# ON IMBEDDING OF A RIEMANNIAN SPACE IN A CONFORMALLY EUCLIDEAN SPACE

BY KENTARO YANO AND RICHARD BLUM

## § 0. Introduction.

Suppose that, in an $n$-dimensional Riemannian space $V_n$ with positive definite fundamental metric tensor $g_{ji}(\eta)$, there are given $N-n$ $(>0)$ symmetric tensors $H_{jix}(\eta)$ and $\frac{1}{2}(N-n)(N-n-1)$ vectors $L_{jxy}(\eta)=-L_{jyx}(\eta)$, where Latin indices $h, i, j, \cdots$ run over the range $1, 2, \cdots, n$ and $x, y, z$ over the range $n+1$, $n+2, \cdots, N$. Yano and Muto [4, 5] have found necessary and sufficient conditions for the Riemannian space $V_n$ to be imbedded in an $N$-dimensional *Euclidean space* $E_N$ in such a way that $\rho^2 g_{ji}$, $\rho M_{jix}$ $\left(M_{jix}=H_{jix}-\frac{1}{n}g^{cb}H_{cbx}g_{ji}\right)$ and $L_{jxy}$ are respectively first, second and third *conformal fundamental quantities* of the imbedded subspace $V_n$, $\rho(\eta)$ being a certain scalar function of $V_n$.

Blum [1, 2] also studied conditions for the Riemannian space to be imbedded in an $N$-dimensional *conformally Euclidean space* in such a way that $g_{ji}$, $H_{jix}$ and $L_{jxy}$ are respectively first, second and third *fundamental quantities* of the imbedded subspace $V_n$.

The purpose of the present paper is to give a complete solution to Blum's problem and to show that, if certain conditions are satisfied, a Riemannian space $V_n$ can be imbedded either in an $N$-dimensional *Euclidean space* in such a way that $\rho^2 g_{ji}$, $\rho M_{jix}$ and $L_{jxy}$ are respectively first, second and third *conformal fundamental quantities* of the imbedded subspace or in an $N$-dimensional *conformally Euclidean space* in such a way that $g_{ji}$, $H_{jix}$ and $L_{jxy}$ are respectively first, second and third *fundamental quantities* of the imbedded subspace.

## § 1. Preliminaries.

Let $V_N$ be an $N$-dimensional Riemannian space of class $C^\omega$ with positive definite fundamental metric

$$ds^2 = g_{\mu\lambda}(\xi)d\xi^\mu d\xi^\lambda, \tag{1.1}$$

where Greek indices $\kappa, \lambda, \mu, \cdots$ run over the range $a_1, a_2, \cdots, a_N$. We denote by

$$\left\{ {\kappa \atop \mu\ \lambda} \right\} = \frac{1}{2} g^{\kappa\alpha}(\partial_\mu g_{\lambda\alpha} + \partial_\lambda g_{\mu\alpha} - \partial_\alpha g_{\mu\lambda}) \tag{1.2}$$

the Christoffel symbols formed with $g_{\mu\lambda}$, $g^{\kappa\alpha}$ being the fundamental contra-

Received October 13, 1960.

variant tensor and $\partial_\mu = \partial/\partial\xi^\mu$. We denote by

$$\nabla_\mu v^\kappa = \partial_\mu v^\kappa + \left\{ {\kappa \atop \mu\ \lambda} \right\} v^\lambda \tag{1.3}$$

the covariant derivative of a contravariant vector $v^\kappa$, and by

$$K_{\nu\mu\lambda}{}^\kappa = \partial_\nu \left\{ {\kappa \atop \mu\ \lambda} \right\} - \partial_\mu \left\{ {\kappa \atop \nu\ \lambda} \right\} + \left\{ {\kappa \atop \nu\ \alpha} \right\} \left\{ {\alpha \atop \mu\ \lambda} \right\} - \left\{ {\kappa \atop \mu\ \alpha} \right\} \left\{ {\alpha \atop \nu\ \lambda} \right\} \tag{1.4}$$

the Riemann-Christoffel curvature tensor of $V_N$.

We now consider an $n$-dimensional subspace $V_n$ of $V_N$ defined by

$$\xi^\kappa = f^\kappa(\eta^1, \eta^2, \cdots, \eta^n), \tag{1.5}$$

where the functions $f^\kappa(\eta)$ are supposed to be of class $C^\omega$ and the matrix whose elements are

$$B_i{}^\kappa = \partial_i \xi^\kappa \qquad\qquad (\partial_i = \partial/\eta^i)$$

is of rank $n$. The fundamental tensor of $V_n$ is given by

$$g_{ji} = B_j{}^\mu B_i{}^\lambda g_{\mu\lambda}. \tag{1.6}$$

We choose $N-n$ mutually orthogonal unit vectors $C_x{}^\kappa$ which are orthogonal to $V_n$ and oriented in such a way that

$$| B_i{}^\kappa, C_x{}^\kappa | > 0.$$

Then we have

$$B_j{}^\mu C_x{}^\lambda g_{\mu\lambda} = 0, \quad C_y{}^\mu C_x{}^\lambda g_{\mu\lambda} = \delta_{yx}, \quad | B_i{}^\kappa, C_x{}^\kappa | = \sqrt{\mathfrak{g}} > 0, \tag{1.7}$$

where $\delta_{yx}$ is Kronecker's delta and $\mathfrak{g}$ the determinant formed by $g_{ji}$.

The Christoffel symbols

$$\left\{ {h \atop j\ i} \right\} = \frac{1}{2} h^{ha}(\partial_j g_{ia} + \partial_i g_{ja} - \partial_a g_{ji}) \tag{1.8}$$

of $V_n$ are given by

$$\left\{ {h \atop j\ i} \right\} = B^h{}_\kappa \left( B_j{}^\mu B_i{}^\lambda \left\{ {\kappa \atop \mu\ \lambda} \right\} + \partial_j B_i{}^\kappa \right), \tag{1.9}$$

where we have put

$$B^h{}_\kappa = B_i{}^\lambda g^{ih} g_{\lambda\kappa}.$$

If we put $C_{x\kappa} = C_x{}^\lambda g_{\lambda\kappa}$, it is easily seen that two matrices

$$(B_i{}^\kappa, C_x{}^\kappa) \quad \text{and} \quad (B^i{}_\lambda, C_{x\lambda})$$

are inverse to each other.

The van der Waerden-Bortolotti covariant derivative of $B_i{}^\kappa$ is given by

$$\nabla_j B_i{}^\kappa = \partial_j B_i{}^\kappa + B_j{}^\mu B_i{}^\lambda \left\{ {\kappa \atop \mu\ \lambda} \right\} - B_h{}^\kappa \left\{ {h \atop j\ i} \right\}. \tag{1.10}$$

Equation (1.9) shows that $\nabla_j B_i{}^\kappa$ are, as vectors in $V_N$, orthogonal to $V_n$ and consequently we have equations of the form

(1.11) $$\nabla_j B_i^{\kappa} = H_{jix} C_x^{\kappa},$$

where $H_{jix}$ are the second fundamental quantities of $V_n$. (1.11) are equations of Gauss for $V_n$.

Differentiating $B_j^{\mu} C_x^{\lambda} g_{\mu\lambda} = 0$ and $C_y^{\mu} C_x^{\lambda} g_{\mu\lambda} = \delta_{yx}$ covariantly we find that $\nabla_j C_x^{\kappa}$ must be of the form

(1.12) $$\nabla_j C_x^{\kappa} = -H_j{}^i{}_x B_i^{\kappa} + L_{jxy} C_y^{\kappa},$$

where $H_j{}^i{}_x = H_{jax} g^{ai}$ and $L_{jxy}$ are the third fundamental quantities of $V_n$. (1.12) are equations of Weingarten for $V_n$.

Now, substituting (1.11) and (1.12) into the Ricci formula:

(1.13) $$\nabla_k \nabla_j B_i^{\kappa} - \nabla_j \nabla_k B_i^{\kappa} = B_{kji}^{\nu\mu\lambda} K_{\nu\mu\lambda}{}^{\kappa} - B_h^{\kappa} K_{kji}{}^h,$$

we find

(1.14) $$B_{kji}^{\nu\mu\lambda} K_{\nu\mu\lambda}{}^{\kappa} - B_h^{\kappa} K_{kji}{}^h = -B_h^{\kappa}(H_k{}^h{}_x H_{jix} - H_j{}^h{}_x H_{kix}) + C_x^{\kappa}(\nabla_k H_{jix} - \nabla_j H_{kix} + H_{kiy} L_{jxy} - H_{jiy} L_{kxy}),$$

where $B_{kji}^{\nu\mu\lambda}$ is an abbreviation of $B_k^{\nu} B_j^{\mu} B_i^{\lambda}$ and $K_{kji}{}^h$ the curvature tensor of $V_n$.

Next substituting (1.11) and (1.12) into the Ricci formula:

(1.15) $$\nabla_k \nabla_j C_x^{\kappa} - \nabla_j \nabla_k C_x^{\kappa} = B_{kj}^{\nu\mu} C_x^{\lambda} K_{\nu\mu\lambda}{}^{\kappa},$$

we find

(1.16) $$\begin{aligned} B_{kj}^{\nu\mu} C_x^{\lambda} K_{\nu\mu\lambda}{}^{\kappa} = &-B_h^{\kappa}(\nabla_k H_j{}^h{}_x - \nabla_j H_k{}^h{}_x + H_k{}^h{}_y L_{jxy} - H_j{}^h{}_y L_{kxy}) \\ &+ C_y^{\kappa}(\nabla_k L_{jxy} - \nabla_j L_{kxy} + H_k{}^a{}_x H_{jay} - H_j{}^a{}_x H_{kay} \\ &+ L_{kxz} L_{jyz} - L_{jxz} L_{kyz}) \end{aligned}$$

where $B_{kj}^{\nu\mu} = B_k^{\nu} B_j^{\mu}$.

When the enveloping space $V_N$ is locally Euclidean, we choose a rectangular coordinate system in $V_N$, then we have $\left\{ {\kappa \atop \mu\ \lambda} \right\} = 0$, $K_{\nu\mu\lambda}{}^{\kappa} = 0$. Thus equations (1.11) and (1.12) become respectively

(1.17) $$\nabla_j B_i^{\kappa} = \partial_j B_i^{\kappa} - B_h^{\kappa} \left\{ {h \atop j\ i} \right\} = H_{jix} C_x^{\kappa},$$

(1.18) $$\nabla_j C_x^{\kappa} = \partial_j C_x^{\kappa} = -H_j{}^h{}_x B_h^{\kappa} + L_{jxy} C_y^{\kappa},$$

and equations (1.14) and (1.16) give

(1.19) $$K_{kji}{}^h = H_k{}^h{}_x H_{jix} - H_j{}^h{}_x H_{kix},$$

(1.20) $$0 = \nabla_k H_{jix} - \nabla_j H_{kix} + H_{kiy} L_{jxy} - H_{jiy} L_{kxy},$$

(1.21) $$0 = \nabla_k H_j{}^h{}_x - \nabla_j H_k{}^h{}_x + H_k{}^h{}_y L_{jxy} - H_j{}^h{}_y L_{kxy},$$

(1.22) $$0 = \nabla_k L_{jxy} - \nabla_j L_{kxy} + H_k{}^a{}_x H_{jay} - H_j{}^a{}_x H_{kay} + L_{kxz} L_{jyz} - L_{jxz} L_{kyz}.$$

(1.19) are equations of Gauss, (1.20) and (1.21), being equivalent, are equations of Mainardi-Codazzi and (1.22) are equations of Ricci-Kühne.

Equations (1.17) and (1.18) may be regarded as a system of simultaneous partial differential equations with unknown functions $B_i^{\kappa}$ and $C_x^{\kappa}$ and then equations (1.19), (1.20) and (1.22) are found to be complete integrability conditions of this system of partial differential equations.

Since the condition

$$\partial_j B_i^\kappa = \partial_i B_j^\kappa$$

is automatically satisfied, from

$$\partial_j \xi^\kappa = B_i^\kappa,$$

we can find functions $\xi^\kappa = f^\kappa(\eta)$ with $N$ additional arbitrary constants. Moreover, we can prove that if the conditions

$$B_j^\lambda B_i^\lambda = g_{ji}, \quad B_j^\lambda C_x^\lambda = 0, \quad C_y^\lambda C_x^\lambda = \delta_{yx} \quad \text{and} \quad |B_i^\lambda, C_x^\lambda| = \sqrt{\mathfrak{g}}$$

are satisfied as initial conditions for $B_i^\kappa$ and $C_x^\kappa$, then they are also satisfied along the solution. Thus the functions $\xi^\kappa = f^\kappa(\eta)$ define an $n$-dimensional subspace $V_n$ whose first, second and third fundamental quantities are respectively $g_{ji}$, $H_{jix}$ and $L_{jxy}$. Moreover, since a figure formed by $B_i^\kappa$ and $C_x^\kappa$ satisfying above conditions at a point is congruent to another figure formed by $B_i^\kappa$ and $C_x^\kappa$ satisfying the same conditions at a different point, the subspace $V_n$, is completely determined up to a motion. This is what we call fundamental theorem of the theory of subspaces.

What we are going to do in the present paper is to see what will happen when we assume that the enveloping space $V_N$ is a conformally Euclidean space $C_N$.

## § 2. Subspaces in $C_N$.

Suppose that our $N$-dimensional space $V_N$ be a conformally Euclidean space $C_N$ and choose a coordinate system such that the fundamental tensor $g_{\mu\lambda}$ has the components

$$g_{\mu\lambda} = e^{2\rho(\xi)}\delta_{\mu\lambda}, \tag{2.1}$$

where $\rho(\xi)$ is a function of $\xi^\kappa$ of class $C^\omega$.

In this case, the Christoffel symbols $\left\{ {\kappa \atop \mu\ \lambda} \right\}$ of $C_N$ take the form

$$\left\{ {\kappa \atop \mu\ \lambda} \right\} = \rho_\mu A_\lambda^\kappa + \rho_\lambda A_\mu^\kappa - \rho^\kappa g_{\mu\lambda}, \tag{2.2}$$

where

$$\rho_\mu = \partial_\mu \rho, \quad \rho^\kappa = \rho_\lambda g^{\lambda\kappa}.$$

Substituting (2.2) into (1.4), we find

$$K_{\nu\mu\lambda}{}^\kappa = -A_\nu^\kappa \rho_{\mu\lambda} + A_\mu^\kappa \rho_{\nu\lambda} - \rho_\nu{}^\kappa g_{\mu\lambda} + \rho_\mu{}^\kappa g_{\nu\lambda}, \tag{2.3}$$

where

$$\rho_{\mu\lambda} = \nabla_\mu \rho_\lambda + \rho_\mu \rho_\lambda - \frac{1}{2}\rho^\alpha \rho_\alpha g_{\mu\lambda}, \quad \rho_\nu{}^\kappa = \rho_{\nu\mu} g^{\mu\kappa}. \tag{2.4}$$

Now we consider an $n$-dimensional subspace $V_n$ defined by $\xi^\kappa = f^\kappa(\eta)$, and put

$$\sigma(\eta) = \rho(f(\eta)), \tag{2.5}$$

$$\sigma_i = \nabla_i \sigma = B_i^\lambda \nabla_\lambda \rho, \quad \sigma_x = C_x^\lambda \nabla_\lambda \rho. \tag{2.6}$$

Then we have

$$\nabla_j \sigma_i = B_j^\mu B_i^\lambda \nabla_\mu \rho_\lambda + H_{jix}\sigma_x, \tag{2.7}$$

$$\nabla_j \sigma^h = B_j^\mu B^h{}_\kappa \nabla_\mu \rho^\kappa + H_j{}^h{}_x \sigma_x, \tag{2.8}$$

$$\nabla_j \sigma_x = B_j^\mu C_x^\lambda \nabla_\mu \rho_\lambda - H_j{}^a{}_x \sigma_a + L_{jxy}\sigma_y. \tag{2.9}$$

Substituting (2.3) into (1.14) and (1.16) we find

$$\begin{aligned} &- B_h^\kappa (K_{kji}{}^h + A^h_k \sigma_{ji} - A^h_j \sigma_{ki} + \sigma_k{}^h g_{ji} - \sigma_j{}^h g_{ki}) - C_x^\kappa(\sigma_{kx} g_{ji} - \sigma_{jx} g_{ki}) \\ &= - B_h^\kappa (H_k{}^h{}_x H_{jix} - H_j{}^h{}_x H_{kix}) + C_x^\kappa(\nabla_k H_{jix} - \nabla_j H_{kix} + H_{kiy} L_{jxy} - H_{jiy} L_{kxy}), \end{aligned} \tag{2.10}$$

$$\begin{aligned} &- B_h^\kappa (A^h_k \sigma_{jx} - A^h_j \sigma_{kx}) = - B_h^\kappa(\nabla_k H_j{}^h{}_x - \nabla_j H_k{}^h{}_x + H_k{}^h{}_y L_{jxy} - H_j{}^h{}_y L_{kxy}) \\ &+ C_y^\kappa(\nabla_k L_{jxy} - \nabla_j L_{kxy} + H_k{}^a{}_x H_{jay} - H_j{}^a{}_x H_{kay} + L_{kxz} L_{jyz} - L_{jxz} L_{kyz}) \end{aligned} \tag{2.11}$$

respectively, where

$$\sigma_{ji} = B_j^\mu B_i^\lambda \rho_{\mu\lambda} = \nabla_j \sigma_i - H_{jix}\sigma_x + \sigma_j \sigma_i - \frac{1}{2}(g^{cb}\sigma_c\sigma_b + \sigma_x\sigma_x) g_{ji}, \tag{2.12}$$

$$\sigma_j{}^h = B_j^\mu B^h{}_\kappa \rho_\mu{}^\kappa = \nabla_j \sigma^h - H_j{}^h{}_x \sigma_x + \sigma_j \sigma^h - \frac{1}{2}(g^{cb}\sigma_c\sigma_b + \sigma_x\sigma_x) A^h_j, \tag{2.13}$$

$$\sigma_{jx} = B_j^\mu C_x^\lambda \rho_{\mu\lambda} = \nabla_j \sigma_x + H_j{}^a{}_x \sigma_a - L_{jxy}\sigma_y + \sigma_j \sigma_x \tag{2.14}$$

by virtue of the relations (2.7), (2.8) and (2.9).

From (2.10) and (2.11) we find

$$K_{kji}{}^h - (H_k{}^h{}_x H_{jix} - H_j{}^h{}_x H_{kix}) + A^h_k \sigma_{ji} - A^h_j \sigma_{ki} + \sigma_k{}^h g_{ji} - \sigma_j{}^h g_{ki} = 0, \tag{2.15}$$

$$\nabla_k H_{jix} - \nabla_j H_{kix} + H_{kiy} L_{jxy} - H_{jiy} L_{kxy} + \sigma_{kx} g_{ji} - \sigma_{jx} g_{ki} = 0, \tag{2.16}$$

$$\nabla_k H_j{}^h{}_x - \nabla_j H_k{}^h{}_x + H_k{}^h{}_y L_{jxy} - H_j{}^h{}_y L_{kxy} - (A^h_k \sigma_{jx} - A^h_j \sigma_{kx}) = 0, \tag{2.17}$$

$$\nabla_k L_{jxy} - \nabla_j L_{kxy} + H_k{}^a{}_x H_{jay} - H_j{}^a{}_x H_{kay} + L_{kxz} L_{jyz} - L_{jxz} L_{kyz} = 0. \tag{2.18}$$

Equations (2.16) and (2.17) are equivalent. If we put

$$P_{kji}{}^h = K_{kji}{}^h - (H_k{}^h{}_x H_{jix} - H_j{}^h{}_x H_{kix}), \tag{2.19}$$

then (2.15) takes the form

$$P_{kji}{}^h + A^h_k \sigma_{ji} - A^h_j \sigma_{ki} + \sigma_k{}^h g_{ji} - \sigma_j{}^h g_{ki} = 0, \tag{2.20}$$

frow which

$$\sigma_{ji} = Q_{ji}, \tag{2.21}$$

where

$$Q_{ji} = - \frac{P_{ji}}{n-2} + \frac{P g_{ji}}{2(n-1)(n-2)} \tag{2.22}$$

and

(2.23) $$P_{ji} = P_{aji}{}^{a} \quad \text{and} \quad P = g^{ji}P_{ji}.$$

Substituting (2.21) into (2.20), we obtain

(2.24) $$P_{kji}{}^{h} + A^{h}_{k}Q_{ji} - A^{h}{}_{j}Q_{ki} + Q_{k}{}^{h}g_{ji} - Q_{j}{}^{h}g_{ki} = 0.$$

Equations (2.15) or (2.20) and the set of (2.21) and (2.24) are equivalent. If we put

(2.25) $$S_{kjix} = \nabla_{k}H_{jix} - \nabla_{j}H_{kix} + H_{kiy}L_{jxy} - H_{jiy}L_{kxy},$$

then (2.16) takes the form

(2.26) $$S_{kjix} + \sigma_{kx}g_{ji} - \sigma_{jx}g_{ki} = 0,$$

from which

(2.27) $$\sigma_{kx} = -\frac{1}{n-1}S_{kx},$$

where

(2.28) $$S_{kx} = S_{kjix}g^{ji}.$$

Substituting (2.27) into (2.26), we obtain

(2.29) $$S_{kjix} - \frac{1}{n-1}(S_{kx}g_{ji} - S_{jx}g_{ki}) = 0.$$

Equation (2.16) or (2.26) and the set of (2.27) and (2.29) are equivalent.

## § 3. Imbedding in a conformally Euclidean space.

We now consider the following problem: In an $n$-dimensinal Riemannian space with fundamental metric tensor $g_{ji}(\eta)$, there are given $N-n$ ($>0$) symmetric tensors $H_{jix}(\eta)$ and $\frac{1}{2}(N-n)(N-n-1)$ vectors $L_{jxy}(\eta) = -L_{jyx}(\eta)$. What are the conditions for the $n$-dimensional Riemannian space to be imbedded in an $N$-dimensional conformally Euclidean space in such a way that the first, second and third fundamental quantities are respectively $g_{ji}(\eta)$, $H_{jix}(\eta)$ and $L_{jxy}(\eta)$?

In order to have such an imbedding, we must find functions $\xi^{\kappa}(\eta)$, $B_{i}{}^{\kappa}(\eta)$ and $C_{x}{}^{\kappa}(\eta)$ satisfying

$$B_{j}{}^{\mu}B_{i}{}^{\lambda}g_{\mu\lambda} = g_{ji}, \quad B_{j}{}^{\mu}C_{x}{}^{\lambda}g_{u\lambda} = 0, \quad C_{y}{}^{\mu}C_{x}{}^{\lambda}g_{\mu\lambda} = \delta_{yx}, \quad |B_{j}{}^{\lambda}, C_{x}{}^{\lambda}| = \sqrt{\mathfrak{g}}$$

and

$$\partial_{i}\xi^{\kappa} = B_{i}{}^{\kappa}, \quad \nabla_{j}B_{i}{}^{\kappa} = H_{jix}C_{x}{}^{\kappa}, \quad \nabla_{j}C_{x}{}^{\kappa} = -H_{j}{}^{i}{}_{x}B_{i}{}^{\kappa} + L_{jxy}C_{y}{}^{\kappa}.$$

But first three equations contain the function $\rho(\xi)$ evaluated on the subspace:

$$\rho(\xi(\eta)) = \sigma(\eta),$$

and the last two equations contain $\sigma_{i} = \nabla_{i}\sigma$ and $\sigma_{x}$. In fact, the last three equations are written as

$$\partial_i \xi^\kappa = B_i{}^\kappa, \tag{3.1}$$

$$\nabla_j B_i{}^\kappa = \partial_j B_i{}^\kappa + \sigma_j B_i{}^\kappa + \sigma_i B_j{}^\kappa - \rho^\kappa g_{ji} - B_h{}^\kappa \left\{ {h \atop j\ i} \right\} = H_{jix} C_x{}^\kappa, \tag{3.2}$$

$$\nabla_j C_x{}^\kappa = \partial_j C_x{}^\kappa + \sigma_j C_x{}^\kappa + \sigma_x B_j{}^\kappa = - H_j{}^i{}_x B_i{}^\kappa + L_{jxy} C_y{}^\kappa, \tag{3.3}$$

where $\rho^\kappa$ has the form

$$\rho^\kappa = B_i{}^\kappa \sigma^i + C_x{}^\kappa \sigma_x.$$

On the other hand, we know that $\sigma$, $\sigma_i$ and $\sigma_x$ satisfy the equations

$$\nabla_i \sigma = \sigma_i, \tag{3.4}$$

$$\nabla_j \sigma_i - H_{jix}\sigma_x + \sigma_j \sigma_i - \frac{1}{2}(g^{cb}\sigma_c\sigma_b + \sigma_x\sigma_x) g_{ji} = Q_{ji}, \tag{3.5}$$

$$\nabla_j \sigma_x + H_j{}^i{}_x \sigma_i - L_{jxy}\sigma_y + \sigma_j \sigma_x = - \frac{1}{n-1} S_{jx}. \tag{3.6}$$

Thus equations (3.1), (3.2), (3.3), (3.4), (3.5) and (3.6) give a system of partial differential equations with unknown functions $\xi^\kappa$, $B_i{}^\kappa$, $C_x{}^\kappa$, $\sigma$, $\sigma_i$ and $\sigma_x$. We are now going to examine the integrability conditions of this system.

The integrability conditions of (3.1) are

$$\partial_j B_i{}^\kappa = \partial_i B_j{}^\kappa,$$

but these are satisfied as we see from (3.2).

The integrability conditions of (3.2) are, as was shown in § 2, given by (2.21), (2.24), (2.27) and (2.29). The integrability conditions of (3.3) are given by (2.27), (2.29) and (2.18). But, (2.21) and (2.27) are included in the system.

Thus to get the integrability conditions of the system, we have only to study, in addition to the equations

$$P_{kji}{}^h + A_k^h Q_{ji} - A_j^h Q_{ki} + Q_k{}^h g_{ji} - Q_j{}^h g_{ki} = 0, \tag{3.7}$$

$$S_{kjix} - \frac{1}{n-1}(S_{kx} g_{ji} - S_{jx} g_{ki}) = 0, \tag{3.8}$$

$$\nabla_k L_{jxy} - \nabla_j L_{kxy} + H_k{}^a{}_x H_{jay} - H_j{}^a{}_x H_{kay} + L_{kxz} L_{jyz} - L_{jxz} L_{kyz} = 0, \tag{3.9}$$

the integrability conditions of (3.5) and (3.6).

Equations (3.5) and (3.6) are respectively written as

$$\nabla_j \sigma_i = Q_{ji} + H_{jix}\sigma_x - \sigma_j \sigma_i + \frac{1}{2}(g^{cb}\sigma_c\sigma_b + \sigma_x\sigma_x) g_{ji}, \tag{3.10}$$

$$\nabla_j \sigma_x = - \frac{1}{n-1} S_{jx} - H_j{}^a{}_x \sigma_a + L_{jxy}\sigma_y - \sigma_j \sigma_x. \tag{3.11}$$

To find the integrability conditions of (3.10), we substitute (3.10) into Ricci formula:

$$\nabla_k \nabla_j \sigma_i - \nabla_j \nabla_k \sigma_i = - K_{kji}{}^h \sigma_h,$$

then we find

$$\nabla_k Q_{ji} - \nabla_j Q_{ki} - \frac{1}{n-1}(S_{kx}H_{jix} - S_{jx}H_{kix}) + (P_{kji}{}^h + A^h_k Q_{ji} - A^h_j Q_{ki} + Q_k{}^h g_{ji} - Q_j{}^h g_{ki})\sigma_h + \left[S_{kjix} - \frac{1}{n-1}(S_{kx}g_{ji} - S_{jx}g_{ki})\right]\sigma_x = 0,$$

or

$$\nabla_k Q_{ji} - \nabla_j Q_{ki} - \frac{1}{n-1}(S_{kx}H_{jix} - S_{jx}H_{kix}) = 0 \tag{3.12}$$

by virtue of (3.7) and (3.8).

To find the integrability conditions of (3.11), we substitute (3.11) in formula:

$$\nabla_k \nabla_j \sigma_x - \nabla_j \nabla_k \sigma_x = 0,$$

then we find

$$-\frac{1}{n-1}(\nabla_k S_{jx} - \nabla_j S_{kx} + S_{ky}L_{jxy} - S_{jy}L_{kxy}) - H_k{}^a{}_x Q_{aj} + H_j{}^a{}_x Q_{ak} - \left[S_{kj}{}^i{}_x - \frac{1}{n-1}(S_{kx}A^i_j - S_{jx}A^i_k)\right]\sigma_i + (\nabla_k L_{jxy} - \nabla_j L_{kxy} + H_k{}^a{}_x H_{ajy} - H_j{}^a{}_x H_{aky} + L_{kzy}L_{jxz} - L_{jzy}L_{kxz})\sigma_y = 0$$

or

$$-\frac{1}{n-1}(\nabla_k S_{jx} - \nabla_j S_{kx} + S_{ky}L_{jxy} - S_{jy}L_{kxy}) - H_k{}^a{}_x Q_{aj} + H_j{}^a{}_x Q_{ak} = 0 \tag{3.13}$$

by virtue of (3.8) and (3.9).

Thus we have shown that the integrability conditions of the system of partial differential equations (3.1), (3.2), (3.3), (3.4), (3.5) and (3.6) are given by (3.7), (3.8), (3.9), (3.12) and (3.13) and consequently if these conditions are satisfied, the system is completely integrable and admits solutions $\xi^\kappa = f^\kappa(\eta)$, $B_i{}^\kappa(\eta)$, $C_x{}^\kappa(\eta)$, $\sigma(\eta)$, $\sigma_i(\eta)$ and $\sigma_x(\eta)$.

We now show that there exists a function $\rho(\xi)$ such that

$$\rho(f(\eta)) = \sigma, \quad \rho_\lambda(f(\eta))B_i{}^\lambda = \sigma_i, \quad \rho_\lambda(f(\eta))C_x{}^\lambda = \sigma_x$$

where $\rho_\lambda(\xi) = \partial_\lambda \rho(\xi)$.

To see this, we consider the equations

$$\xi^\kappa = f^\kappa(\eta) + z^x C_x{}^\kappa(\eta). \tag{3.14}$$

Since

$$\frac{\partial \xi^\kappa}{\partial \eta^i} = \frac{\partial f^\kappa}{\partial \eta^i} + z^x \frac{\partial C_x{}^\kappa}{\partial \eta^i}, \quad \frac{\partial \xi^\kappa}{\partial z^x} = C_x{}^\kappa,$$

$$\left| \frac{\partial \xi^\kappa}{\partial \eta^i}, \frac{\partial \xi^\kappa}{\partial z^x} \right|_{z=0} = |B_i{}^\kappa, C_x{}^\kappa| = \sqrt{\mathfrak{g}} \neq 0,$$

we can solve equations (3.14) with respect to $\eta^i$ and $z^x$ and get

$$\eta^i = \eta^i(\xi), \quad z^x = z^x(\xi) \tag{3.15}$$

in the neighborhood of $z^x = 0$. If we consider (3.15) as a coordinate transformation, then the original subspace can be represented by equations $z^x=0$ in new coordinate system. Now put

$$\rho(\xi)=\sigma(\eta(\xi))+z^x(\xi)\sigma_x(\eta(\xi)), \tag{3.16}$$

then we have

$$[\rho(\xi)]_{z=0}=\rho(f(\eta))=\sigma(\eta).$$

Differentiating (3.16) with respect to $\eta^i$ and evaluating at $z^x=0$, we get

$$\rho_\lambda(f(\eta))B_i{}^\lambda=\sigma_i.$$

Next differentiating (3.16) with respect to $z^x$ and evaluating at $z^x=0$, we get

$$\rho_\lambda(f(\eta))C_x{}^\lambda=\sigma_x(\eta).$$

Thus the function $\rho(\xi)$ given by (3.16) is a required one.

We shall next show that, these solutions satisfy

$$B_j{}^\mu B_i{}^\lambda g_{\mu\lambda}=g_{ji},\quad B_j{}^\mu C_x{}^\lambda g_{\mu\lambda}=0,\quad C_y{}^\mu C_x{}^\lambda g_{\mu\lambda}=\delta_{yx}, \tag{3.17}$$

whenever their initial conditions satisfy them. By a straightforward computation, we find

$$\nabla_k(B_j{}^\mu B_i{}^\lambda g_{\mu\lambda}-g_{ji})=H_{kjx}(C_x{}^\mu B_i{}^\lambda g_{\mu\lambda})+H_{kix}(B_j{}^\mu C_x{}^\lambda g_{\mu\lambda}),$$
$$\nabla_k(B_j{}^\mu C_x{}^\lambda g_{\mu\lambda})=H_{kjy}(C_y{}^\mu C_x{}^\lambda g_{\mu\lambda}-\delta_{yx})-H_k{}^i{}_x(B_j{}^\mu B_i{}^\lambda g_{\mu\lambda}-g_{ji})+L_{kxy}(B_j{}^\mu C_y{}^\lambda g_{\mu\lambda}),$$
$$\begin{aligned}\nabla_k(C_y{}^\mu C_x{}^\lambda g_{\mu\lambda}-\delta_{yx})=&-H_k{}^i{}_y(B_i{}^\mu C_x{}^\lambda g_{\mu\lambda})+L_{kyz}(C_z{}^\mu C_x{}^\lambda g_{\mu\lambda}-\delta_{zx})\\&-H_k{}^i{}_x(C_y{}^\mu B_i{}^\lambda g_{\mu\lambda})+L_{kxz}(C_y{}^\mu C_z{}^\lambda g_{\mu\lambda}-\delta_{yz}).\end{aligned}$$

These equations show that if we choose initial conditions in such a way that (3.17) are satisfied at a fixed point of the space, then they are satisfied along the solution. Thus we have proved:

THEOREM. *Suppose that there are given, in an n-dimensional Riemannian space* $V_n$ *of class* $C^\omega$ *with fundamental metric tensor* $g_{ji}(\eta)$, $N-n$ $(>0)$ *symmetric tensors* $H_{jix}(\eta)$ *and* $\frac{1}{2}(N-n)(N-n-1)$ *vectors* $L_{jxy}(\eta)=-L_{jyx}(\eta)$. *A necessary and sufficient condition for* $V_n$ *to be imbedded in an N-dimensional conformally Euclidean space* $C_N$ *as a subspace with the first, second and third fundamental quantities* $g_{ji}(\eta)$, $H_{jix}(\eta)$ *and* $L_{jxy}(\eta)$ *respectively is that they satisfy* (3.7), (3.8), (3.9), (3.12) *and* (3.13).

## §4. Other forms of integrability conditions.

We now introduce tensors

$$M_{jix}=H_{jix}-\frac{1}{n}g^{cb}H_{cbx}g_{ji} \tag{4.1}$$

which are invariant under a conformal transformation of the enveloping space and are called conformal second fundamental tensors [3]. It is known that the third fundamental vectors $L_{jxy}$ are also invariant under such a conformal transformation.

If we denote the mean curvature by

$$H_x = \frac{1}{n} g^{cb} H_{cbx},$$

equation (4.1) can be written as

$$H_{jix} = M_{jix} + g_{ji} H_x. \tag{4.2}$$

Substituting this into (2.19), we find

$$\begin{aligned} P_{kji}{}^h = K_{kji}{}^h &- (M_k{}^h{}_x M_{jix} - M_j{}^h{}_x M_{kix}) \\ &- M_k{}^h{}_x g_{ji} H_x - A_k^h M_{jix} H_x - A_k^h g_{ji} H_x H_x \\ &+ M_j{}^h{}_x g_{ki} H_x + A^h_j M_{kix} H_x + A^h_j g_{ki} H_x H_x, \end{aligned} \tag{4.3}$$

from which

$$P_{ji} = K_{ji} + M_j{}^a{}_x M_{aix} - (n-2) M_{jix} H_x - (n-1) g_{ji} H_x H_x$$

and

$$P = K + M_b{}^a{}_x M_a{}^b{}_x - n(n-1) H_x H_x,$$

by virtue of $g^{ji} M_{jix} = 0$, where $K_{ji} = K_{aji}{}^a$ and $K = g^{ji} K_{ji}$. Hence

$$Q_{ji} = L_{ji} - \frac{M_j{}^a{}_x M_{aix}}{n-2} + \frac{M_b{}^a{}_x M_a{}^b{}_x g_{ji}}{2(n-1)(n-2)} + M_{jix} H_x + \frac{1}{2} g_{ji} H_x H_k, \tag{4.4}$$

where

$$L_{ji} = -\frac{K_{ji}}{n-2} + \frac{K g_{ji}}{2(n-1)(n-2)}.$$

Substituting (4.3) and (4.4) into (3.7), we find

$$C_{kji}{}^h + A_k^h M_{ji} - A^h_j M_{ki} + M_k{}^h g_{ji} - M_j{}^h g_{ki} = 0, \tag{4.5}$$

where

$$C_{kji}{}^h = K_{kji}{}^h + A_k^h L_{ji} - A^h_j L_{ki} + L_k{}^h g_{ji} - L_j{}^h g_{ki} \tag{4.6}$$

is the Weyl conformal curvature tensor and

$$M_{ji} = -\frac{M_j{}^a{}_x M_{aix}}{n-2} + \frac{M_b{}^a{}_x M_a{}^b{}_x g_{ji}}{2(n-1)(n-2)}. \tag{4.7}$$

Substituting next (4.2) into (2.25), we find

$$\begin{aligned} S_{kjix} = \nabla_k M_{jix} - \nabla_j M_{kix} &+ (\nabla_k H_x) g_{ji} - (\nabla_j H_x) g_{ki} \\ &+ M_{kiy} L_{jxy} - M_{jiy} L_{kxy} + g_{ki} H_y L_{jxy} - g_{ji} H_y L_{kxy}, \end{aligned} \tag{4.8}$$

from which

$$-\frac{1}{n-1} S_{kx} = \frac{1}{n-1} (\nabla_a M_k{}^a{}_x + M_k{}^a{}_y L_{axy}) - (\nabla_k H_x - L_{kxy} H_y). \tag{4.9}$$

Substituting (4.8) and (4.9) into (3.8), we find

$$\nabla_k M_{jix} - \nabla_j M_{kix} + M_{kiy} L_{jxy} - M_{jiy} L_{kxy} + M_{kx} g_{ji} - M_{jx} g_{ki} = 0, \tag{4.10}$$

where

$$M_{kx} = \frac{1}{n-1}(\nabla_a M_k{}^a{}_x + M_k{}^a{}_y L_{axy}). \tag{4.11}$$

Substituting (4.2) into (3.9), we find

$$\nabla_k L_{jxy} - \nabla_j L_{kxy} + M_k{}^a{}_x M_{jay} - M_j{}^a{}_x M_{kay} + L_{kxz} L_{jyz} - L_{jxz} L_{kyz} = 0. \tag{4.12}$$

Now, (4.4) and (4.9) may respectively be written as

$$Q_{ji} = L_{ji} + M_{ji} + M_{jix} H_x + \frac{1}{2} g_{ji} H_x H_x \tag{4.13}$$

and

$$-\frac{1}{n-1} S_{kx} = M_{kx} - \nabla_k H_x + L_{kxy} H_y. \tag{4.14}$$

Substituting (4.2), (4.13), and (4.14) into (3.12), we find

$$\begin{aligned} &\nabla_k L_{ji} - \nabla_j L_{ki} + \nabla_k M_{ji} - \nabla_j M_{ki} + M_{kx} M_{jix} - M_{jx} M_{kix} \\ &\qquad + (\nabla_k M_{jix} - \nabla_j M_{kix} + M_{kiy} L_{jxy} - M_{jiy} L_{kxy} + M_{kx} g_{ji} - M_{jx} g_{ki}) H_x = 0 \end{aligned}$$

or

$$\nabla_k L_{ji} - \nabla_j L_{ki} + \nabla_k M_{ji} - \nabla_j M_{ki} + M_{kx} M_{jix} - M_{jx} M_{kix} = 0 \tag{4.15}$$

by virtue of (4.10).

We substitute finally (4.2), (4.13) and (4.14) into (3.13) and find

$$\begin{aligned} &\nabla_k M_{jx} - \nabla_j M_{kx} - M_k{}^a{}_x L_{aj} + M_j{}^a{}_x L_{ak} - M_k{}^a{}_x M_{aj} + M_j{}^a{}_x M_{ak} + M_{ky} L_{jxy} - M_{jy} L_{kxy} \\ &\qquad + (\nabla_k L_{jxy} - \nabla_j M_{kxy} + M_k{}^a{}_x M_{ajy} - M_j{}^a{}_x M_{aky} + L_{kzy} L_{jxz} - L_{jzy} L_{kxz}) H_y = 0 \end{aligned}$$

or

$$\begin{aligned} &\nabla_k M_{jx} - \nabla_j M_{kx} - M_k{}^a{}_x L_{aj} + M_j{}^a{}_x L_{ak} - M_k{}^a{}_x M_{aj} + M_j{}^a{}_x M_{ak} \\ &\qquad + M_{ky} L_{jxy} - M_{jy} L_{kxy} = 0 \end{aligned} \tag{4.16}$$

by virtue of (4.12).

Thus we have seen that the set of equations (3.7), (3.8), (3.9), (3.12) and (3.13) is equivalent to the set of equations (4.5), (4.10), (4.12), (4.15) and (4.16). But the set of equations (4.5), (4.10), (4.12), (4.15) and (4.16) is the condition that an $n$-dimensional Riemannian space with tensors $g_{ji}$ $M_{jix}$ and $L_{jxy}$ is imbedded in an $N$-dimensional Euclidean space in such a way that $\rho^2 g_{ji}$, $\rho M_{jix}$ and $L_{jxy}$ are respectively the first, second and third fundamental quantities [4,5]. Thus we have

THEOREM. *Suppose that there are given, in an n-dimensional Riemannian space with fundamental tensor* $g_{ji}$, $N-n$ *symmetric tensors* $H_{jix}$ *and* $\frac{1}{2}(N-n)(N-n-1)$ *vectors* $L_{jxy} = -L_{jyx}$ *which satisfy the conditions* (3.7), (3.8), (3.9), (3.12) *and* (3.13) *or the conditions* (4.5), (4.10), (4.12), (4.15) *and* (4.16), *then the Riemannian space* $V_n$ *can be imbedded either in an N-dimensional*

*Euclidean space in such a way that* $\rho^2 g_{ji}$, $\rho M_{jix} = \rho\left(H_{jix} - \frac{1}{n} g^{cb} H_{cbx} g_{ji}\right)$ *and* $L_{jxy}$ *are respectively first, second and third conformal fundamental quantities of the imbedded space, or in an N-dimensional conformally Euclidean space in such a way that* $g_{ji}$, $H_{jix}$ *and* $L_{jxy}$ *are respectively first, second and third fundamental quantities of the imbedded space.*

## Bibliography

[1] Blum, R., The metric of a conformally euclidean space referred to a subspace. Trans. Royal Soc. Canada, Third Ser., Sec. III, 49 (1955).

[2] ——, The fundamental equations of a Riemannian space imbedded in a conformally euclidean space. To appear.

[3] Yano, K., Sur quelques propriétés conformes de $V_l$ dans $V_m$ dans $V_n$. Proc. Imp. Acad. Tokyo **16** (1940), 83–86.

[4] Yano, K., and Y. Muto, Sur le théorème fondamental dans la géométrie conforme des sous-espaces riemanniens. Proc. Phys.-Math. Soc. Japan 24 (1942), 437–449.

[5] ——, Note sur le théorème fondamental dans la géométrie conforme des sous-espaces riemanniens. Proc. Japan Acad. 12 (1946), 338–342.

Department of Mathematics, Tokyo Institute of Technology, and
Department of Mathematics, University of Saskatchewan, Saskatoon, Canada.