A REMARK ON AN IRREGULAR SINGULAR POINT OF THE SECOND ORDER LINEAR DIFFERENTIAL EQUATION

By Tosiya Saito

1. Preliminaries. Let the system

$$\frac{dY}{dx} = Y \sum_{p=-s}^{\infty} T_p x^p \tag{1.1}$$

be given, where Y is an unknown n-n matrix and T_p are constant n-n matrices. Without loss of generality, we may suppose that the radius of convergence of the series in the right-hand member of (1.1) is greater than unity.

Let $\mathfrak{B}$ be the domain in the complex x-plane such that

$$0 < \rho_1 \leqq |x| \leqq \rho_2 < 1$$

and b be an arbitrary point of this domain. Then, after the notation of Lappo-Danilevsky [1], we denote by $Y(b|x)$ a solution of (1.1) satisfying the initial condition

$$Y = E \qquad \text{for} \qquad x = b,$$

where, as usual, E denotes a unit matrix.

Let C be a contour starting from b and turning around the origin in positive sense in the domain $\mathfrak{B}$. Carrying out the procedure of analytic continuation along this curve, $Y(b|x)$ will turn into a matrix $V(b)$ which is, in general, different from E.

To obtain the analytical expression of the solution of (1.1) valid in $\mathfrak{B}$, we must first obtain the eigenvalues of $V(b)$. If $x=0$ is a regular singular point of (1.1), these eigenvalues can easily be calculated by a well-known algebraic method. However, when $x=0$ is an irregular singular point, this method is inapplicable, and, in most of the cases, we have no means to calculate these eigenvalues as yet.

Lappo-Danilevsky [1] has tried to obtain the explicit form of the matrix $V(b)$, and succeeded to express it in power series of b which can be written in the form

$$V(b) = E + \sum_{\nu=1}^{\infty} \sum_{p_1,\cdots,p_\nu=-s}^{\infty} T_{p_1}\cdots T_{p_\nu} b^{p_1+\cdots+p_\nu+\nu} \sum_{\mu=0}^{\nu} \overset{*}{\alpha}{}^{(0)}_{p_1\cdots p_\mu} \sum_{\kappa=0}^{\nu-\mu} \alpha^{(\kappa)}_{p_{\mu+1}\cdots p_\nu} (2\pi i)^\kappa, \tag{1.2}$$

where $\alpha^{(\mu)}_{p_1\cdots p_\nu}$ and $\overset{*}{\alpha}{}^{(\mu)}_{p_1\cdots p_\nu}$ are defined by following complicated recurrence formulae:

$$\alpha^{(0)}_{p_1} = \frac{1}{p_1+1}, \quad \overset{*}{\alpha}{}^{(0)}_{p_1} = -\frac{1}{p_1+1} \quad (p_1+1 \neq 0),$$

$$\alpha^{(1)}_{p_1} = \begin{cases} 0 & (p_1+1 \neq 0), \\ 1 & (p_1+1=0), \end{cases} \quad \overset{*}{\alpha}{}^{(1)}_{p_1} = \begin{cases} 0 & (p_1+1 \neq 0), \\ -1 & (p_1+1=0), \end{cases}$$

Received August 17, 1960.

$$
\begin{aligned}
&\alpha^{(\nu)}_{p_1\cdots p_\nu} = \overset{*}{\alpha}{}^{(\nu)}_{p_1\cdots p_\nu} = 0 \quad (p_1+\cdots+p_\nu+\nu \neq 0),\\
&\alpha^{(\mu)}_{p_1\cdots p_\nu} = \frac{1}{p_1+\cdots+p_\nu+\nu}\Bigg[\alpha^{(\mu)}_{p_1\cdots p_{\nu-1}} - \frac{\mu+1}{p_1+\cdots+p_\nu+\nu}\alpha^{(\mu+1)}_{p_1\cdots p_{\nu-1}} + \cdots \\
&\qquad\qquad + (-1)^{\nu-\mu-1}\frac{(\mu+1)(\mu+2)\cdots(\nu-1)}{(p_1+\cdots+p_\nu+\nu)^{\nu-\mu-1}}\alpha^{(\nu-1)}_{p_1\cdots p_{\nu-1}}\Bigg],\\
&\overset{*}{\alpha}{}^{(\mu)}_{p_1\cdots p_\nu} = \frac{-1}{p_1+\cdots+p_\nu+\nu}\Bigg[\overset{*}{\alpha}{}^{(\mu)}_{p_2\cdots p_\nu} - \frac{\mu+1}{p_1+\cdots+p_\nu+\nu}\overset{*}{\alpha}{}^{(\mu+1)}_{p_2\cdots p_\nu} + \cdots \\
(1.3)\quad &\qquad\qquad + (-1)^{\nu-\mu-1}\frac{(\mu+1)(\mu+2)\cdots(\nu-1)}{(p_1+\cdots+p_\nu+\nu)^{\nu-\mu-1}}\overset{*}{\alpha}{}^{(\nu-1)]}_{p_2\cdots p_\nu}\Bigg]\\
&\qquad\qquad (\mu=0,\ 1,\ \cdots,\ \nu-1;\ p_1+\cdots+p_\nu+\nu \neq 0),\\
&\alpha^{(\mu)}_{p_1\cdots p_\nu} = \frac{1}{\mu}\alpha^{(\mu-1)}_{p_1\cdots p_{\nu-1}},\quad \overset{*}{\alpha}{}^{(\mu)}_{p_1\cdots p_\nu} = -\frac{1}{\mu}\overset{*}{\alpha}{}^{(\mu-1)}_{p_2\cdots p_\nu}\\
&\qquad\qquad (\mu=1,\ 2,\ \cdots,\ \nu;\ p_1+\cdots+p_\nu+\nu=0),\\
&\alpha^{(0)}_{p_1\cdots p_\nu}:\ \text{arbitrary}\quad (p_1+\cdots+p_\nu+\nu=0),\\
&\sum_{\mu=0}^{\nu}\overset{*}{\alpha}{}^{(0)}_{p_1\cdots p_\mu}\alpha^{(0)}_{p_{\mu+1}\cdots p_\nu} = 0 \quad (p_1+\cdots+p_\nu+\nu=0),\quad \text{where}\\
&\overset{*}{\alpha}{}^{(0)}_{p_1\cdots p_\mu} = 1 \quad \text{for}\quad \mu=0 \quad \text{and}\quad \alpha^{(0)}_{p_{\mu+1}\cdots p_\nu} = 1 \quad \text{for}\quad \mu=\nu.
\end{aligned}
$$

In what follows, we try to draw some conclusion about the irregular singular point of the second order linear differential equation availing the above stated result of Lappo-Danilevsky.

2. **Matrix $V(b)$ associated with the second order equation.** The equation we are to consider in this paper is the second order linear equation of the form

$$
\frac{d^2y}{dx^2} = f(x)y, \qquad f(x) = \sum_{p=-s}^{\infty} f_p x^p,
$$

where the radius of convergence of $\sum f_p x^p$ is supposed to be greater than unity. Putting

$$
y = y_1, \quad \frac{dy}{dx} = y_2, \quad \vec{y} = (y_1,\ y_2),
$$

this equation will be transformed into a linear system

$$
\frac{d\vec{y}}{dx} = \vec{y}\sum_{p=-s}^{\infty} T_p x^p, \quad T_p = \begin{bmatrix} 0 & f_p \\ \delta_{0p} & 0 \end{bmatrix}, \tag{2.1}
$$

where δ_{0p} denotes a Kronecker's symbol. An associated matrix equation will be written as

$$
\frac{dY}{dx} = Y\sum_{p=-s}^{\infty} T_p x^p. \tag{2.2}
$$

We shall apply to this system the result of Lappo-Danilevsky stated in the preceding section.

Let

$$(y_1^{(i)},\ y_2^{(i)}),\quad i=1,\ 2,$$

be the fundamental system of solutions of (2.1) such that

$$y_1^{(1)}(b)=1,\quad y_2^{(1)}(b)=0,\quad y_1^{(2)}(b)=0,\quad y_2^{(2)}(b)=1.$$

Then, naturally, the solution $Y(b|x)$ of (2.2) will be

$$Y(b|x)=\begin{bmatrix} y_1^{(1)} & y_2^{(1)} \\ y_1^{(2)} & y_2^{(2)} \end{bmatrix}.$$

Since

$$\begin{aligned}\frac{d|Y(b|x)|}{dx} &= (y_1^{(1)}y_2^{(2)}-y_1^{(2)}y_2^{(1)})', \\ &= y_1^{(1)\prime}y_2^{(2)}+y_1^{(1)}y_2^{(2)\prime}-y_1^{(2)\prime}y_2^{(1)}-y_1^{(2)}y_2^{(1)\prime} \\ &= y_2^{(1)}y_2^{(2)}+fy_1^{(1)}y_1^{(2)}-y_2^{(2)}y_2^{(1)}-fy_1^{(2)}y_1^{(1)}=0,\end{aligned}$$

and

$$|Y(b|b)|=|E|=1,$$

we have

$$|Y(b|x)|\equiv 1.$$

Therefore, from the definition of $V(b)$, we have

$$|V(b)|=1,$$

which shows that the product of the eigenvalues of $V(b)$ is equal to 1. Hence these eigenvalues can be written as

$$e^{2\pi i\lambda},\quad e^{-2\pi i\lambda}.$$

Thus, to calculate the eigenvalues of $V(b)$, it is sufficient to know the value of

$$\operatorname{trace} V(b)=e^{2\pi i\lambda}+e^{-2\pi i\lambda}.$$

3. Calculation of trace $V(b)$. From formula (1.2), we immediately have

$$\begin{aligned}\operatorname{trace} V(b) = 2 + \sum_{\nu=1}^{\infty}\sum_{p_1,\cdots,p_\nu=-s}^{\infty} &\operatorname{trace}(T_{p_1}\cdots T_{p_\nu})b^{p_1+\cdots+p_\nu+\nu} \\ &\times \sum_{\mu=0}^{\nu}{}^{*}\alpha^{(0)}_{p_1\cdots p_\mu}\sum_{\kappa=0}^{\nu-\mu}\alpha^{(\kappa)}_{p_{\mu+1}\cdots p_\nu}(2\pi i)^{\kappa}.\end{aligned}$$

As is well known, the eigenvalues of $V(b)$ and, consequently, trace $V(b)$ must be independent of b. Therefore, in the expression of trace $V(b)$ written above, all the terms containing the positive powers of b must vanish. Hence we have

$$\operatorname{trace} V(b)=2+\sum_{\nu=1}^{\infty}\sum_{p_1+\cdots+p_\nu=-\nu}\operatorname{trace}(T_{p_1}\cdots T_{p_\nu})\sum_{\mu=0}^{\nu}{}^{*}\alpha^{(0)}_{p_1\cdots p_\mu}\sum_{\kappa=0}^{\nu-\mu}\alpha^{(\kappa)}_{p_{\mu+1}\cdots p_\nu}(2\pi i)^{\kappa}.$$

As can easily be shown by induction, product of n matrices of the form $\begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix}$ is of the form

$$\begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix} \text{ if } n \text{ is even,}\quad \begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix} \text{ if } n \text{ is odd.}$$

Therefore, if ν is an odd number, we have

$$\operatorname{trace}(T_{p_1}\cdots T_{p_\nu})=0.$$

So we have only to consider the case when $\nu=2n$.

Since

$$T_{p_j}T_{p_{j+1}}=\begin{bmatrix}0 & f_{p_j}\\ \delta_{0p_j} & 0\end{bmatrix}\begin{bmatrix}0 & f_{p_{j+1}}\\ \delta_{0p_{j+1}} & \end{bmatrix}=\begin{bmatrix}\delta_{0p_{j+1}}f_{p_j} & 0\\ 0 & \delta_{0p_j}f_{p_{j+1}}\end{bmatrix},$$

we have

$$\begin{aligned}T_{p_1}\cdots T_{p_{2n}}&=(T_{p_1}T_{p_2})\cdots(T_{p_{2n-1}}\ T_{p_{2n}})\\ &=\begin{bmatrix}\delta_{0p_2}f_{p_1} & 0\\ 0 & \delta_{0p_1}f_{p_2}\end{bmatrix}\cdots\begin{bmatrix}\delta_{0p_{2n}}f_{p_{2n-1}} & 0\\ 0 & \delta_{0p_{2n-1}}f_{p_{2n}}\end{bmatrix}\\ &=\begin{bmatrix}\delta_{0p_2}\delta_{0p_4}\cdots\delta_{0p_{2n}}f_{p_1}f_{p_3}\cdots f_{p_{2n-1}} & 0\\ 0 & \delta_{0p_1}\delta_{0p_3}\cdots\delta_{0p_{2n-1}}f_{p_2}f_{p_4}\cdots f_{p_{2n}}\end{bmatrix}.\end{aligned}$$

Hence

$$\begin{aligned}&\operatorname{trace}(T_{p_1}\cdots T_{p_{2n}})\\ &=\delta_{0p_2}\delta_{0p_4}\cdots\delta_{0p_{2n}}f_{p_1}f_{p_3}\cdots f_{p_{2n-1}}+\delta_{0p_1}\delta_{0p_3}\cdots\delta_{0p_{2n-1}}f_{p_2}f_{p_4}\cdots f_{p_{2n}}.\end{aligned}$$

Since the summation is to be made for the indices $p_1,\cdots,p_{2n}$ for which

$$p_1+\cdots+p_{2n}=-2n,$$

$p_1,\cdots,p_{2n}$ can never vanish simultaneously. In other words, at least one of the indices $p_1,\cdots,p_{2n}$, say p_j, is different from zero. Then

$$\operatorname{trace}(T_{p_1}\cdots T_{p_{2n}})=\begin{cases}\delta_{0p_2}\delta_{0p_4}\cdots\delta_{0p_{2n}}f_{p_1}f_{p_3}\cdots f_{p_{2n-1}} & \text{if } j \text{ is odd,}\\ \delta_{0p_1}\delta_{0p_3}\cdots\delta_{0p_{2n-1}}f_{p_2}f_{p_4}\cdots f_{p_{2n}} & \text{if } j \text{ is even.}\end{cases}$$

Therefore the non-vanishing value of $\operatorname{trace}(T_{p_1}\cdots T_{p_{2n}})$ is obtained only if one of the following situations is realized:

1) $p_2=p_4=\cdots=p_{2n}=0$, 2) $p_1=p_3=\cdots=p_{2n-1}=0$.

Consequently

$$\begin{aligned}&\operatorname{trace} V(b)\\ &=2+\sum_{n=1}^{\infty}\Big[\sum_{p_1+p_3+\cdots+p_{2n-1}=-2n}f_{p_1}f_{p_3}\cdots f_{p_{2n-1}}\sum_{\mu=0}^{2n}{}^{*}\alpha^{(0)}_{p_10p_3\cdots p_\mu}\sum_{\kappa=0}^{2n-\mu}\alpha^{(\kappa)}_{p_{\mu+1}\cdots p_{2n-1}0}(2\pi i)^\kappa\\ &\qquad+\sum_{p_2+p_4+\cdots+p_{2n}=-2n}f_{p_2}f_{p_4}\cdots f_{p_{2n}}\sum_{\mu=0}^{2n}{}^{*}\alpha^{(0)}_{0p_20\cdots p_\mu}\sum_{\kappa=0}^{2n-\mu}\alpha^{(\kappa)}_{p_{\mu+1}\cdots 0p_{2n}}(2\pi i)^\kappa\Big].\end{aligned}\tag{3.1}$$

4. **Further reduction.** Written in detail, the term

$$\sum_{\mu=0}^{2n}{}^{*}\alpha^{(0)}_{p_1\cdots p_\mu}\sum_{\kappa=0}^{2n-\mu}\alpha^{(\kappa)}_{p_{\mu+1}\cdots p_{2n}}(2\pi i)^\kappa$$

in the expression of $V(b)$ will be

$$\begin{aligned}&(\alpha^{(0)}_{p_1\cdots p_{2n}}+\overset{*}{\alpha}{}^{(0)}_{p_1}\alpha^{(0)}_{p_2\cdots p_{2n}}+\overset{*}{\alpha}{}^{(0)}_{p_1p_2}\alpha^{(0)}_{p_3\cdots p_{2n}}+\cdots+\overset{*}{\alpha}{}^{(0)}_{p_1\cdots p_{2n}})\\ &+(2\pi i)(\alpha^{(1)}_{p_1\cdots p_{2n}}+\overset{*}{\alpha}{}^{(0)}_{p_1}\alpha^{(1)}_{p_2\cdots p_{2n}}+\overset{*}{\alpha}{}^{(0)}_{p_1p_2}\alpha^{(1)}_{p_3\cdots p_{2n}}+\cdots+\overset{*}{\alpha}{}^{(0)}_{p_1\cdots p_{2n-1}}\alpha^{(1)}_{p_{2n}})\\ &+\cdots\\ &+(2\pi i)^{2n-1}(\alpha^{(2n-1)}_{p_1\cdots p_{2n}}+\overset{*}{\alpha}{}^{(0)}_{p_1}\alpha^{(2n-1)}_{p_2\cdots p_{2n}})\\ &+(2\pi i)^{2n}\alpha^{(2n)}_{p_1\cdots p_{2n}}.\end{aligned}$$

Since $p_1+\cdots+p_n=-2n$, we have

$$\alpha^{(0)}_{p_1\cdots p_{2n}}+\overset{*}{\alpha}{}^{(0)}_{p_1}\alpha^{(0)}_{p_2\cdots p_{2n}}+\cdots+\overset{*}{\alpha}{}^{(0)}_{p_1\cdots p_{2n}}=0$$

according to the last relation of (1.3). Hence, renumbering the indices, we have

$$\text{trace } V(b)=2+\sum_{n=1}^{\infty}\sum_{p_1+\cdots+p_n=-2n}A_{p_1\cdots p_n}f_{p_1}\cdots f_{p_n},$$

$$A_{p_1\cdots p_n}=\sum_{k=1}^{2n}\alpha^{(k)}_{p_1\cdots p_n}\,(2\pi i)^k,$$

$$\alpha^{(1)}_{p_1\cdots p_n}=\alpha^{(1)}_{p_10\cdots p_n0}+\overset{*}{\alpha}{}^{(0)}_{p_1}\alpha^{(1)}_{0\cdots p_n0}+\cdots+\overset{*}{\alpha}{}^{(0)}_{p_10\cdots p_n}\alpha^{(1)}_{0}$$

$$+\alpha^{(1)}_{0p_1\cdots 0p_n}+\overset{*}{\alpha}{}^{(0)}_{0}\alpha^{(1)}_{p_1\cdots 0p_n}+\cdots+\overset{*}{\alpha}{}^{(0)}_{0p_1\cdots 0}\alpha^{(1)}_{p_n},$$

$$\alpha^{(2)}_{p_1\cdots p_n}=\alpha^{(2)}_{p_10\cdots p_n0}+\alpha^{(0)}_{p_1}\overset{*}{\alpha}{}^{(2)}_{0\cdots p_n0}+\cdots+\overset{*}{\alpha}{}^{(0)}_{p_10\cdots 0}\alpha^{(2)}_{p_n0}$$

$$+\alpha^{(2)}_{0p_1\cdots 0p_n}+\overset{*}{\alpha}{}^{(0)}_{0}\alpha^{(2)}_{p_1\cdots 0p_n}+\cdots+\overset{*}{\alpha}{}^{(0)}_{0p_1\cdots p_{n-1}}\alpha^{(2)}_{0p_n},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$\alpha^{(2n-1)}_{p_1\cdots p_n}=\alpha^{(2n-1)}_{p_10\cdots p_n0}+\overset{*}{\alpha}{}^{(0)}_{p_1}\alpha^{(2n-1)}_{0\cdots p_n0}+\alpha^{(2n-1)}_{0p_1\cdots 0p_n}+\overset{*}{\alpha}{}^{(0)}_{0}\alpha^{(2n-1)}_{p_1\cdots 0p_n},$$

$$\alpha^{(2n)}_{p_1\cdots p_n}=\alpha^{(2n)}_{p_10\cdots p_n0}+\alpha^{(2n)}_{0p_1\cdots 0p_n}.$$

According to (1.3), the relation $p_1+\cdots+p_n=-2n$ implies

$$\alpha^{(2n)}_{p_10\cdots p_n0}=\frac{1}{2n}\alpha^{(2n-1)}_{p_10\cdots p_n}=0,$$

$$\alpha^{(2n)}_{0p_1\cdots 0p_n}=\frac{1}{2n}\alpha^{(2n-1)}_{0p_1\cdots 0p_{n-1}}$$

$$=\begin{cases}0 & \text{if } p_1+\cdots+p_{n-1}\neq-(2n-1),\\ \dfrac{1}{2n(2n-1)}\alpha^{(2n-2)}_{0p_1\cdots 0p_{n-1}}=0 & \text{if } p_1+\cdots+p_{n-1}=-(2n-1).\end{cases}$$

Whence follows that

$$\alpha^{(2n)}_{p_1\cdots p_n}=0. \tag{4.2}$$

Further, from the relation

$$\text{trace } V(b)=e^{2\pi i\lambda}+e^{-2\pi i\lambda}=e^{-2\pi i\lambda}+e^{2\pi i\lambda}=\text{trace } V(b)^{-1},$$

we have

$$\text{trace } V(b)=\frac{1}{2}(\text{trace } V(b)+\text{trace } V(b)^{-1}).$$

If we notice that the expression for trace $V(b)^{-1}$ can be obtained by replacing $(2\pi i)$ in the expression (3.1) by $(-2\pi i)$,

$$\text{trace } V(b)=\frac{1}{2}(\text{trace } V(b)+\text{trace } V(b)^{-1})$$

$$=2+\sum_{n=1}^{\infty}\sum_{p_1+\cdots p_n=-2n}\left[\sum_{k=1}^{2n}\alpha^{(k)}_{p_1\cdots p_n}\frac{(2\pi i)^k+(-2\pi i)^k}{2}\right]f_{p_1}\cdots f_{p_n}.$$

Since

$$\frac{1}{2}((2\pi i)^k+(-2\pi i)^k)=\begin{cases}(2\pi i)^k & \text{if } k \text{ is even},\\ 0 & \text{if } k \text{ is odd},\end{cases}$$

we have, by taking the relation (4.2) into account,

$$\text{trace } V(b) = 2 + \sum_{n=1}^{\infty} \sum_{p_1+\cdots p_n=-2n} \left[\sum_{r=1}^{n-1} a^{(2r)}_{p_1\cdots p_n} (2\pi i)^{2r} \right] f_{p_1} \cdots f_{p_n}. \tag{4.3}$$

From (4.3) we can immediately see that, if

1) $p_j \geqq -2$, i.e. $x=0$ is a regular singular point,

or if

2) $p_j \leqq -2$, i.e. $x=\infty$ is a regular singular point,

trace $V(b)$ is a function of f_{-2} only, because, in these cases, the relation

$$p_1 + \cdots + p_n = -2n$$

can be realized when and only when

$$p_1 = \cdots = p_n = -2.$$

Except these well-known cases, eigenvalues of $V(b)$ seem to depend upon all the coefficients f_p. Explicit calculation shows

$$\begin{aligned} \text{trace } V(b) = {} & 2 + (2\pi i)^2 (f_{-2}^2 - f_{-1} f_{-3}) \\ & + (2\pi i)^2 \Big(-2 f_{-2}^3 + 3 f_{-1} f_{-2} f_{-3} - \sum_{j \neq -2,-3} \frac{2}{(j+2)(j+3)} f_{-1} f_j f_{-5-j} \\ & + \sum_{j \neq -1,-2,-3} \frac{4}{(j+3)(j+1)} f_{-2} f_j f_{-4-j} - \sum_{j \neq -1,-2} \frac{2}{(j+1)(j+2)} f_{-3} f_j f_{-3-j} \Big) + \cdots . \end{aligned}$$

In the forthcoming paper, we are intending to treat the simplest case

$$f(x) = \frac{f_{-3}}{x^3} + \frac{f_{-2}}{x^2} + \frac{f_{-1}}{x}$$

in further detail.

REFERENCE

[1] LAPPO-DANILEVSKY, J. A., Mémoires sur la théorie des systèmes des équations différentielles linéaires. Chelsea, 1953; Mém. III.

DEPARTMENT OF MATHEMATICS,
TOKYO INSTITUTE OF TECHNOLOGY.