ON FREDHOLM EIGEN VALUE PROBLEM FOR PLANE DOMAINS

BY NOBUYUKI SUITA

1. Fredholm eigen value problem was mostly investigated in the case of basic domains bounded by a finite number of sufficiently smooth curves ([2], [4], [5]). Recently Ozawa dealt with the problem for a general domain ([3]). In the present paper we shall derive some properties of basic domains with special eigen values.

2. We now explain the notations. Let D be a planar domain and $L^2(D)$ a Hilbert space consisting of all analytic functions with finite norm and single-valued indefinite integral. Here the inner product and norm are defined by

$$(f, g) = \iint_D f\bar{g}d\tau \qquad \text{and} \qquad \| f \| = \sqrt{(f, f)},$$

respectively. There exist Bergman's kernel function $K(z, \bar{\zeta})$ and its adjoint $l(z, \zeta)$ in $L^2(D)$. If $L^2(D) = \phi$, both kernels are defined to be identically equal to zero. It is known that they have the properties enumerated as follows:

(i) $K(z, \bar{\zeta}) = \overline{K(\zeta, \bar{z})}, \qquad l(z, \zeta) = l(\zeta, z),$

(ii) $\displaystyle\iint_D K(z, \bar{\zeta})f(\zeta)d\tau_\zeta = f(z) \qquad$ for any $f \in L^2(D)$,

(iii) $\displaystyle\iint_D l(z, t)\overline{l(t, \zeta)}d\tau_t = K(z, \bar{\zeta}) - \Gamma(z, \bar{\zeta}),$

and

(iv) $K(z, \bar{\zeta}) - \Gamma(z, \bar{\zeta})$ and $\Gamma(z, \bar{\zeta})$ are positive definite, where $\Gamma(z, \bar{\zeta})$ is defined by the integral

$$\Gamma(z, \bar{\zeta}) = \frac{1}{\pi^2} \iint_{D^c} \frac{d\tau_t}{(t-z)^2\overline{(t-\zeta)^2}}$$

and belongs to $L^2(D)$ ([3]).

Moreover they are closely connnected with canonical slit mapping functions. In fact, let

$$p(z, \zeta) = \frac{1}{z-\zeta} + \text{regular part} \tag{1}$$

and

$$q(z, \zeta) = \frac{1}{z-\zeta} + \text{regular part} \tag{2}$$

Received October 22, 1960.

be univalent in D and map the domain D onto the whole plane slit along straight segments parallel to the real and imaginary axes, respectively. We remark that, in general, $p(z,\zeta)$ and $q(z,\zeta)$ are the limiting functions of the sequences $p_n(z,\zeta)$ and $q_n(z,\zeta)$ corresponding to the n-th exhaustion D_n of D in the usual sense. Then, in terms of these mapping functions, the kernel functions are represented by the formulas

$$K(z,\bar{\zeta})=\frac{1}{2\pi}(p'(z,\zeta)-q'(z,\zeta)) \tag{3}$$

and

$$l(z,\zeta)=\frac{1}{2\pi}\left(p'(z,\zeta)+q'(z,\zeta)+\frac{1}{(z-\zeta)^2}\right). \tag{4}$$

Now we consider the integral equation

$$f(z)=\lambda\iint_D(K(z,\bar{\zeta})-\Gamma(z,\bar{\zeta}))f(\zeta)d\tau_\zeta, \tag{5}$$

or

$$Hf=\rho f(z) \quad \text{with } \rho=1/\lambda \tag{5'}$$

where Hf is an integral transform of the right hand side of (5) except the factor λ. The notions of spectrum, eigen value and eigen function are defined in the usual manner as in the operator theory. The problem of seeking the eigen values of the integral equation (5) is called the Fredholm eigen value problem for D. The operator H is hermitian self-adjoint, positive and half-bounded ([3]). Moreover, it is bounded, that is $\|H\|\leqq 1$. To show this, we set

$$Tf(z)=\iint_D l(z,\zeta)\overline{f(\zeta)}d\tau_\zeta. \tag{6}$$

Then we get $\|Tf\|^2=(Hf,f)\leqq\|f\|^2$, in view of (iii) and (iv), and therefore

$$\|Hf\|^2=\|TTf\|^2\leqq\|Tf\|^2\leqq\|f\|^2.$$

The spectra are all positive and not larger than one. However, there may be continuous spectra. By the theory of bounded operators, we then have a unique spectral decomposition

$$H=\int_{-0}^{1}dE(\rho), \tag{7}$$

where $E(\rho)$ is a resolution of the identity corresponding to H.

3. We consider a special case in which all spectra concentrate at zero. We denote the complement of D by E. Ozawa showed that when a closed disc U is removed from D and if all spectra of the Fredholm eigen value problem for $D-U$ are equal to zero and with two dimensional measure $m(E)=0$, then E is of class $N_{\mathfrak{D}}$.

We shall now show that the condition $m(E)=0$ can be omitted here.

Put $D_1=U^c$. Let the kernels and the associated mapping functions for D_1 be denoted by K_1, l_1, p_1 and q_1. If all spectra of the problem for $D-U$ are equal to zero, then we have $Hf=0$ for any $f\in L^2(D-U)$ by (7), whence follows $\| Tf \|^2 = (Hf, f) = 0$. This implies that

$$\iint_{D-U} l(z, \zeta)\overline{f(\zeta)}d\tau_\zeta = 0$$

for any $f \in L^2(D-U)$, and hence $l(z, \zeta) \equiv 0$. On the other hand, it is readily verified that $l_1(z, \zeta) \equiv 0$, and hence, by (4), $p'(z, \zeta) + q'(z, \zeta) = p_1'(z, \zeta) + q_1'(z, \zeta)$. It follows that

$$d(p(z, \zeta) - p_1(z, \zeta)) + d(q(z, \zeta) - q_1(z, \zeta)) = 0. \tag{8}$$

Surely $D-U$ has a free boundary arc. Since the first term in (8) is real and the second is purely imaginary on this analytic arc, we get $p'(z, \zeta) = p_1'(z, \zeta)$ and $q'(z,\zeta) = q_1'(z, \zeta)$, and hence, by (3), $K(z, \bar\zeta) = K_1(z, \bar\zeta)$. For any $f(z) \in L^2(D-U)$, we define

$$f^*(z) = \begin{cases} f(z) & \text{in } D-U, \\ 0 & \text{on } E. \end{cases}$$

Then

$$f^{**}(z) = \iint_{D_1} K(z, \bar\zeta) f^*(\zeta) d\tau_\zeta = f(z)$$

in $D-U$, and $f^{**}(z)$ is analytic also on E. This implies $E \in N_{\mathfrak{D}}$ ([1]). Thus, we obtain the following theorem.

THEOREM 1. *If, by removing a closed disc U from D, all spectra of the Fredholm eigen value problem for $D-U$ are equal to zero, then the complement of D is of class $N_{\mathfrak{D}}$.*

Next we consider a problem of deciding the basic domain when all spectra are equal to zero. As above this implies $l(z, \zeta) = 0$. We have failed to solve the problem in the most general case. We add here a condition that there exists an isolated boundary component which does not reduce to a point. We integrate both sides of (4) and take zero as the value of integration constant. We then get

$$\frac{1}{2}(p(z, \zeta) + q(z, \zeta)) = \frac{1}{z-\zeta}$$

for all z and ζ in D. On the other hand, the function $w = (p(z, \zeta) + q(z, \zeta))/2$ maps the above boundary component onto a closed convex analytic curve γ ([1]). We shall show that γ is a circle. Assume that it were not the case. Then, we can draw a circle κ which intersects γ at not less than four points. We take a point τ in the image of D which lies on κ. Let t and k be the inverse image of τ and κ, respectively, by $w = (p(z, \zeta) + q(z, \zeta))/2 \equiv 1/(z-\zeta)$. Now by the function

$$\omega = \frac{1}{2}(p(z, t) + q(z, t)) \equiv \frac{1}{z-t},$$

the circle k is mapped onto a straight line while the image curve of the boundary of D is cut by this line at not less than four points. Hence this curve cannot be convex, which is a contradiction. Since the mapping considered is a linear transformation, the isolated boundary must be a circle. Theorem 1 shows that the remaining boundary components are of class $N_{\mathfrak{D}}$. Thus we get the following theorem.

THEOREM 2. *If all spectra of the Fredholm eigen value problem for D are equal to zero and there exists a continuum as an isolated boundary, then this continuum is a circle and the remaining components are of class* $N_{\mathfrak{D}}$.

REFERENCES

[1] AHLFORS, L. V., AND A. BEURLING, Conformal invariants and function-theoretic null-sets. Acta Math. 83 (1950), 101–129.

[2] BERGMAN, S., AND M. SCHIFFER, Kernel functions and conformal mapping. Comp. Math. 8 (1951), 205–249.

[3] OZAWA, M., Fredholm eigen value problem for general domains. Kōdai Math. Sem. Rep. 12 (1960), 38–44.

[4] SCHIFFER, M., Fredholm eigen values of plane domain. Pacific Journ. Math. 7 (1957), 1187–1225.

[5] SCHIFFER, M., Fredholm eigen values of multiply-connected domains. Pacific Journ. Math. 9 (1959), 211–269.

DEPARTMENT OF MATHEMATICS,
TOKYO METROPOLITAN UNIVERSITY.