Evidently the condition (A) does not hold.

References

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Department of Mathematics,
Tokyo Institute of Technology.