ON THE SYSTEM OF INTEGRAL EQUATIONS OF VOLTERRA TYPE WITH INFINITELY MANY UNKNOWN FUNCTIONS

BY TAKAO SUZUKI

The purpose of this paper is to consider the continuous solutions of an enumerably infinite system of integral equations of Volterra type with singularity; that is, the equations

$$\text{(A)}\qquad \begin{aligned} xU_j(x) &= \int_0^x \sum_k a_{jk}(x, t) U_k(t)\, dt + b_j(x), \quad \text{or equivalently,} \\ &= \int_0^x F_j(x, t, U_1(t), U_2(t), \cdots)\, dt + b_j(x) \quad (j = 1, 2, \cdots, \infty). \end{aligned}$$

Here we shall define, whenever

$$xU(x) = \int_0^x F(x, t, U(t))\, dt,$$

that

$$U(0) = \lim_{x \to 0} \frac{1}{x} \int_0^x F(x, t, U(t))\, dt.$$

In this paper we shall discuss the existence of continuous solutions of the above system by an argument similar to that used by Pogorzelski in [2], and apply the result to differential and integral equations.

In the first place, we shall state the following Property-N of normal-determinant and the theorem on which we base our argument.

Normal-determinant (N-determinant) [3]. An infinite determinant

$$|(\mathrm{A})| = |(\delta_{jk} + a_{jk})| \quad (j, k = 1, 2, \cdots),$$

where δ_{jk} is the Kronecker symbol, is called an normal- or an N-determinant if $S = \sum_{j,k} |a_{jk}|$ converges. The fundamental theorem on the solution of an infinite system of linear equations reads as follows:

Property-N: In the infinite system of linear equations

$$\sum_k (\delta_{jk} + a_{jk}) x_k = b_j \quad (j = 1, 2, \cdots),$$

suppose that the determinant $|(\mathrm{A})|$ is normal and distinct from zero, and that $|b_j| < b$ $(0 < b < \infty;\ j = 1, 2, \cdots)$. Then among all bounded sequences of numbers $(x_1, x_2, \cdots)$ there exists one and only one solution given by

$$x_j = \sum_k b_k |(D_{kj})| / |(\mathrm{A})| \quad (j = 1, 2, \cdots),$$

where $|(D_{kj})|$ is the co-factor of $\delta_{kj} + a_{kj}$ in $|(\mathrm{A})|$ for every k.

Received September 2, 1960.

Schauder-Theorem [5]. If, in a Banach space, a continuous operation transforms a bounded, closed and convex set of points into its compact subset, then there exists at least one point invariant by this operation.

§ 1. The main theorem.

THEOREM 1. *Let the system* (A) *be given.*

Hypotheses. (i) $a_{jk}(x, t)$ $(j, k=1, 2, \cdots)$ *are given by power series expansions* $a_{jk}(x, t)=\sum_{\alpha,\beta\geqq 0}a_{jk}^{\alpha\beta}x^{\alpha}t^{\beta}$ *convergent for* $|x|<r$, $|t|<r$. (i′) $\sum_k|a_{jk}(x, t)|<\Gamma$, ($\Gamma$: *a finite constant*). (ii) $F_j(x, t, u_1, u_2, \cdots)$ *are continuous in a region* $|x|\leqq r$, $|t|\leqq r$, $|u_1|\leqq R$, $|u_2|\leqq R$, $\cdots$, *i.e., given any* $\varepsilon>0$, *we can find a natural number* $N(\varepsilon)$ *and a positive number* $\eta(\varepsilon)$ *both depending on* ε *but not on* x, t, u_1, u_2, $\cdots$ *such that*

$$|F_j(x, t, u_1, u_2, \cdots)-F_j(x_0, t_0, u_1^0, u_2^0, \cdots)|<\varepsilon$$

for all F_j, *if* $|x-x_0|<\eta$, $|t-t_0|<\eta$, $|u_\nu-u_\nu^0|<\eta$ $(\nu=1, 2, \cdots, N)$; *and* $b_j(x)$ *are given by power series expansions* $\sum_{n=0}^{\infty}b_j^{(n)}x^n$ *convergent for* $|x|<r$. (iii) *There exist finite positive constants* $K^{(n)}$ *and* $C^{(n)}$ *such that* $\sum_k\sum_{\alpha+\beta=n}|a_{jk}^{\alpha\beta}|<K^{(n)}$, $|b_j^{(n)}|<C^{(n)}$. (iv) $|(\delta_{jk}-a_{jk}^{00})|$ *is normal and* $|(\delta_{jk}-a_{jk}^{00}/n)|$ $(n=1, 2,\cdots)$, *are distinct from zero.*

Conclusion. There exists one and only one continuous solution $U(x)=(U_1(x), U_2(x), \cdots)$ *of the system* (A) *and each element of* $U(x)$ *is given by a power series*

$$(*)\qquad U_j(x)=\sum_{n=0}^{\infty}p_j^{(n)}x^n \quad (j=1, 2, \cdots)$$

convergent for $|x|<r^*$, *where* r^* *is a certain positive constant.*

Proof. The existence proof proceeds as follows: Firstly, we show by an actual construction that there exists a uniquely determined formal power series satisfying (A). Then, we prove that this formal solution is convergent for $|x|<r^*$. We shall search for such a power series (*). Now the formal series for $U(x)$ substituted into the system (A) yields

$$(\mathrm{E})\qquad \begin{aligned} &p_j^{(0)}x+p_j^{(1)}x^2+\cdots+p_j^{(n)}x^{n+1}+\cdots \\ &=\sum_k a_{jk}^{00}p_k^{(0)}x+\sum_k(a_{jk}^{00}p_k^{(1)}/2+a_{jk}^{10}p_k^{(0)}+a_{jk}^{01}p_k^{(1)}/2)x^2+\cdots \\ &\quad+(\sum_k a_{jk}^{00}p_k^{(n)}/(n+1))x^{n+1}+\mathfrak{p}_j^{(n)}(a_{jk}^{\alpha\beta};\ p_k^{(\nu)})x^{n+1}+\cdots \\ &\quad+b_j^{(1)}x+b_j^{(2)}x^2+\cdots+b_j^{(n)}x^n+\cdots, \end{aligned}$$

where $\mathfrak{p}_j^{(n)}$ is a polynomial with respect to $a_{jk}^{\alpha\beta}$ $(0\leqq\alpha+\beta<n)$ and $p_k^{(\nu)}$ $(0\leqq\nu<n)$. For (E) to hold formally, we must have

$$(\mathrm{E}0)\qquad \sum_k(\delta_{jk}-a_{jk}^{00})p_k^{(0)}=b_j^{(1)}$$

and

$$(\mathrm{E}n)\qquad \sum_k(\delta_{jk}-a_{jk}^{00}/(n+1))p_k^{(n)}=\mathfrak{p}_j^{(n)}(a_{jk}^{\alpha\beta};\ p_k^{(\nu)})+b_j^{(n+1)}.$$

These linear equations with respect to $p_j^{(n)}$ form a recursive system which can be solved uniquely for $p_j^{(0)}, p_j^{(1)}, \cdots$ by the Property-N and the hypothesis (iv); and an easy induction proves, by (iii), (iv) and Property-N, that there exist constants $\overline{K}^{(n)}$ such that

$$|p_j^{(n)}| < \overline{K}^{(n)} \quad (j = 1, 2, \cdots; \ n = 0, 1, 2, \cdots).$$

Hence the formal solution of (A) exists and is unique.

We are now to prove the continuity of the formal power series. Put for $n \geqq \nu$ ($[\Gamma] + 1 \geqq$ integer $\nu > \Gamma$),

$$P_{jn}(x) = p_j^{(0)} + p_j^{(1)} x + \cdots + p_j^{(n-1)} x^{n-1} \quad (j = 1, 2, \cdots), \tag{1}$$

so that

$$U_j(x) = P_{jn}(x) + z_j \quad (j = 1, 2, \cdots). \tag{2}$$

Changing the variables $U_j(x)$ to z_j, we have the system of equations

$$x z_j(x) = \int_0^x \sum_k a_{jk}(x, t) z_k(t)\, dt + b_{jn}(x) \quad (j = 1, 2, \cdots), \tag{3}$$

where

$$b_{jn}(x) = \int_0^x \sum_k a_{jk}(x, t) P_{kn}(t)\, dt + b_j(x) - x P_{jn}(x);$$

and $b_{jn}(x)$ is clearly continuous for $|x| \leqq r$ and $b_{jn}(0) = 0$. By the above assumption, the equation (3) possesses a formal solution

$$z_j(x) \sim p_j^{(n)} x^n + p_j^{(n+1)} x^{n+1} + \cdots + p_j^{(m)} x^m + \cdots \quad (j = 1, 2, \cdots).$$

When we substitute $z_j(x)$ in (3), each term of (3), except $b_{jn}(x)$, has no term whose order is lower than $n+1$. Hence $b_{jn}(x)$ itself also enjoys the same property. Therefore there exists a constant B_n such that $|b_{jn}(x)| \leqq B_n |x|^{n+1}$ for $|x| \leqq \bar{r}$.

Now we shall consider an infinite system of integral equations of infinitely many unknown functions

$$\varphi_1(x), \ \varphi_2(x), \ \cdots, \ \varphi_n(x), \ \cdots \tag{4}$$

of Volterra type as follows:

$$\begin{aligned} x\varphi_j(x) &= \int_0^x \sum_k a_{jk}(x, t) \varphi_k(t)\, dt + b_{jn}(x) \\ &= \int_0^x F_j(x, t, \varphi_1(t), \varphi_2(t), \cdots)\, dt + b_{jn}(x) \quad (j = 1, 2, \cdots). \end{aligned} \tag{3'}$$

By hypothesis (ii), the functions $F_j(x, t, u_1, u_2, \cdots)$ are continuous at every point $(x, t, u_1, u_2, \cdots)$ of the region

$$\Omega: \ |x| \leqq r, \ |t| \leqq r, \ |u_1| \leqq R, \ |u_2| \leqq R, \ \cdots. \tag{5}$$

And, moreover, by (i′), the functions F_j are uniformly bounded, i.e., there exists a positive number M such that all the functions F_j satisfy the inequalities

$$|F_j| \leqq M \quad (j = 1, 2, \cdots)$$

on Ω. Let a space Π be composed of points $A(x, t, u_1, u_2, \cdots)$ where $(x, t, u_1, u_2, \cdots)$ is an arbitrary point of Ω, and define the distance $\rho(A, A')$ between the points

$$A(x, t, u_1, u_2, \cdots), \quad A'(x', t', u_1', u_2', \cdots) \tag{6}$$

of the space Π by

$$\rho(A, A') = |x - x'| + |t - t'| + \sum_\nu \varepsilon_\nu |u_\nu - u_\nu'|$$

where $\{\varepsilon_\nu\}$ is a sequence of positive numbers such that $\sum_\nu \varepsilon_\nu$ converges; this sequence will be fixed once for all. It is clear that the set (5) is bounded in the space Π. Then, under these assumptions, the results of the Pogorzelski's paper [2] read as follows:

P-1°. The set (5) is compact in the space Π.

P-2°. Each function F_j is uniformly continuous in Ω; that is, given $\varepsilon > 0$, every pair of elements, A and A' in Ω, satisfies the inequality $|F_j(A) - F_j(A')| < \varepsilon$ if $\rho(A, A') < \eta$, where η is dependent only on j and ε.

In the study of the system (3′), we shall consider the function-space E^∞ whose points are all the sequences $(\varphi_1(x), \varphi_2(x), \varphi_3(x), \cdots)$, in symbols $U(\varphi_\nu)$ or $V(\varphi_\nu)$, where $\varphi_j(x)$ is defined and continuous for $|x| \leqq r$ and $|\varphi_j(x)| \leqq R'_U$. Now define the distance $\delta(U, V)$ between two points, $U(\varphi_\nu)$ and $V(g_\nu)$, of the space E^∞ by a formula

$$\delta(U, V) = \sum_\nu \varepsilon_\nu \sup |\varphi_\nu(x) - g_\nu(x)|. \tag{7}$$

The norm of the point $U(\varphi_\nu)$ is defined by the distance between $U(\varphi_\nu)$ and the origin $\Theta(0, 0, 0, \cdots)$:

$$\|U\| = \delta(U, \Theta) = \sum_\nu \varepsilon_\nu \sup |\varphi_\nu(x)|. \tag{8}$$

The definition (7) clearly satisfies all the metric-properties.

P-3°. The space E^∞ is complete, and linear if we define addition and scalar multiplication in E^∞ as follows: When $U(\varphi_\nu)$ and $V(g_\nu)$ belong to E^∞, respectively

Sum: $U + V = \{\varphi_\nu + g_\nu\}$, Product: $\lambda U = \{\lambda \varphi_\nu\}$, ($\lambda$ being a real number).

P-4°. If S, in E^∞, is a closed set of points $U(\varphi_\nu)$ with $|\varphi_\nu(x)| \leqq A|x|^n \leqq R$, then S is convex. (Here assume $(n+1)B_n/(n+1-\Gamma) \leqq A$).

Now we shall consider the solution of the system (3):

$$x z_j(x) = \int_0^x F_j(x, t, z_1(t), z_2(t), \cdots)\, dt + b_{jn}(x) \quad (j = 1, 2, \cdots, \infty).$$

We define a correspondence $U(\varphi_\nu) \in S \rightarrow U'(\psi_\nu)$ by the formula

$$x \psi_j(x) = \int_0^x F_j(x, t, \varphi_1(t), \varphi_2(t), \cdots)\, dt + b_{jn}(x) \quad (j = 1, 2, \cdots, \infty). \tag{9}$$

Then the point $U'(\psi_\nu)$ satisfies

$$|\psi_j(x)| \leqq (\Gamma A/(n+1) + B_n)|x|^n \leqq A|x|^n$$

by (i′) and the assumption in P-4°. To prove, by Schauder-Theorem, the existence of a fixed point of the transformation (9) we must show that (9) is a continuous transformation of S into S' and that the transformed set S' is compact. Now suppose that a sequence of points $U_\nu(\varphi_1^\nu, \varphi_2^\nu, \cdots)$ of S converges in S to a point $U(\varphi_1, \varphi_2, \cdots)$. Then we have

$$\delta(U_\nu, U) = \sum_n \varepsilon_n \sup |\varphi_n^\nu - \varphi_n| \to 0 \text{ when } \nu \to \infty. \tag{10}$$

If $\{U'_\nu(\psi_1^\nu, \psi_2^\nu, \cdots)\}$ is the sequence of points corresponding to the points $\{U_\nu\}$ by (9) and if $U'(\psi_1, \psi_2, \cdots)$ corresponds to its limit point U, we obtain, for all positive integer (j, ν),

$$\psi_j^\nu(x) - \psi_j(x) = \frac{1}{x}\int_0^x (F_j(x, t, \varphi_1^\nu(t), \cdots) - F_j(x, t, \varphi_1(t), \cdots))\,dt.$$

By the assumption (10), all the differences $\varphi_n^\nu - \varphi_n$ tend uniformly to zero in Ω when $\nu \to \infty$; on the other hand, since every function F_j is uniformly continuous, we can assign to a given $\varepsilon > 0$ an index $N_j(\varepsilon)$ such that $|F_j(x, t, \varphi_1^\nu(t), \cdots) - F_j(x, t, \varphi_1(t), \cdots)| < \varepsilon$ if $\nu > N_j(\varepsilon)$, where $N_j(\varepsilon)$ may depend on j. Hence we have

$$|\psi_j^\nu(x) - \psi_j(x)| < \varepsilon \text{ if } \nu > N_j(\varepsilon). \tag{11}$$

Here notice that, for every j and ν, $|\varphi_j^\nu(x)| \leqq A|x|^n \leqq R$. For the moment, we assume that $\varphi_j^\nu(x)$ and $\psi_j(x)$ are continuous in x, that is, $U'_\nu = \{\psi_j^\nu(x)\}$ and $U' = \{\psi_j(x)\}$ belong to S; then we can choose an index $n_0(\varepsilon)$ which depends on ε only and is sufficiently large such that

$$\sum_{n=n_0(\varepsilon)+1}^{\infty} \varepsilon_n \sup |\psi_n^\nu - \psi_n| \leqq \varepsilon \text{ for every } \nu. \tag{12}$$

If we fix the index $n_0(\varepsilon)$, then we have, by (11),

$$\sum_{n=1}^{n_0(\varepsilon)} \varepsilon_n \sup|\psi_n^\nu - \psi_n| \leqq A\varepsilon, \quad A = \sum_{n=1}^{\infty} \varepsilon_n, \tag{13}$$

for $\nu > \bar{N}(\varepsilon)$, where $\bar{N}(\varepsilon) \geqq \max_{1 \leqq j \leqq n_0(\varepsilon)} N_j(\varepsilon)$. Therefore it follows, by (12) and (13), that

$$\delta(U'_\nu, U') = \sum_{n=1}^{\infty} \varepsilon_n \sup|\psi_n^\nu - \psi_n| \leqq (1+A)\varepsilon$$

if $\nu > \bar{N}(\varepsilon)$; consequently $\delta(U'_\nu, U') \to 0$ when $\nu \to \infty$, and the functional-transformation defined by (9), is continuous in the set S.

It now remains to prove that U'_ν and U' both $\in S$ and that the transformed set S' is compact. For this purpose, it suffices to verify that the components $\psi_j(x)$ of the points $\{\psi_j\}$ of S' are, for fixed j, equicontinuous; namely, that for each positive number ε there exists a positive number $\eta = \eta(j)$ such that $|x - x_0| < \eta$ implies

$$|\psi_j(x) - \psi_j(x_0)| < \varepsilon \text{ uniformly on } S'. \tag{14}$$

Now decompose the difference (14) into three parts such that

$$\begin{aligned}\psi_j(x) - \psi_j(x_0) = &\frac{1}{x}\int_0^x (F_j(x, t, \varphi_1(t), \cdots) - F_j(x_0, t, \varphi_1(t), \cdots))\,dt \\ &+ \frac{1}{x}\int_0^x F_j(x_0, t, \varphi_1(t), \cdots)\,dt - \frac{1}{x_0}\int_0^{x_0} F_j(x_0, t, \varphi_1(t), \cdots)\,dt \\ &+ b_{jn}(x)/x - b_{jn}(x_0)/x_0.\end{aligned} \tag{15}$$

By the continuity property of the function F_j, we have

$$|F_j(x, t, \varphi_1(t), \cdots) - F_j(x_0, t, \varphi_1(t), \cdots)| < \varepsilon/3$$

if $|x - x_0| < \eta_1 = \eta_1(\varepsilon)$. Hence we have easily

$$\left|\frac{1}{x}\int_0^x (F_j(x, t, \varphi_1(t), \cdots) - F_j(x_0, t, \varphi_1(t), \cdots))dt\right| < \varepsilon/3$$

if $|x - x_0| < \eta_1$; here we shall remark that

$$(1^\circ) \qquad \left|\frac{1}{x}\int_0^x F_j(x_0, t, \varphi_1(t), \cdots)dt - \frac{1}{x_0}\int_0^{x_0} F_j(x_0, t, \varphi_1(t), \cdots)dt\right| \leqq \Gamma A(|x|^n + |x_0|^n)/(n+1)$$

and

$$\left|\frac{1}{x}\int_0^x F_j(x_0, t, \varphi_1(t), \cdots)dt - \frac{1}{x_0}\int_0^{x_0} F_j(x_0, t, \varphi_1(t), \cdots)dt\right|$$

$$(2^\circ) \qquad \leqq \left|\frac{1}{x}\int_{x_0}^x F_j(x_0, t, \varphi_1(t), \cdots)dt\right| + \left|\left(\frac{1}{x} - \frac{1}{x_0}\right)\int_0^{x_0} F_j(x_0, t, \varphi_1(t), \cdots)dt\right|$$

$$\leqq 2M|x - x_0|/|x| \quad \text{if } x \neq 0.$$

Let $\eta_2(\varepsilon) = \eta_2$ be defined by $\Gamma A|2\eta_2|^n/(n+1) = \varepsilon/6$ and let x_0 stay within the interval $(-\eta_2, \eta_2)$. Then, by (1°),

$$\Gamma A(|x|^n + |x_0|^n)/(n+1) < \varepsilon/3 \quad \text{if } |x - x_0| < \eta_2.$$

On the other hand, in case x_0 lies outside of $(-\eta_2, \eta_2)$, we can choose, by (2°), a positive number $\eta_3 = \eta_3(\varepsilon) = \min(\eta_2/2, \varepsilon\eta_2/12M)$ satisfying the condition $2M|x - x_0|/|x| < \varepsilon/3$ when $|x - x_0| < \eta_3$. Accordingly we can assign to any positive number ε a positive number η_3 such that

$$\left|\frac{1}{x}\int_0^x F_j(x_0, t, \varphi_1(t), \cdots)dt - \frac{1}{x_0}\int_0^{x_0} F_j(x_0, t, \varphi_1(t), \cdots)dt\right| < \varepsilon/3$$

if $|x - x_0| < \eta_3$. And, furthermore, since $b_{jn}(x)/x$ is continuous, there exists a positive number $\eta_4 = \eta_4(j, \varepsilon)$ such that

$$|b_{jn}(x)/x - b_{jn}(x_0)/x_0| < \varepsilon/3 \quad \text{if } |x - x_0| < \eta_4.$$

Consequently we obtain a conclusion: Let $\eta = \eta(j) = \min(\eta_1, \eta_3, \eta_4)$. Then $|\psi_j(x) - \psi_j(x_0)| < \varepsilon$ if $|x - x_0| < \eta$, for every pair of point (x, x_0) in the interval $|x| \leqq \bar{r}$. Hence, for fixed j, the function $\psi_j(x)$ is equicontinuous. To every point $U'(\psi_1, \psi_2, \cdots)$ of the transformed set S' we shall assign a point $U''_\varepsilon(\psi_1, \psi_2, \cdots, \psi_{N_\varepsilon}, 0, 0, \cdots)$ such that

$$(16) \qquad \delta(U', U''_\varepsilon) = \sum_{n=N_\varepsilon+1}^{\infty} \varepsilon_n \sup|\psi_n| < \varepsilon$$

where ε is a positive number chosen arbitrarily. The set of all the points U''_ε is compact by the property (14) and Arzelà-Theorem. Hence we conclude, by Fréchet-Theorem [1], that the transformed set S' is compact. Therefore all the condition of Schauder-Theorm are varified, and so there exists at least one invariant point associated with (9).

Finally, we must prove that $U_j(x) = P_{jn}(x) + \varphi_j(x)$, where $\varphi_j(x)$ is an invariant function of (9), coincides with the formal solution. For the purpose, it suffices to prove that the system (3) has one and only one solution $\varphi_j(x)$ such that (1*) $\varphi_j(x)$ is continuous in the neighborhood of the origin and (2*) $\varphi_j(x) = O(x^\nu)$. Suppose that there exist two solutions $\varphi_j(x)$ and $\bar{\varphi}_j(x)$. Then putting $\phi(t) = \sup_j\{|\varphi_j(t) - \bar{\varphi}_j(t)|\}$, $\phi(t)$ is continuous for $t > 0$ and satisfies an inequality

$$(17) \qquad t\phi(t) \leqq \nu\int_0^t \phi(t)\,dt, \quad t > 0,\ \nu > 0.$$

Since (17) has one and only one solution $\phi(t) \equiv 0$ such that $\phi(t)$ is continuous and $0 \leqq \phi(t) = o(t^{\nu-1})$; (Cf. [4]). We must have

$$\varphi_j(x) \equiv \bar{\varphi}(x) \quad (j = 1, 2, \cdots).$$

Therefore we conclude that the system (3) has one and only one solution $\varphi_j(x)$ which is continuous in x in the neighborhood of the origin and $\varphi_j(x) = O(x^\nu)$; thus $U_j(x) = P_{jn}(x) + \varphi_j(x)$ is clearly continuous in x ($|x| \leqq r^*$). Since n ($\geqq \nu$) is arbitrary, the expansion of $U_j(x)$ coincides with the formal solution.

Thus the existence of continuous solution of the system (A) is completely proved.

§ 2. Applications to differential and integral equations.

We shall show how this result can be applied to an infinite system of differential and integral equations with an isolated singularity.

APPLICATION 1. Firstly we shall apply this result to the system

$$(\mathrm{a}) \qquad xy_j' = \sum_k a_{jk}(x)y_k + b_j(x) \quad (j = 1, 2, \cdots, \infty), \quad ('= d/dx),$$

with given initial conditions $y_j(0) = 0$; here $b_j(0) = 0$.

It is clear that the system (a) together with $y_j(0) = 0$ is equivalent to the system of integral equations

$$(\mathrm{A}^\circ) \qquad xy_j(x) = \int_0^x \sum_k (a_{jk}(t) + \delta_{jk})y_k(t)\,dt + b_j^*(x) \quad (j = 1, 2, \cdots, \infty),$$

with $y_j(0) = 0$, where

$$b_j^*(x) = \int_0^x b_j(t)\,dt = \sum_{n=2}^{\infty} b_j^{*(n)} x^n.$$

It is clear that the form of (A°) is a special form of (A). Hence we can apply the above result to the system (a). Formulating precisely, our theorem reads as follows:

THEOREM D. *We assume, for the system* (a), *that* (i) $a_{jk}(x)$ *are given by power series expansions* $\sum_{n=0}^\infty a_{jk}^{(n)}x^n$ *convergent for* $|x| < r$; (ii) $b_j(x)$ $(j = 1, 2, \cdots)$ *are given by power series expansions* $b_j(x) = \sum_{n=1}^\infty b_j^{(n)}x^n$ *convergent for* $|x| < r$; (iii) *there exist positive constants* Γ, $K^{(n)}$ *and* $C^{(n)}$ *such that* $\sum_k |a_{jk}(x)| < \Gamma$, $\sum_k |a_{jk}^{(n)}| < K^{(n)}$ $(n = 0, 1, 2, \cdots)$, $|b_j^{(n)}| < C^{(n)}$; (iv) $\sum_k (a_{jk}(x) + \delta_{jk})u_k$ $(j = 1, 2, \cdots)$

are continuous in a closed region Ω: $|x|\leqq r$, $|t|\leqq r$, $|u_1|\leqq R,\cdots$; (v) $|(\delta_{jk}-a_{jk}^{(0)})|$ *is normal and* $|(\delta_{jk}-a_{jk}^{(0)}/n)|\neq 0$ $(n=1,2,\cdots)$. *Then, there exists one and only one solution* $(y_1, y_2, \cdots)$ *of the system* (a), *and each element* y_j *is given by a power series*

$$(**)\qquad y_j=\sum_{n=1}^{\infty}p_j^{(n)}x^n$$

convergent for $|x|<r^*$, *where* r^* *is a certain positive constant.*

Proof. The argument is essentially similar to that in the proof of the main theorem. First of all, we observe that, by these assumptions, $a_{jk}(x)+\delta_{jk}$ and $\mathrm{b}_j^*(x)$ are given by certain power series expansions convergent for $|x|<r$, and hence are continuous in x $(|x|\leqq r)$, and that there exist positive finite constants Γ^*, $\bar{K}^{(n)}$ and $\bar{C}^{(n)}$ such that

$$\sum_k|a_{jk}(x)+\delta_{jk}|<\Gamma^*,\ \sum_k|a_{jk}^{(n)}+\delta_{jk}|<\bar{K}^{(n)},\ \text{and}\ |b_j^{*(n)}|<C^{(n)}.$$

Next, we shall remark that the above conditions imply the uniqueness of the formal solution as follows: The formal power series (**) for y_j substituted into the system (A°) yields

$$\text{(E)}\qquad \begin{aligned} &p_j^{(1)}x^2+p_j^{(2)}x^3+\cdots+p_j^{(n)}x^{n+1}+\cdots\\ &=\sum_k((a_{jk}^{(0)}+\delta_{jk})p_k^{(1)}/2)x^2+\sum_k((a_{jk}^{(0)}+\delta_{jk})p_k^{(2)}/3+a_{jk}^{(0)}p_k^{(1)}/3)x^3+\cdots\\ &\quad+(\sum_k(a_{jk}^{(0)}+\delta_{jk})p_k^{(n)}/(n+1)+\mathfrak{p}_j^{(n)}(a_{jk}^{(\nu)};\ p_k^{(\nu)}))x^{n+1}+\cdots\\ &\quad+(b_j^{(1)}/2)x^2+(b_j^{(2)}/3)x^3+\cdots+(b_j^{(n)}/(n+1))x^{n+1}+\cdots,\end{aligned}$$

where $\mathfrak{p}_j^{(n)}$ is a polynominal with respect to $a_{jk}^{(\nu)}$ $(0\leqq\nu<n)$ and $p_k^{(\nu)}$ $(0\leqq\nu<n)$. For (E) to hold formally, we must have

$$\text{(E1)}\qquad \sum_k(\delta_{jk}-(a_{jk}^{(0)}+\delta_{jk})/2)p_k^{(1)}=b_j^{(1)}/2,\ \text{or equivalently,}\ \sum_k(\delta_{jk}-a_{jk}^{(0)})p_k^{(1)}=b_j^{(1)}$$

and

$$\text{(E}n\text{)}\qquad \sum_k(\delta_{jk}-(a_{jk}^{(0)}+\delta_{jk})/(n+1))\,p_k^{(n)}=\mathfrak{p}_j^{(n)}(a_{jk}^{(\nu)};\ p_k^{(\nu)})+b_j^{(n)}/(n+1),$$

or equivalently,

$$\sum_k(\delta_{jk}-a_{jk}^{(0)}/n)p_k^{(n)}=(n+1)\mathfrak{p}_j^{(n)}(a_{jk}^{(\nu)};\ p_k^{(\nu)})/n+b_j^{(n)}/n.$$

These linear equations with respect to $p_j^{(n)}$ form a recursive system, which can be solved uniquely for $p_j^{(1)}$, $p_j^{(2)}$, $\cdots$ by the Property-N and the hypothesis (v); moreover, there exist constants $K^{*(n)}>0$ such as $|p_j^{(n)}|<K^{*(n)}$. All the conditions which assure the existence of a unique solution of (A°) are verified, and so the proof may be obtained by an argument similar to that of THEOREM 1.

REMARK. This result includes, as a special case, that of a finite system of the same form; thus, let, in the system

$$\text{(D–f)}\qquad xy_j'=\sum_{k=1}^{n}a_{jk}(x)y_k+b_j(x)\quad(j=1,2,\cdots,n),$$

$a_{jk}(x)$ and $b_j(x)$ be regular at the origin and let there exist at least one formal solution $y(x)$ whose elements are given by power series expansions

$$y_j = p_j^{(1)}x + p_j^{(2)}x^2 + \cdots + p_j^{(n)}x^n + \cdots \quad (j=1, 2, \cdots, n).$$

Then, these y_j $(j=1, 2, \cdots, n)$ are the solutions of (D–f), convergent for $|x| < r^*$, where r^* is a certain positive constant.

APPLICATION 2. We shall discuss the non-linear system

$$\text{(B)} \qquad xU_j(x) = \int_0^x f_j(x, t, U_1(t), U_2(t), \cdots)dt + b_j(x) \quad (j=1, 2, \cdots, \infty)$$

by an entirely similar argument as in THEOREM 1.

THEOREM 2. *Concerning the system* (B) *we make the following hypotheses.* (i) *The function* $f_j(x, t, u_1, u_2, \cdots)$ *and* $b_j(x)$ *are defined and continuous in a closed region* Ω: $|x| \leqq r$, $|t| \leqq r$, $|u_1| \leqq R, \cdots$. (ii) f_j $(j=1, 2, \cdots)$ *are of analytic type, i.e.,* $f_j = a_j^{(0)} + f_j^{(1)} + f_j^{(2)} + \cdots$, *where* $f_j^{(k)}$ *is the totality of the homogeneous terms in* $x, t, u_1, \cdots$ *of degree* k; *and let* $f_j^{(1)} = a_j^{100}x + a_j^{010}t + \sum_k a_{jk}^{001}u_k$. (iii) *There exists a constant* $\Gamma > 0$ *such that* $|f_j(x, t, u_1, \cdots) - f_j(x, t, \bar{u}_1, \cdots)| \leqq \Gamma \sup(|u_k - \bar{u}_k|)$ *in* Ω. (iv) *There exist finite positive constants* K, $K^{(n)}$ *and* $C^{(n)}$ *such that*

$$|a_j^{(0)}| < K, \quad \sum_k \sum_{\alpha+\beta+\cdots+\gamma=n} |a_{jk}^{\alpha\beta\cdots\gamma}| < K^{(n)}, \quad |b_j^{(n)}| < C^{(n)}.$$

(v) $|(\delta_{jk} - a_{jk}^{001})|$ *is normal and* $|(\delta_{jk} - a_{jk}^{001}/n)|$ $(n=1, 2, \cdots)$, *are distinct from zero.*

Then, there exists a unique solution $U(x)$ *of* (B) *in* $|x| \leqq r^*$, *where* r^* *is a positive constant, and each element of* $U(x)$ *is given by a power series*

$$(*) \qquad U_j(x) = \sum_{n=0}^{\infty} p_j^{(n)}x^n \quad (j=1, 2, \cdots)$$

convergent for $|x| < r^*$.

The proof of this theorem can be carried out by an argument similar to that of THEOREM 1. Here we shall only check the essential parts. Firstly, by a similar calculation as in THEOREM 1, we conclude that the formal solution of (B) exists and is unique; and that there exists a positive constant $K^{(n)}$ such as $|p_j^{(n)}| < K^{(n)}$. Next, put, for $n \geqq \nu$ $([\Gamma]+1 \geqq \nu > \Gamma)$,

$$(1) \qquad P_{jn}(x) = p_j^{(0)} + p_j^{(1)}x + \cdots + p_j^{(n-1)}x^{n-1} \quad (j=1, 2, \cdots),$$

so that

$$(2) \qquad U_j(x) = P_{jn}(x) + z_j \ (j=1, 2, \cdots) \text{ or } U(x) = P_n(x) + z.$$

By changing the variable $U_j(x)$ to z_j, we have the system of equations

$$(3) \qquad xz_j(x) = \int_0^x f_j(x, t, P_n(t) + z(t))dt + b_j(x) - xP_{jn}(x),$$

or equivalently,

$$xz_j(x)=\int_0^x(f_j(x,\ t,\ P_n(t)+z(t))-f_j(x,\ t,\ P_n(t)))\,dt$$
$$+\int_0^x f_j(x,\ t,\ P_n(t))\,dt+b_j(x)-xP_{jn}(x)$$
$$=\int_0^x F_j(x,\ t,\ z_1(t),\ z_2(t),\ \cdots)\,dt+b_{jn}(x),$$
$$b_{jn}(x)=\int_0^x f_j(x,\ t,\ P_n(t))\,dt+b_j(x)-xP_{jn}(x).$$

By our assumptions, $b_{jn}(x)$ is obviously continuous for $|x|\leqq r$ and $b_{jn}(0)=0$. Since the equation (3), by the assumption, possesses a formal solution

$$z_j(x)\sim p_j^{(n)}x^n+p_j^{(n+1)}x^{n+1}+p_j^{(m)}x^m+\cdots\quad (j=1,\ 2,\ \cdots),$$

when we substitute $z_j(x)$ into (3), each term of (3), except $b_{jn}(x)$, has no term whose order is lower than $n+1$. Hence $b_{jn}(x)$ itself also enjoys the same property. By taking into account the inequalities of the assumptions (i) and (iv), there exists a constant B_n (>0) such that

$$|b_{jn}(x)|\leqq B_n|x|^{n+1}\ \text{ for }\ |x|\leqq \bar{r}.$$

As in the proof of THEOREM 1, we shall consider an infinite system of integral equations of infinitely many unknown functions

$$(4)\qquad \varphi_1(x),\ \varphi_2(x),\ \varphi_3(x),\ \cdots,\ \varphi_n(x),\ \cdots$$

of Volterra type as follows:

$$(3^*)\qquad x\varphi_j(x)=\int_0^x F_j(x,\ t,\ \varphi_1(t),\ \varphi_2(t),\ \cdots)\,dt+b_{jn}(x)\quad (j=1,\ 2,\ \cdots),$$

where $F_j(x,\ t,\ u_1,\ u_2,\ \cdots)$ are the functions of infinitely many variables defined in a closed region

$$\Omega:\quad |x|\leqq r,\ \ |t|\leqq r,\ \ |u_1|\leqq R,\ \cdots.$$

By the hypothesis (i), if $P_n(\cdot)+u$, $P_n(\cdot)$ and u belong to Ω the functions $F_j(x,$ $t,\ u_1,\ u_2,\ \cdots)$ are continuous at every point of Ω; and, in view of the hypothesis (iv), the functions $F_j(x,t,u_1,u_2,\cdots)$ are bounded in Ω. Moreover, if we define the space Π, E^∞, S, S' and the metric in Π, E^∞ as are defined in THEOREM 1, it is clear that the Pogorzelski's results P-1°, P-2°, P-3° and P-4° are true as well. To each point $U(\varphi_\nu)\in S$ we shall assign a point $U'(\psi_\nu)\in E^\infty$ whose components are determined by the formula

$$(5)\qquad x\psi_j(x)=\int_0^x F_j(x,\ t,\ \varphi_1(t),\ \varphi_2(t),\ \cdots)\,dt+b_{jn}(x)\quad (j=1,\ 2,\ \cdots).$$

Then the transformed set S' is in S. In fact, by hypothesis (iii) and the assumption of P-4° (i.e., $(n+1)B_n/(n+1-\Gamma)\leqq A$), by use of (3)

$$|\psi_j(x)|\leqq\left|\frac{1}{x}\left(\int_0^x F_j(x,\ t,\ \varphi_1(t),\ \cdots)\,dt+b_{jn}(x)\right)\right|$$
$$\leqq\frac{1}{|x|}\left(\int_0^{|x|}\Gamma A|t|^n dt+B_n|x|^{n+1}\right)\leqq A|x|^n,$$

and, moreover, we can easily prove the continuity of $\psi_j(x)$; hence we have $U'(\psi_\nu) \in S$. The existence of a fixed point of the functional-transformation (5) will be derived, as in THEOREM 1, by an entirely similar argument. For, the argument based upon the linearity of $F_j(x, t, u_1, \cdots)$ in the proof of THEOREM 1, will be carried out in this case by the assumption (iii) and the formula (3). Furthermore, by the same process of methods as used in THEOREM 1, we conclude that there exists a unique continuous solution $U(x)$ of (B), each element of which is given by a power series

$$U_j(x) = p_j^{(0)} + p_j^{(1)}x + p_j^{(2)}x^2 + \cdots + p_j^{(n)}x^n + \cdots \quad (j = 1, 2, \cdots)$$

convergent for $|x| < r^*$ $(r^* > 0)$.

This theorem implies

COROLLARY. *Let the system*

$$\text{(b)} \qquad xU_j(x) = \int_0^x (F_j(t, U_1(t), U_2(t), \cdots) + U_j(t))\,dt \quad (j = 1, 2, \cdots)$$

be given. Let the hypothesis, except that pertaining to $b_j(x)$, similar to those described in THEOREM 2 *be satisfied. Then such a solution $U_j(x)$ as is defined in* THEOREM 2 *exists and is unique.*

This corollary is a special case of THEOREM 2; therefore we can easily prove, with minor changes in the proof of THEOREM 2, the existence and the uniqueness of continuous solution of (b).

APPLICATION 3. The above corollary can be applied to an infinite system of differential equations of Briot-Bouquet type; that is, to the equations

$$\text{(b*)} \qquad xy'_j = f_j(x, y_1, y_2, \cdots) \quad (j = 1, 2, \cdots, \infty),$$

with given initial conditions.

REMARK. In case of a finite system

$$\text{(D–B)} \qquad xy'_j = f_j(x, y_1, y_2, \cdots, y_n) \quad (j = 1, 2, \cdots, n),$$

if we assume that $f_j(x, y_1, y_2, \cdots)$ are analytic with respect to $x, y_1, y_2, \cdots, y_n$, $f_j(0, 0, \cdots, 0) = 0$ and that there exists a formal solution $y(x)$ whose elements $y_j(x)$ are given by power series

$$y_j(x) = \sum_{n=1}^{\infty} p_j^{(n)} x^n,$$

then the formal solutions are regular at the origin.

In conclusion, the author desires to express his hearty gratitude to Prof. K. Yoshida and Prof. M. Hukuhara who guided him in the preparation of this paper by many valuable suggestions and criticisms. The author is much obliged to Prof. T. Saito for his taking care of the publication of this paper.

REFERENCES

[1] FRÉCHET, M., Quelques propriétés des ensembles abstraits. Fund. Math. 12 (1928), 289-310.

[2] POGORZELSKI, W., Sur le système d'équations intégrales à une infinité de fonctions inconnues. Ann. Soc. Math. Polon. 2 (1955), 106-117.

[3] RIESZ, F., Les systèmes d'équations linéaires à une infinité d'inconnues. Paris, 1952, pp. 24-29.

[4] SATŌ, T., Sur l'équation intégrale $xu(x)=f(x)+\int_0^x K(x, t, u)\,dt$. Journ. Math. Soc. Japan 5 (1953), 145-153.

[5] SCHAUDER, J., Der Fixpunktsatz in Funktionalräumen. Studia Math. 2 (1930), 171-180.

[6] TAKESADA, T., On the singular point of integral equations of Volterra type. Journ. Math. Soc. Japan 7 (1955), 123-136.

DEPARTMENT OF MATHEMATICS, FACULTY OF ARTS AND SCIENCES,
IBARAKI UNIVERSITY.