THE SPACE OF NON-NEGATIVE SOLUTIONS OF THE EQUATION $\Delta u = pu$ ON A RIEMANN SURFACE

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Introduction.

Consider a form $p(z)|dz|^2$ on a Riemann surface R. This defines a self-adjoint elliptic partial differential equation $\Delta u = pu$ on R. The so called global theory of this equation aims to investigate the structures of the spaces of some distinguished solutions of this equation after the theory of harmonic functions. The investigation of this direction was begun first by Ozawa [10] and continued by himself [11], [12] and L. Myrberg [4], [5] and later by Royden [13] and the present author [7].

The aim of this paper is to study the space $\mathfrak{P}(R)$ of non-negative solutions of this equation from the aspect of minimal solutions of $\mathfrak{P}(R)$. After Martin we say that a function u in $\mathfrak{P}(R)$ is a minimal solution of the above equation if $u \neq 0$ and $u \geqq v$ for some v in $\mathfrak{P}(R)$ implies the existence of a constant c_v such that $v = c_v u$ on R. It is not hard to see that $\mathfrak{P}(R)$ has sufficiently many minimal solutions of the above equation, i.e. the totality of linear combinations of minimal solutions with non-negative coefficients is dense in $\mathfrak{P}(R)$ with respect to the compact convergence topology. Our problem is to determine the shapes of minimal solutions and to determine the standard way to approximate the functions in $\mathfrak{P}(R)$. This problem was raised essentially by Ozawa in [11]. We shall see that the situation is quite similar to that of harmonic case as was treated by Martin [3].

In chapter I we state among some known fundamental results the continuity of Green's function and precise nature of it around the pole. We also introduce a class of functions which plays the similar role to super- and sub-harmonic functions in the theory of harmonic functions. The similar investigation is found in a recent work of L. Myrberg [6].

In chapter II we develope the theory of Green potentials. These are extensively used in the following chapter. The Green potentials are quite similar to the harmonic Green potentials. We believe that the potential theoretic method will be a powerful tool for the future study of global theory of $\Delta u = pu$.

In the final chapter III we reproduce the Martin theory for $\Delta u = pu$. We shall state that every minimal solution of this equation is obtained by the limiting process

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$$\lim_n \left(\frac{G(z, \zeta_n)}{G(z_0, \zeta_n)} \right)$$

for some appropriate sequence $\{\zeta_n\}$ of points in R which has no point of accumulation in R and by multiplying a non-negative constant, where $G(z, \zeta)$ is Green's function of $\Delta u = pu$ with respect to R and with pole ζ and z_0 is a fixed point in R. The totality $\{K(z, \zeta); \zeta \in M\}$ of minimal solutions of $\Delta u = pu$ normalized by the condition that $K(z_0, \zeta) = 1$ generates $\mathfrak{P}(R)$ in the following manner: for any $u(z)$ in $\mathfrak{P}(R)$ there exists a unique Borel measure μ on M such that

$$u(z) = \int_M K(z, \zeta)\, d\mu(\zeta),$$

where M is topologized so as to be a metric space by which $K(z, \zeta)$ is continuous on $R \times M$. The method of proof used here is quite due to Martin [3].

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I. Preliminaries.

1.1. General notions and notations. Throughout this paper we denote by R a Riemann surface in the sense of Weyl-Radó. By a *density* $p(z)$ on R we mean a non-negative continuously differentiable function of local parameter z such that the expression $p(z)\,|dz|^2$ is invariant under the change of local parameters. Moreover we always assume that $p(z) \not\equiv 0$ on R. Then we can consider the elliptic partial differential equation

$$\Delta u = pu, \qquad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \tag{1.1}$$

on R. By a *solution* u of (1.1) on an open subset D of R we mean that u is twice continuously differentiable function satisfying (1.1) on D.

An open subset D of R is said to be *nice* if its relative boundary ∂D consists of countably many piecewise analytic Jordan curves which do not cluster in R and each of which has no end point in R. A *subdomain* D of R is a connected open subset of R and a *compact subdomain* is a subdomain whose closure is compact in R. An *exhaustion* $\{R_n\}_{n=1}^{\infty}$ of R is a sequence of nice compact subdomains R_n of R such that $\bar{R}_n \subset R_{n+1}$ $(n = 1, 2, \cdots)$ and $R = \cup_{n=1}^{\infty} R_n$. For a point z_0 in R, $K_r(z_0)$ denotes the neighborhood disc of local parameter z valid for $|z - z_0| \leqq 1$ such that $K_r(z_0) = (z; |z - z_0| < r)$ for $0 < r \leqq 1$.

1.2. Fundamental properties of solutions. For the sake of convenience, we describe here the known fundamental properties concerning the solutions of the equation (1.1).

Functions considered in this paper are all assumed to be real-valued.

(I) *Maximum principle.* Let p and q be two densities on R and D be a nice compact subdomain of R. Suppose that $p \geqq q$ on D and u, v and w be solutions of $\Delta u = pu$, $\Delta v = qv$ and $\Delta w = 0$ respectively and continuous on $\bar{D}$

satisfying $w \geqq v \geqq u \geqq 0$ on ∂D. Then the same inequality holds on D.

(II) *Harnack inequality.* Let D be a nice compact subdomain of R. Then there exists a positive constant $c = c(D)$ such that for any pair of points z and z' in D and for each non-negative solutian u of (1.1) on R it holds

$$c^{-1}u(z) \leqq u(z') \leqq cu(z).$$

(III) *Dirichlet problem.* Let D be a nice compact subdomain of R and φ be a continuous function on ∂D. Then there exists a unique continuous function u on $\bar{D}$ such that $u = \varphi$ on ∂D and u is a solution of (1.1) in D.

(IV) *Completeness.* Suppose that $\{u_n\}$ is a sequence of solutions u_n of (1.1) on R and converges uniformly on each compact subset of R to a function u on R. Then u is also a solution of (1.1) on R.

(V) *Monotone compactness.* Let $\{u_n\}$ be a monotone sequence of solutions u_n of (1.1) on R such that $\{u_n(z_0)\}$ is bounded for some point z_0 in R. Then $\{u_n\}$ converges uniformly on each compact subset of R to a solution of (1.1) on R.

(VI) *Bounded compactness.* Let $\{u_n\}$ be a sequence of solutions u_n of (1.1) such that $|u_n| \leqq M \lneqq \infty$. Then there exists a subsequence of $\{u_n\}$ which converges uniformly on each compact subset of R to a solution of (1.1) on R.

As for the proofs of these propositions, refer to Myrberg [4].

(VII) *Removable singularity.* Let D be a compact subdomain of R, E be a compact subset of D and u be a bounded solution of (1.1) in $D-E$. In order that u is continued to D so as to be a solution of (1.1) on D it is necessary and sufficient that E is of logarithmic capacity zero (cf. [7]).

1.3. Supersolutions. A continuous function f on an open subset D of R is said to be a *supersolution* (resp. *subsolution*) of (1.1) on D if for any point z_0 in D there exists a positive constant $r(z_0)$ such that $\bar{K} \subset D$ and $f(z) \geqq u(z)$ (resp. $f(z) \leqq u(z)$) on K, where $K = K_r(z_0)$ for arbitrary r in $0 < r \leqq r(z_0)$ and u is the solution of (1.1) on K with boundary value f on ∂K. Clearly a solution of (1.1) is a supersolution and subsolution of (1.1) and vice versa. It is also clear that f is a supersolution if and only if $-f$ is subsolution.

LEMMA 1.1. (i) *Let f_1 and f_2 be supersolutions of* (1.1) *and c_i ($i=1, 2$) be non-negative constants. Then c_i, $c_1f_1 + c_2f_2$ and* $\min(f_1, f_2)$ *are also supersolutions of* (1.1).

(ii) *Let $\{f_n\}$ be a sequence of supersolutions of* (1.1) *in an open subset D of R and converges uniformly on each compact subset of D to a function f on D. Then f is also a supersolution of* (1.1) *in D.*

(iii) *A supersolution $f(z)$ of* (1.1) *on a nice compact subdomain D of R and continuous on $\bar{D}$ with $f \geqq 0$ on ∂D is non-negative on D.*

(iv) *Let f be a supersolution of R and D be a nice subdomain relatively compact in R and f_D be obtained from f by replacing by u inside D, where u is the solution of* (1.1) *in D with boundary value f on ∂D. Then f_D is again supersolution of* (1.1) *on R and $f_D \leqq f$ on R.*

These are easy consequences of our definition and preceding paragraph.

Here we remark that we can apply *Perron's method* using super- and sub-solutions to solve the Dirichlet problem which sharpens (III) in §1.2. Let D be a nice compact subdomain of R and φ be a bounded function on ∂D. We put $\mathfrak{U}_\varphi$ (or $\mathfrak{B}_\varphi$) be the class of all super- (or sub-) solutions of (1.1) such that

$$\varliminf_{z\to\zeta} f(z) \geqq \varphi(\zeta) \text{ (or } \varlimsup_{z\to\zeta} f(z) \leqq \varphi(\zeta))$$

at any point ζ in ∂D. We put

$$\bar{H}_\varphi(z) = \inf(f(z);\ f \in \mathfrak{U}_\varphi)$$

and

$$\underline{H}_\varphi(z) = \sup(f(z);\ f \in \mathfrak{B}_\varphi).$$

Then as in the harmonic case we easily see that $\bar{H}_\varphi$ and $\underline{H}_\varphi$ are solutions of (1.1) on D and $\bar{H}_\varphi \geqq \underline{H}_\varphi$ on D. Moreover if φ is continuous at ζ in ∂D, then we have

$$\lim_{z\to\zeta} \bar{H}_\varphi(z) = \lim_{z\to\zeta} \underline{H}_\varphi(z) = \varphi(\zeta).$$

1.4. Green's function. Let ζ be an arbitrary fixed point in R. Now consider the family F_ζ of functions $v(z)$ on $R-\zeta$ such that

(G. 1) *$v(z)$ is a non-negative solution of* (1.1) *in $R-\zeta$;*

(G. 2) *there exists a positive constant M_ζ such that*

$$-M_\zeta \leqq v(z) + \log|z-\zeta| \leqq M_\zeta$$

on $K_1(\zeta)-\zeta$.

Moreover if R is a nice subdomain of a larger surface R' with relative boundary γ, we suppose that

(G. 3) *$v(z)$ vanishes continuously at each point of γ.*

Myrberg [5] proved that the family F_ζ is non-empty for any point ζ in R. We put

$$G(z, \zeta) = \inf(v(z);\ v \in F_\zeta)$$

and we call $G(z, \zeta)$ is *Green's function* of (1.1) with respect to R. It is easily seen from §1.2 that $G(z, \zeta)$ possesses the properties (G. 1), (G. 2) and (G. 3). Let $\{R_n\}$ be an exhaustion of R and $G_n(z, \zeta)$ be Green's function of (1.1) with respect to R_n. Then $\{G_n(z, \zeta)\}$ is an increasing sequence and

$$G(z, \zeta) = \lim G_n(z, \zeta), \tag{1.2}$$

where convergence is uniform on each compact subset of R. Let $H(z, \zeta)$ be Green's function of $\Delta u = qu$ with respect to R and $g(z, \zeta)$ be harmonic Green's function with respect to R, where we promise that $g(z, \zeta) \equiv \infty$ if R is a parabolic Riemann surface. If $0 \leqq q \leqq p$ on R, then from §1.2 and by using exhaustion it is easily seen that

$$g(z, \zeta) \geqq H(z, \zeta) \geqq G(z, \zeta) \tag{1.3}$$

holds on $R-\zeta$.

By defining $G(\zeta, \zeta)=+\infty$, we get a function $G(z, \zeta)$ defined on $R \times R$. Now we derive important properties of $G(z, \zeta)$ considered on $R \times R$. From (G. 1) and (II) in § 1.2, we get

(G. 4) *$G(z, \zeta)>0$ on $R \times R$ and $G(z, \zeta)=+\infty$ if and only if $z=\zeta$.*

By Green's formula we easily see that $G_n(z, \zeta)=G_n(\zeta, z)$ on $R_n \times R_n$ and by (1.2) we get

$$\text{(G. 5)} \qquad G(z, \zeta)=G(\zeta, z) \quad \text{on} \quad R \times R.$$

Next we prove

(G. 6) *$G(z, \zeta)$ is continuous on $R \times R$.*

This is a consequence of the following (G. 6 a) and (G. 6 b) which will be used repeatedly.

(G. 6 a) *$G(z, \zeta)$ is finitely continuous at (z_0, ζ_0) in $R \times R$ with $z_0 \neq \zeta_0$.*

Proof. We take a small disc $K=K_1(z_0)$ in R such that $\zeta_0 \notin \bar{K}$. Let $\bar{G}(z, \zeta)$ be Green's function of (1.1) with respect to K. Then from (1.3)

$$0<\bar{G}(z, \zeta) \leqq \log \left| \frac{1-\overline{(\zeta-z_0)}(z-z_0)}{z-\zeta} \right| \tag{1.4}$$

on $K \times K$. By Green's formula

$$G(z, \zeta)=\frac{1}{2\pi} \int_{\partial K} G(t, \zeta) \frac{\partial}{\partial n} \bar{G}(z, t)\, ds.$$

Here $\partial / \partial n$ denotes the inner normal differentiation and ds denotes the line element of ∂K. From this

$$|G(z, \zeta)-G(z, \zeta_0)| \leqq \frac{1}{2\pi} \int_{\partial K} |G(t, \zeta)-G(t, \zeta_0)| \frac{\partial}{\partial n} \bar{G}(z, t)\, ds.$$

From this, by (1.4) and Schwarz's inequality, we obtain

$$|G(z, \zeta)-G(z, \zeta_0)| \leqq a(z)\left(\int_{\partial K} |G(t, \zeta)-G(t, \zeta_0)|^2 ds\right)^{1/2}, \tag{1.5}$$

where $a(z)=(2\pi)^{-1/2}(1+|z-z_0|)/(1-|z-z_0|)$. Let D be a neighborhood of ζ_0 such that $\bar{D} \subset R-\bar{K}$. By (II) in § 1.2 we have a positive constant c such that $0<G(t, \zeta) \leqq cG(t, \zeta_0)$ on D for all t in ∂K. Hence we have

$$|G(t, \zeta)-G(t, \zeta_0)|^2 \leqq 2(1+c^2)G(t, \zeta_0)^2$$

for all (t, ζ) in $\partial K \times \bar{D}$ and the right term of the above inequality is ds-integrable on ∂K. We also have $\lim_{\zeta \to \zeta_0} |G(t, \zeta)-G(t, \zeta_0)|^2=0$ for each t in ∂K. Hence by Lebesgue's convergence theorem, for arbitrarily given positive number ε, we can find a neighborhood V_{ζ_0} of ζ_0 contained in D such that

$$\left(\int_{\partial K} |G(t, \zeta)-G(t, \zeta_0)|^2 ds\right)^{1/2}<\pi^{1/2} \frac{\varepsilon}{4} \tag{1.6}$$

on V_{ζ_0}. We can also find a neighborhood V_{z_0} of z_0 contained in K such that

$$a(z)<2(2\pi)^{-1/2} \quad \text{and} \quad |G(z, \zeta_0)-G(z_0, \zeta_0)|<\frac{\varepsilon}{2} \tag{1.7}$$

on V_{z_0}. Hence we get, by (1.5), (1.6) and (1.7),

$$|G(z, \zeta) - G(z_0, \zeta_0)| < \varepsilon$$

on $V_{z_0} \times V_{\zeta_0}$. Q.E.D.

(G. 6b) *$G(z, \zeta)$ is represented in $D = K_{1/2}(z_0) \times K_{1/2}(z_0)$ as follows:*

$$G(z, \zeta) = \log \left| \frac{1 - \overline{(\zeta - z_0)}(z - z_0)}{z - \zeta} \right| + w(z, \zeta),$$

where $w(z, \zeta)$ is finitely continuous on $\bar{D}$.

Proof. Let $K = K_1(z_0)$ and $K' = K_{1/2}(z_0)$. We take Green's function $\bar{G}(z, \zeta)$ of (1.1) with respect to K and harmonic Green's function $g(z, \zeta)$ with respect to K. Let $u(z, \zeta)$ be the solution of (1.1) in K with boundary value $G(z, \zeta)$ on ∂K with arbitrary fixed ζ in $\bar{K}'$. Hence $u(z, \zeta) = G(z, \zeta)$ on $\partial K \times \bar{K}'$. As we have seen that $G(z, \zeta)$ is continuous on $\partial K \times K'$, so $u(z, \zeta)$ is uniformly continuous on $\partial K \times \bar{K}'$. Let ε be an arbitrary positive number. We can find a positive number $\delta = \delta(\varepsilon)$ such that $|u(z, \zeta) - u(z, \zeta')| \leqq \varepsilon/2$ for any z in ∂K and for any ζ and ζ' in $\bar{K}'$ with $|\zeta - \zeta'| < \delta$. The function $u(z, \zeta) - u(z, \zeta')$ is a solution of (1.1) in K for fixed ζ and ζ'. Hence by (I) in §1.2, $|u(z, \zeta) - u(z, \zeta')| \leqq \varepsilon/2$ for any z in K and for any ζ and ζ' in $\bar{K}'$ with $|\zeta - \zeta'| < \delta$. Fix a point (z', ζ') in $\bar{K}' \times \bar{K}'$. There exists a positive number $\eta = \eta(\varepsilon, \zeta')$ such that $|u(z, \zeta') - u(z', \zeta')| \leqq \varepsilon/2$ if z is in $\bar{K}'$ and $|z - z'| < \eta$. Thus we have shown that if (z, ζ) is in $\bar{K}' \times \bar{K}'$ with $|z - z'| < \eta$ and $|\zeta - \zeta'| < \delta$ then $|u(z, \zeta) - u(z', \zeta')| \leqq \varepsilon$ or $u(z, \zeta)$ is continuous on $\bar{K}' \times \bar{K}'$.

Now it is easy to see that $\bar{G}(z, \zeta) = G(z, \zeta) - u(z, \zeta)$ on D. If we put

$$v(z, \zeta) = \frac{1}{2\pi} \iint_{K'} p(z') g(z, z') G(z', \zeta) \, dx' dy'$$

on $\bar{D}$, where $z' = x' + iy'$, then by Green's formula

$$g(z, \zeta) = \bar{G}(z, \zeta) + v(z, \zeta)$$

on $\bar{D}$. From this we see that $v(z, \zeta) = v(\zeta, z)$ on D. As $\bar{G}(z, \zeta) \leqq g(z, \zeta) = \log|(1 - \overline{(\zeta - z_0)}(z - z_0))/(z - \zeta)|$, so by putting $b(z) = (1 + |z - z_0|)/(1 - |z - z_0|)$ it holds

$$\iint_{K'} G(\zeta, z')^2 \, dx' dy' \leqq 4\pi b(\zeta).$$

By Schwarz's inequality,

$$(1.8) \qquad |v(z, \zeta) - v(z_1, \zeta)| \leqq \frac{M}{\pi} (\pi b(\zeta))^{1/2} \left(\iint_{K'} |g(z, z') - g(z_1, z')|^2 \, dx' dy' \right)^{1/2},$$

where $M = \sup(p(z'); z' \in K)$. It is clear that

$$\lim_{z \to z_1} \iint_K |g(z, z') - g(z_1, z')|^2 \, dx' dy' = 0.$$

Thus $v(z, \zeta)$ is continuous in z with fixed ζ and by symmetricity of $v(z, \zeta)$, $v(z, \zeta)$ is continuous in ζ with fixed z. Now fix a point (z_1, ζ_1) in $\bar{D}$. From (1.8) it holds

$$|v(z, \zeta)-v(z_1, \zeta)| \leqq \left(\frac{3}{\pi}\right)^{1/2} M\rho(z, z_1)$$

for (z, ζ) in $\bar{D}$, where $\lim_{z\to z_1}\rho(z, z_1)=0$. We also have

$$\rho'(\zeta, \zeta_1)=|v(z_1, \zeta)-v(z_1, \zeta_1)|\to 0$$

as $\zeta\to\zeta_1$. Hence

$$|v(z, \zeta)-v(z_1, \zeta_1)| \leqq \left(\frac{3}{\pi}\right)^{1/2} M\rho(z, z_1)+\rho'(\zeta, \zeta_1),$$

which shows that $v(z, \zeta)$ is continuous on $\bar{D}$. By putting $u(z, \zeta)-v(z, \zeta) = w(z, \zeta)$, it holds (**G. 6 b**).

II. Potential theory.

2.1. Fundamental properties of Green potential. By a *measure* μ on R we mean a non-negative countably additive set function defined on the Borel field generated by all compact subsets of R and $\mu(K)$ is finite for every compact subset K of R. The *support* S_μ of μ is defined as follows: $S_\mu=\cap F$, where F is closed in R and $\mu(R-F)=0$.

As we have seen in §1.4, Green's function $G(z, \zeta)$ of (1.1) with respect to R is strictly positive, symmetric and continuous on $R\times R$ and $G(z, \zeta)$ is finite unless $z=\zeta$. Hence $G(z, \zeta)$ is a kernel in the sense of potential theory. A *Green potential* $U^\mu(z)$ of a measure μ with kernel $G(z, \zeta)$ is defined by

$$U^\mu(z)=\int_R G(z, \zeta)\,d\mu(\zeta).$$

By using (II) in §1.2 it is easy to see that $U^\mu(z)$ is finite on $R-S_\mu$ if $U^\mu(z_0)$ is finite for some point z_0 in R and that $U^\mu(z)$ is lower semicontinuous on R and continuous on $R-S_\mu$. Hereafter we consider only those potentials $U^\mu(z)$ such that $U^\mu(z)$ is finite on $R-S_\mu$.

LEMMA 2.1. *The potential $U^\mu(z)$ is a solution of* (1.1) *in $R-S_\mu$.*

Proof. Let μ_n be the restriction of μ on R_n, where R_n is an exhaustion of R. Then $U^{\mu_n}(z)$ converges increasingly to $U^\mu(z)$ on $R-S_\mu$. If $U^{\mu_n}(z)$ is a solution of (1.1) in $R-S_{\mu_n}$, then $U^\mu(z)$ is a solution of (1.1) in $R-S_\mu$ by (V) in §1.2. Hence we may assume without loss of generality that S_μ is compact.

We take an arbitrary point z_0 in $R-S_\mu$ and a local parameter z around z_0 such that $\bar{K}\subset R-S_\mu$, where $K=K_1(z_0)$. We have only to show that $U^\mu(z)$ is a solution of (1.1) in K.

By the definition of integral and the uniform continuity of $G(z, \zeta)$ on $\bar{K}\times S_\mu$ we have sequences $\{a_i^{(n)}\}$ of non-negative numbers and $\{\zeta_i^{(n)}\}$ of points in S_μ $(i=1, 2, \cdots, N_n)$ such that

$$U^\mu(z)=\lim_n u_n(z)$$

on K, where

$$u_n(z)=\sum_{i=1}^{N_n} a_i^{(n)} G(z,\ \zeta_i^{(n)})$$

and

$$u_n(z) \leqq U^\mu(z)+\frac{1}{n}$$

on K. Clearly $u_n(z)$ is a solution of (1.1) in K and by the bounded compactness of $\{u_n(z)\}$ on K it converges to a solution of (1.1) on K. Hence $U^\mu(z)$ is a solution of (1.1) in K. Q.E.D.

In § 2.4, the converse of this Lemma will be proved. The following is an easy consequence of (G. 3).

LEMMA 2.2. *If S_μ is compact and R has a piecewise analytic relative boundary γ, then the potential $U^\mu(z)$ vanishes continuously on γ.*

Next we prove

LEMMA 2.3.
$$\varlimsup_{S_\mu \ni z \to z_0} U^\mu(z)=\varlimsup_{R \ni z \to z_0} U^\mu(z).$$

Proof. Let $K=K_{1/2}(z_0)$ and μ' be the restriction of μ on $\bar{K}$ and $\mu''=\mu-\mu'$. Then $U^\mu(z)=U^{\mu'}(z)+U^{\mu''}(z)$ and $U^{\mu''}(z)$ is continuous on K. Hence we may assume that $S_\mu \subset K$ and z is in K. By (G. 6 b)

$$U^\mu(z)=\int_K \log\left(\frac{1}{|z-\zeta|}\right) d\mu(\zeta)+\int_K u(z,\ \zeta)\, d\mu(\zeta)$$

on K, where $u(z,\ \zeta)$ is finitely continuous on $\bar{K}\times\bar{K}$. Clearly the second term of the right hand side of the above is continuous on K and it is well known and easily verified that

$$\varlimsup_{S_\mu \ni z \to z_0} \int_K \log\frac{1}{|z-\zeta|} d\mu(\zeta)=\varlimsup_{K \ni z \to z_0} \int_K \log\frac{1}{|z-\zeta|} d\mu(\zeta).$$

Hence we get the required identity. Q.E.D.

LEMMA 2.4. *Let K be a compact subset of R and φ be a non-negative continuous function admitting the value ∞ such that φ is a supersolution of* (1.1) *in $R-S_\mu \cup K$. Suppose that $\varphi(z) \geqq U^\mu(z)$ on $S_\mu \cup K$, where S_μ is compact, then the same inequality holds on the whole space R.*

Proof. Let $\{R_n\}$ be an exhaustion of R and $G_n(z,\ \zeta)$ be Green's function of (1.1) with respect to R_n and we put

$$U_n^\mu(z)=\int_{R_n} G_n(z,\ \zeta)\, d\mu(\zeta),$$

where $S_\mu \subset R_n$. From (1.2) we see that $U_n^\mu(z) \nearrow U^\mu(z)$ on R. From this we may assume that R has the relative boundary γ and $R \cup \gamma$ is compact. Let D be a component of $R-S_\mu \cup K$. Then from Lemma 2.2 and 2.3

$$\overline{\lim_{D\ni z\to z_0}} U^{\mu}(z)\leq\varphi(z_0),$$

where z_0 is an arbitrary point in ∂D. Hence by Lemma 1.1 and 2.1, $U^{\mu}(z) \leqq \varphi(z)$ on D. So this inequality holds on R. Q.E.D.

The *mutual energy* (μ, ν) of measures μ and ν is defined by

$$(\mu, \nu)=\iint_{R\times R} G(z, \zeta)\, d\mu(z)\, d\nu(\zeta).$$

The *energy* $\|\mu\|^2$ of a measure μ is, by definition, $\|\mu\|^2=(\mu, \mu)$. Let K be a compact subset of R. The *capacity* $C(K)$ of K is defined by

$$C(K)=\frac{1}{\inf(\|\mu\|^2;\ \mu(S_\mu)=1,\ S_\mu\subset K)}.$$

For an arbitrary subset X of R, its *inner capacity* $C(X)$ is defined by

$$C(X)=\sup(C(K);\ K \text{ is compact in } R \text{ and } K\subset X).$$

Hereafter capacity means always inner capacity.

LEMMA 2.5. *The kernel $G(z, \zeta)$ is regular*[1], *that is, for any ζ_0 in R and any neighborhood V of ζ_0, there exist a measure μ of total mass* 1 *with finite energy and $S_\mu\subset V$ satisfying*

$$U^{\mu}(z)\leqq 2G(z, \zeta_0)$$

on the whole space R.

Proof. Take a local parameter such that $K_1(\zeta_0)\subset V$. By (G. 6b), there exists a positive constant c such that

$$-c-\log|z-\zeta|\leqq G(z, \zeta)\leqq c-\log|z-\zeta| \tag{2.1}$$

holds on $K_{1/2}(\zeta_0)\times K_{1/2}(\zeta_0)$. Let ε be in $(0, 1/2)$ and put

$$d\mu_\varepsilon(\zeta)=\begin{cases}(2\pi)^{-1}d\arg(\zeta-\zeta_0) & \text{if } |\zeta-\zeta_0|=\varepsilon,\\ 0 & \text{elsewhere.}\end{cases}$$

Clearly the total mass of μ_ε is 1 and the support of μ_ε is contained in $K_{1/2}(\zeta_0)$. From (2.1)

$$U^{\mu_\varepsilon}(z)\leqq c+\min(-\log\varepsilon,\ -\log|z-\zeta_0|) \tag{2.2}$$

on $K_{1/2}(\zeta_0)$. From this we have $\|\mu_\varepsilon\|^2\leqq c-\log\varepsilon<\infty$. From (2.1) and (2.2), $U^{\mu_\varepsilon}(z)\leqq 2G(z, \zeta_0)$ on $K_{1/2}(\zeta_0)$ if ε is sufficiently small. By Lemma 2.4, this inequality holds on the whole space R. Q.E.D.

An important consequence of the regularity of $G(z, \zeta)$ is the following

LEMMA 2.6. *Let X be an arbitrary subset of R and φ_i $(i=1, 2)$ be continuous functions defined on X. If*

$$\varphi_1(z)\leqq U^{\mu}(z)\leqq\varphi_2(z)$$

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1) Terminology due to Ninomiya [8].

holds on X except a set of capacity zero, then the same inequality holds at every inner point of X.

Proof. Let z_0 be an inner point of X and let $\{\Omega_n\}$ be a fundamental base of neighborhoods of z_0 such that $\bar{\Omega}_{n+1} \subset \Omega_n \subset X$. By the regularity of $G(z, \zeta)$, we get a sequence $\{\mu_n\}$ of measures such that $S_{\mu_n} \subset \Omega_n$ and $\|\mu_n\| < \infty$ and $\mu_n(S_{\mu_n}) = 1$ and $U^{\mu_n}(z) \leqq 2G(z_0, z)$ on R. Then it is easy to see that

$$U^{\mu}(z_0) = \lim_n \int_R U^{\mu}(z)\, d\mu_n(z)$$

and

$$\varphi_i(z_0) = \lim_n \int_R \varphi_i(z)\, d\mu_n(z) \qquad (i = 1, 2).$$

Noticing that μ_n-measure of a set of capacity zero is zero, we have

$$\int_R \varphi_1(z)\, d\mu_n(z) \leqq \int_R U^{\mu}(z)\, d\mu_n(z) \leqq \int_R \varphi_2(z)\, d\mu_n(z).$$

Letting n tend to infinity we get the required inequality. Q.E.D.

2.2. **Potential theoretic principles.** Green potentials enjoy almost all important potential theoretic principles from which we can derive the analogous properties as that of harmonic Green potentials. They are listed in the following

THEOREM 2.1. *Green kernel $G(z, \zeta)$ satisfies the following principles.*

(i) **Continuity principle.** *If the restriction of $U^{\mu}(z)$ on S_{μ} is finitely continuous on S_{μ}, then $U^{\mu}(z)$ is continuous on R as the function on R.*

(ii) **Frostman's maximum principle.** *If the inequality $U^{\mu}(z) \leqq 1$ holds on compact S_{μ}, then the same inequality holds on the whole space R.*

(iii) **Cartan's maximum principle.** *If the inequality $U^{\mu}(z) \leqq U^{\nu}(z)$ on compact S_{μ} with $\|\mu\| < \infty$, then the same inequality holds on R.*

(iv) **Unicity principle.** *If $U^{\mu}(z) = U^{\nu}(z)$ on R except a set of capacity zero, then $\mu = \nu$.*

(v) **Equilibrium principle.** *For an arbitrary compact subset K of R there always exists a unique measure called equilibrium measure of K satisfying $S_{\mu} \subset K$ and $U^{\mu}(z) = 1$ on K except a subset of ∂K of capacity zero and $U^{\mu}(z) \leqq 1$ on R.*

(vi) **Balayage principle.** *For an arbitrary compact subset K of R and a measure ν there always exists a unique measure μ called balayaged measure of ν on K satisfying $S_{\mu} \subset K$ and $U^{\mu}(z) = U^{\nu}(z)$ on K except a set of inner capacity zero and $U^{\mu}(z) \leqq U^{\nu}(z)$ on R.*

(vii) **Energy principle.** *For any measures μ and ν with compact supports such that $\sigma = \mu - \nu \not\equiv 0$ it holds $\iint_{R\times R} G(z, \zeta)\, d\sigma(z)\, d\sigma(\zeta) > 0$.*

Proof. (i) follows at once from Lemma 2.3 and the lower semicontinuity of $U^{\mu}(z)$ on R. (ii) is a consequence of Lemma 2.4. To prove (iii) and (iv), we notice that for kernel $G(z, \zeta)$ (iii) and (iv) are equivalent to the following (iii)′ and (iv)′ respectively (cf. Ninomiya [8], Théorèmes 7 and 9):

(iii)′ Suppose that ζ_0 is a point in R and λ is a measure with compact support S_λ not containing ζ_0. If $U^\lambda(z) \leqq G(z, \zeta_0)$ holds on S_λ, then the same inequality holds on the whole space R.

(iv)′ Suppose that ζ_0 is a point in R and λ is a measure with compact support S_λ not containing ζ_0. If $U^\lambda(z) \leqq G(z, \zeta_0)$ holds on S_λ, then $U^\lambda(z) < G(z, \zeta_0)$ on a neighborhood of ζ_0.

These are easy consequences of Lemma 2.4 and the fact $G(\zeta_0, \zeta_0) = \infty$. (vii) is a consequence of (ii) (or (iii)) and (iv) (cf. Ninomiya [8], Lemma 6). Next we concern ourselves with (v) and (vi). Existence of equilibrium measure on K (resp. balayaged measure of ν on K) is equivalent to (ii) (resp. (iii)) (cf. Ninomiya [8], Théorème 4 (resp. 5)). Unicity of them follows from (iv) (cf. Ninomiya, ibid.). Non existence of exceptional points in inner point of K for equality $U^\mu(z) = 1$ follows from Lemma 2.6.

2.3. Sets of capacity zero. It is quite easy to see that a subset X of R is of inner capacity zero if and only if $C(X \frown K_1(z)) = 0$ for all z in R. We say that a subset X of R is of *inner logarithmic capacity zero* if inner logarithmic capacity of $X \frown K_1(z)$ is zero for all z in R. We can prove

THEOREM 2.2. *For a subset X of R, $C(X) = 0$ if and only if X is a set of inner logarithmic capacity zero.*

Proof. From the above remark and the definition of $C(X)$ and inner logarithmic capacity, we may assume without loss of generality that X is a compact subset of a parameter disc $K = K_{1/2}(z_0)$. We put

$$C_0(X) = 1/\inf\left(\|\mu\|_0^2 = \iint_{K \times K} g(z, \zeta)\, d\mu(z)\, d\mu(\zeta);\ \mu(S_\mu) = 1,\ S_\mu \subset X\right),$$

where $g(z, \zeta)$ is harmonic Green's function of $K_1(z_0)$. It is well known that X is of logarithmic capacity zero if and only if $C_0(X) = 0$. Hence we have only to prove the equivalence of $C_0(X) = 0$ and $C(X) = 0$. From (**G. 6b**) we can find a positive constant c such that

$$g(z, \zeta) \leqq G(z, \zeta) + c \leqq g(z, \zeta) + 2c$$

on $\bar{K} \times \bar{K}$. If μ is a measure such that $S_\mu \subset X$ and $\mu(S_\mu) = 1$. Then

$$\|\mu\|_0^2 \leqq \|\mu\|^2 + c \leqq \|\mu\|_0^2 + 2c$$

and so

$$\frac{1}{C_0(X)} \leq \frac{1}{C(X)} + c \leq \frac{1}{C_0(X)} + 2c.$$

This shows the equivalence of $C_0(X) = 0$ and $C(X) = 0$. Q.E.D.

It is known that from continuity principle it follows the following (cf., for example, Ugaheri [14]).

THEOREM 2.3. *Let E be an F_σ-set of capacity zero in R. Then there exists a measure μ satisfying $\mu(R-E)=0$ and $U^\mu(z)=\infty$ at each point z in E.*

This measure μ (or potential $U^\mu(z)$) is called *Evans' measure* (or *Evans' potential*) of E with respect to R.

2.4. Gauss' variation. A sequence $\{\mu_n\}$ of measures is said to converge to a measure μ *vaguely* if $\left\{\int_R f(z)\,d\mu_n(z)\right\}$ converges to $\int_R f(z)\,d\mu(z)$ for all continuous functions $f(z)$ with compact support on R. The following is well known and can be proved easily by using Riesz-Markoff-Kakutani's representation theorem of continuous linear functionals (cf., for example, Halmos [1], at page 243).

SELECTION THEOREM. *If S_{μ_n} are contained in a compact set in R and $\{\mu_n(R)\}$ is uniformly bounded, there exists a subsequence of μ_n which converges to a measure vaguely.*

Let K be a compact subset of R. By Stone-Weierstrass theorem, we can easily see that the closed subalgebra generated by $C(K)$ of $C(K\times K)$ coincides with $C(K\times K)$, where $C(K)$ and $C(K\times K)$ are the totality of continuous functions on K and $K\times K$ respectively. Using this fact, we get at once

LEMMA 2.7. *Suppose that S_{μ_n} are contained in a fixed compact set in R and $\{\mu_n\}$ converges to μ vaguely. Then* $\underline{\lim}_n(\mu_n, \mu_n)\geqq(\mu, \mu)$.

The following theorem is very useful in the potential theory and proved easily by using Lusin's theorem. For the proof, see, for example, Kishi [2], Lemma 2.

REGULARIZATION THEOREM. *Let K be a compact set and μ be a measure with $S_\mu\subset K$ and $\|\mu\|<\infty$. Then there exists a sequence $\{\mu_n\}$ of measures with $\mu_n\leqq\mu$ satisfying 1° $\{\mu_n\}$ converges to μ vaguely and $\lim_n\|\mu_n-\mu\|=0$, 2° $U^{\mu_n}(z)$ are all continuous on R, 3° $\{U^{\mu_n}(z)\}$ converges increasingly to $U^\mu(z)$ pointwise in R.*

The following is also well known. For the proof, see, for example, Ninomiya [8], Lemma 5.

LEMMA 2.8. *Suppose that $\{\mu_n\}$ converges to μ vaguely and S_{μ_n} are contained in a fixed compact set in R. Then*

$$U^\mu(z)=\underline{\lim}_n U^{\mu_n}(z)$$

on R except a set of capacity zero.

Let K be a compact set in R and φ be a continuous function on K. For a measure μ with $S_\mu\subset K$ we put

$$G_\varphi(\mu) = (\mu, \mu) - 2\int_K \varphi(z)\, d\mu(z)$$

and

$$m_\varphi = \inf(G_\varphi(\mu);\ S_\mu \subset K).$$

As $G_\varphi(0) = 0$, so $m_\varphi \leqq 0$. *Gauss' variational problem* with respect to the system (K, φ) is to find a measure μ such that $m_\varphi = G_\varphi(\mu)$. We say that $\{\mu_n\}$ is a *minimal sequence* if $G_\varphi(\mu_n) \searrow m_\varphi$.

LEMMA 2.9. *Gauss' variational problem is always solved.*

Proof. Let $\{\mu_n\}$ be a minimal sequence. We put $\Lambda = \{\lambda;\ S_\lambda \subset K,\ \lambda(K) = 1\}$ and $a = \inf(\|\lambda\|^2;\ \lambda \in \Lambda)$. Let λ_n be in Λ and $\|\lambda_n\|^2 \searrow a$. By selection theorem we may assume that λ_n converges vaguely to a measure λ in Λ. By Lemma 2.7,

$$a \leqq \|\lambda\|^2 \leqq \underline{\lim}_n \|\lambda_n\|^2 = a.$$

Thus $a = \|\lambda\|^2$ and by energy principle $a > 0$. Now it is easy to see that $G_{\varphi/t}(\lambda) > 0$ if $t > 2\|\varphi\|/a$, where $\lambda \in \Lambda$ and $\|\varphi\| = \sup(|\varphi(z)|;\ z \in K)$. As $G_\varphi(t\lambda) = t^2 G_{\varphi/t}(\lambda)$, so if $\mu(K) > 2\|\varphi\|/a$, then $G_\varphi(\mu) > 0$. Hence $\mu_n(K) \leqq 2\|\varphi\|/a$ for all sufficiently large n since $m_\varphi \leqq 0$. By selection theorem we may assume that $\{\mu_n\}$ converges vaguely to a measure μ. By Lemma 2.7,

$$m_\varphi \leqq G_\varphi(\mu) \leqq \underline{\lim}_n (\mu_n, \mu_n) + 2\int_R \varphi(z)\, d\mu(z) = \lim_n G_\varphi(\mu_n) = m_\varphi.$$

Hence $G_\varphi(\mu) = m_\varphi$.

The following is known as fundamental theorem in potential theory.

THEOREM 2.4. *Let K be a compact set and φ be a continuous function defined on K. Then there exists a unique measure μ with $S_\mu \subset K$ and $U^\mu(z) \leqq \varphi(z)$ on S_μ and $U^\mu(z) \geqq \varphi(z)$ on K except an F_σ-set E of ∂K with $C(E) = 0$.*

Proof. Let μ be a solution of Gauss' variational problem with respect to (K, φ). Let $d = \sup(U^\mu(z) - \varphi(z);\ z \in S_\mu)$ and $E = (z \in K;\ U^\mu(z) - \varphi(z) < d)$ and $E_n = (z \in K;\ U^\mu(z) - \varphi(z) \leqq d + 1/n)$. Then it is clear that E_n is compact and $E = \cup_n E_n$ or E is an F_σ-set. First we claim that $C(E) = 0$. Contrary to the assertion we suppose $C(E) > 0$. By the definition of $C(E)$ we can find a compact set F contained in E such that $C(F) > 0$. Let ω be equilibrium measure of F. Then

$$0 < \omega(F) \leqq \textstyle\sum_n \omega(F \frown E_n)$$

and so there exists an n such that $\omega(F \frown E_n) > 0$. Let ω_n be the restriction of ω on E_n. Then $\|\omega_n\| \leqq \|\omega\| < \infty$ by $C(F) > 0$ and so $\omega_n(E_n) > 0$. By regularization theorem we can find a measure β with $S_\beta \subset E_n$ and $\|\beta\| < \infty$ and $U^\beta(z)$ is continuous on R. By the definition of d and the lower semicontinuity of $U^\mu(z) - \varphi(z)$, we can find a neighborhood W of a point z_0 in S_μ such that

$$U^{\beta}(z)-\varphi(z)>d-\frac{1}{2n} \tag{2.3}$$

on $W\frown K$. Now define the set function λ by

$$\lambda=\begin{cases}-\mu & \text{on } W,\\ \dfrac{\mu(W)\beta}{\beta(E_n)}. & \text{on } E_n,\\ 0 & \text{elsewhere on } R.\end{cases} \tag{2.4}$$

Then by $\|\mu\|<\infty$ and the continuity of $U^{\beta}(z)$, we get the finiteness of $\iint_{R\times R} G(z,\zeta)\,d\lambda(z)\,d\lambda(\zeta)$ and $\int_R U^{\mu}(z)\,d\lambda(z)$. If $0<t<1$, then $\mu+t\lambda$ be a measure whose support is in K. Hence $G_{\varphi}(\mu+t\lambda)\geqq G_{\varphi}(\mu)$ and from this $\int_R (U^{\mu}(z)-\varphi(z))\,d\lambda(z)\geqq 0$. On the other hand, from (2.3) and (2.4) we see that $\int_R (U^{\mu}(z)-\varphi(z))\,d\lambda(z)=-\mu(W)/2n<0$. This is a contradiction and so $C(E)=0$.

Thus we have seen that $U^{\mu}(z)-\varphi(z)\geqq d$ on K except an F_{σ}-set E with capacity zero and $U^{\mu}(z)-\varphi(z)=d$ on S_{μ} except a set of capacity zero. Hence we have $(\mu,\mu)-\int_K \varphi(z)\,d\mu(z)=d\cdot\mu(K)$. On the other hand $\mu+t\mu$ with $|t|<1$ is a measure with support contained in K. So we have $G_{\varphi}(\mu+t\lambda)-G_{\varphi}(\mu)\geqq 0$. From this $(\mu,\mu)-\int_K \varphi(z)\,d\mu(z)=0$ or $d=0$. Thus we have $U^{\mu}(z)\leqq\varphi(z)$ on S_{μ} and $U^{\mu}(z)\geqq\varphi(z)$ on K except E of capacity zero. By Lemma 2.6, we see that $E\subset\partial K$.

Finally we prove the unicity of μ. Suppose that μ' also satisfies the assertion of our theorem. Hence in particular $\|\mu'\|<\infty$. So $U^{\mu'}(z)\geqq\varphi(z)$ on K except a set of capacity zero implies $(\mu',\mu)\geqq\int_K \varphi(z)\,d\mu(z)$. Hence $\|\mu-\mu'\|^2=\|\mu\|^2+\|\mu'\|^2-2(\mu',\mu)\leqq\|\mu\|^2+\|\mu'\|^2-2\int_K \varphi(z)\,d\mu(z)=G_{\varphi}(\mu)+\|\mu'\|^2$. As $U^{\mu'}(z)=\varphi(z)$ on $S_{\mu'}$ except a set of capacity zero, so $\|\mu'\|^2=\int_K \varphi(z)\,d\mu'(z)$ or $G(\mu')=-\|\mu'\|^2$. Thus we get $\|\mu-\mu'\|^2\leqq G(\mu)-G(\mu')\leqq 0$. Thus by energy principle, $\mu=\mu'$. Q.E.D.

The following will be extensively used in Chapter III.

THEOREM 2.5. *Let K be a compact set in R and f be a non-negative supersolution on R. Then there exists a unique measure μ with $S_{\mu}\subset K$ and $U^{\mu}(z)\leqq f(z)$ on R and $U^{\mu}(z)=f(z)$ on K except an F_{σ}-set E of ∂K with $C(E)=0$.*

Proof. This follows from Theorem 2.4 and Lemma 2.4. Q.E.D.

As an application of Theorem 2.5 we prove the converse of Lemma 2.1. We state this as follows.

THEOREM 2.6. *$U^{\mu}(z)$ is a solution of* (1.1) *in $K_1(z_0)$ if and only if*

$S_\mu \frown K_1(z_0)$ *is empty.*

Proof. In virtue of Lemma 2.1, we have only to show that $S_\mu \frown K_1(z_0)$ is empty if $U^\mu(z)$ is a solution of (1.1) in $K_1(z_0)$. Let K be a disc in $K_1(z_0)$ such that $\bar{K} \subset K_1(z_0)$. Let μ' be the restriction of μ on K and $\mu'' = \mu - \mu'$. Then

$$U^\mu(z) = U^{\mu'}(z) + U^{\mu''}(z)$$

on R. Clearly $U^{\mu'}(z)$ is continuous on $R - K$. As $U^{\mu'}(z)$ and $U^{\mu''}(z)$ are lower semicontinuous and their sum, i.e. $U^\mu(z)$, is continuous on $K_1(z_0)$, so $U^{\mu'}(z)$ is continuous on $K_1(z_0)$ and so on R. Moreover $U^\mu(z)$ and $U^{\mu''}(z)$ are solutions of (1.1) in K and so $U^{\mu'}(z)$ is a solution of (1.1) in K. If we can prove $\mu' = 0$ on K, we may conclude that $S_\mu \frown K_1(z_0) = 0$ since K is arbitrary in $K_1(z_0)$. From this remark we may assume without loss of generality that $S_\mu \subset \bar{K}_1(z_0)$ and $U^\mu(z)$ is continuous on R and a solution of (1.1) in R except $\partial K_1(z_0)$.

Let K be a disc in $K_1(z_0)$ such that $\bar{K} \subset K_1(z_0)$. By F we denote a compact set $\bar{K}_1(z_0) - K$. By Theorem 2.5 there exists a unique measure λ with $S_\lambda \subset F$ and $U^\lambda(z) = U^\mu(z)$ on F except an F_σ-set E of capacity zero in ∂F and $U^\lambda(z) \leqq U^\mu(z)$ on R. By the lower semicontinuity of $U^\lambda(z)$ and the continuity of $U^\mu(z)$, we easily see that $U^\lambda(z)$ is continuous on $\partial F - E$ and $U^\lambda(z) = U^\mu(z)$ there.

Let $U^\nu(z)$ be Evans' potential of E (cf. Theorem 2.3) and $\{R_n\}$ be an exhaution of R with $R_n \supset \bar{K}_1(z_0)$ and $G_n(z, \zeta)$ be Green's function of (1.1) with respect to R_n. Put $U_n(z) = \int_{S_\mu} G_n(z, \zeta)\, d\mu(\zeta)$. Clearly $U_n{}^\mu(z) \nearrow U^\mu(z)$. For an arbitrary positive number ε, we consider

$$u(z) = U^\lambda(z) + \varepsilon U^\nu(z) - U_n{}^\mu(z),$$

which is a solution of (1.1) in $D_n = R_n - \bar{K}_1(z_0)$. It is easily seen that

$$\varliminf_{z \to \zeta} u(z) \geqq 0$$

for all ζ in ∂D_n (cf. Lemma 2.2). Hence by (I) in §1.2, we have

$$U^\lambda(z) + \varepsilon U^\nu(z) \geqq U_n{}^\mu(z)$$

on D_n. Letting n tend to infinity and next $\varepsilon \searrow 0$, we get

$$U^\lambda(z) \geqq U^\mu(z)$$

on $R - \bar{K}_1(z_0)$. Similarly by considering $U^\lambda(z) + \varepsilon U^\nu(z) - U^\mu(z)$, which is a solution of (1.1) in K, it can be proved that

$$U^\lambda(z) \geqq U^\mu(z)$$

in K. Hence we have proved that $U^\mu(z) = U^\lambda(z)$ on R except a set E of capacity zero. By unicity principle we get $\mu = \lambda$ and so $S_\mu \frown K = S_\lambda \frown K$ is empty. Since K is arbitrary, $S_\mu \frown K_1(z_0)$ is empty. Q.E.D.

2.5. Miscellaneous facts. Before closing this chapter we state two more results. Although they have their own interests but we shall make no use of them, so we omit their proofs.

1. Let $\mathfrak{E}$ be the totality of measures μ with $\|\mu\|<\infty$ and with compact S_μ and $\mathfrak{E}_K$ be the subset of $\mathfrak{E}$ whose measures have compact support contained in a fixed compact set K. Let μ_i be in $\mathfrak{E}$ and put

$$\|\mu_1-\mu_2\|^2=\iint_{R\times R} G(z,\zeta)\,d\sigma(z)\,d\sigma(\zeta), \qquad \sigma=\mu_1-\mu_2.$$

By energy principle this defines a metric on $\mathfrak{E}$. Ohtsuka [9] proved that $\mathfrak{E}_K$ is complete with respect to this metric. The same is also valid for $\mathfrak{E}$.

2. If $f(z)$ is a supersolution of (1.1) on R dominating a solution of (1.1), then we can find a unique measure μ such that $f(z)-U^\mu(z)$ is a solution of (1.1).

III. Minimal solutions of $\Delta u=pu$.

3. 1. Martin's kernel and Martin's boundary. Let $G(z,\zeta)$ be Green's function of (1.1) with respect to R and $K(z,\zeta)$ be defined by

$$K(z,\zeta)=G(z,\zeta)/G(z_0,\zeta)$$

on $R\times R$ if $\zeta\neq z_0$ and $K(z,z_0)=0$ if $z\neq z_0$ and $K(z_0,z_0)=1$. We notice that $K(z,\zeta)$ is continuous in ζ on R with fixed z in R and $K(z,\zeta)$ is a solution of (1.1) in z on $R-\{\zeta\}$ with fixed ζ in R.

We take a fixed exhaustion $\{R_n\}$ of R such that R_1 contains z_0 and put

$$d(\zeta_1,\zeta_2)=\sum_{n=1}^{\infty}2^{-n}\sup_{z\in R_n}\left|\frac{K(z,\zeta_1)}{1+K(z,\zeta_1)}-\frac{K(z,\zeta_2)}{1+K(z,\zeta_2)}\right|. \tag{3.1}$$

Then d defines a metric on R. We denote by R^* the completion of R by this metric. It is easily seen, by using fundamental properties of solutions of (1.1) stated in Chapter I, that R^* is a compact metric space with naturally extended metric d. For a point ζ in R^*-R, we can find a sequence $\{\zeta_n\}$ in R such that $d(\zeta,\zeta_n)\to 0$ and so we can define

$$K(z,\zeta)=\lim_n K(z,\zeta_n),$$

then the extended metric d is also in the form (3.1). By the fundamental properties of solutions of (1.1) described in Chapter I, we see at once that $K(z_0,\zeta)=1$ and $K(z,\zeta)$ is a solution of (1.1) in z on $R-\zeta$ with fixed ζ in R^* and $K(z,\zeta)$ is finitely continuous on $D\times(R^*-D)$, where D is an open subset of R.

We shall say R^* and $K(z,\zeta)$ to be *Martin's compactification* of R with origin z_0 and *Martin's kernel* of (1.1) on $R\times R^*$ with origin z_0 respectively. It is easy to see that the special choice of exhaustion does not affect the structure of R^* and does not change $K(z,\zeta)$. The same is also true for z_0.

We denote by ∂R the set R^*-R and we shall say that ∂R is *Martin's boundary* of R with respect to $\Delta u=pu$. Clearly ∂R is compact.

3.2. Martin potential. By a *measure* μ on R^* we mean a countably additive set function defined on the Borel field generated by compact subset of R^* and $\mu(R^*)<\infty$. Hence the restriction μ' on R of a measure μ on R^* is a measure on R. The meaning of the support and the notation S_μ are similar to that defined in § 2.1.

Selection theorem also holds for a compact subset K of R^* and the sequence of measures whose total measures are uniformly bounded and whose supports are contained in K. Here K may be R^*.

Let D be an open subset of R and μ be a measure on R. For a Borel set X in R we put

$$\mu_D(X) = \int_{D \frown X} G(z_0, \zeta)^{-1} d\mu(\zeta). \tag{3.2}$$

As $G(z_0, \zeta)$ is strictly positive and continuous on D, we see that μ_D is a measure on R with $S_{\mu_D} \subset \bar{D}$.

Using Martin's kernel $K(z, \zeta)$ on $R \times R^*$, we can define a potential $K^\mu(z)$, so called *Martin's potential*, by

$$K^\mu(z) = \int_{R^*} K(z, \zeta)\, d\mu(\zeta).$$

Clearly $K^\mu(z_0) = \mu(R^*)$. Martin potential is, of course, closely related to Green potential and the properties of the former are deduced from those of the latter by using (3.2). For example,

LEMMA 3.1. *$K^\mu(z)$ is a solution of* (1.1) *in $K_1(z_0)$ if and only if $K_1(z_0)$ $\frown S_\mu$* is empty.

Proof. By the similar method as in the proof of Theorem 1.1, $K^\mu(z)$ is a solution of (1.1) in $D = K_1(z_0)$ if $D \frown S_\mu$ is empty.

Conversely assume that $K^\mu(z)$ is a solution of (1.1) in D. Let μ' be the restriction of μ on $R^* - D$. Then $K^\mu(z) = K^{\mu'}(z) + U^{\mu_D}(z)$. As $K^\mu(z)$ and $K^{\mu'}(z)$ are solutions of (1.1) in D, so $U^{\mu_D}(z)$ is a solution of (1.1) in D and hence S_{μ_D} $\frown D$ is empty by Theorem 2.6. Thus $S_\mu \frown D$ is empty. Q.E.D.

3.3. Operator l_A. We denote by $S(R)$ the totality of non-negative supersolutions of (1.1) on R and $N(R^*)$ be the totality of closed subsets A of R^* such that $(A - \partial A) \frown R$ is a nice open subset of R.

Let A be in $N(R^*)$ and f be in $S(R)$. The set $D = R - A$ is again a nice open subset of R. We denote by $U_f{}^D$ the totality of supersolutions of (1.1) on D satisfying

$$\varliminf_{z \to \zeta} g(z) \geqq f(\zeta)$$

for every point ζ in $\partial D \frown R$. We put

$$H_f{}^D(z) = \inf(g(z);\ g \in U_f{}^D) \tag{3.3}$$

on D.

Next consider an exhaustion $\{R_n\}$ of R. Let u_n be the solution of (1.1) on $D \frown R_n$ with the boundary value $f(z)$ on $\partial D \frown R_n$ and 0 on $\partial R_n \frown D$. This boundary value function on $\partial(D \frown R_n)$ is not continuous in general but we can find such a solution as remarked in § 1.3. The boundary value of u_n is f

on $\partial D \frown R_n$ and 0 on $\partial R_n \frown D$ with possible exceptional points in $(\partial D \frown R_n) \frown (\partial R_n \frown D)$. It is easy to see that $\{u_n\}$ forms an increasing sequence dominated by $f(z)$ on D and so

$$(3.4) \qquad E_f{}^D(z) = \lim_n u_n(z)$$

is a solution of (1.1) in D with $0 \leqq E_f{}^D(z) \leqq f(z)$ on D and $E_f{}^D(z) = f(z)$ on $\partial D \frown R$.

Next decompose D into components $\{D_n\}$ and $G_{D_n}(z, \zeta)$ be Green's function of (1.1) with respect to D_n and put $G_{D_n}(z, \zeta) = 0$ for $(z, \zeta) \in D_m \times D_n$ $(n \neq m)$. We define the Green's function $G_D(z, \zeta)$ of (1.1) with respect to the open set D by

$$G_D(z, \zeta) = \sum_n G_{D_n}(z, \zeta).$$

Now set

$$(3.5) \qquad G_f{}^D(z) = \frac{1}{2\pi} \int_{\partial D \frown R} \frac{\partial}{\partial n} G_D(z, \zeta) f(\zeta)\, ds,$$

where $\partial / \partial n$ denotes the inner normal derivation with respect to D and ds denotes the line element of $\partial D \frown R$.

Now we remark the following

LEMMA 3.2. *Let $\{v_m\}$ be the positive solutions of (1.1) in $D = K_1(z_0) \frown \{z; \operatorname{Im}(z - z_0) > 0\}$ and continuous on $\bar{D}$. Suppose that $v_m = 0$ on $\alpha = K_1(z_0) \frown \{z; \operatorname{Im}(z - z_0) = 0\}$ and $\partial v_m / \partial n$ exists on α and $\{v_m\}$ converges to 0 decreasingly and uniformly on each compact subset of $\bar{D} - \bar{\alpha}$. Then*

$$\lim_m (\partial v_m / \partial n)(z) = 0$$

on α and $\{v_m\}$ converges to 0 uniformly on each compact subset of $D \smile \alpha$.

Proof. Let h_m be the harmonic function such that $h_m = v_m$ on ∂D. Then $\{h_m\}$ forms a decreasing sequence converging to 0 on each compact subset of $\bar{D} - \bar{\alpha}$. Moreover $0 \leqq v_m \leqq h_m$ on D and so

$$0 \leqq (\partial v_m / \partial n)(z) \leqq (\partial h_m / \partial n)(z)$$

on α, since $h_m = 0$ on α. Clearly $\lim_m (\partial h_m / \partial n)(z) = 0$ on α and thus the assertion of our Lemma follows. Q.E.D.

LEMMA 3.3. $\qquad H_f{}^D(z) = E_f{}^D(z) = G_f{}^D(z).$

Proof. As $E_f{}^D(z) \in U_f{}^D$, so $E_f{}^D(z) \geqq H_f{}^D(z)$. On the other hand, the function $u_n(z)$ in (3.3) clearly satisfies $u_n \leqq g$ on $R_n \frown D$ for any g in $U_f{}^D$. From this we get $E_f{}^D(z) \leqq H_f{}^D(z)$ and so $E_f{}^D = H_f{}^D$ on D.

Let $G^{(m)}(z, \zeta)$ be Green's function of (1.1) with respect to $D \frown R_m$. By (G. 3) and (1.2), $G_D(z, \zeta) - G^{(m)}(z, \zeta)$ satisfies the assumption of Lemma 3.1 as the function of ζ with fixed z and so $(\partial G^{(m)} / \partial n)(z, \zeta) \nearrow (\partial G_D / \partial n)(z, \zeta)$. By putting $(\partial G^{(m)} / \partial n)(z, \zeta) = 0$ on $\partial D - \partial D \frown R_m$ and by Green's formula

$$u_m(z)=\frac{1}{2\pi}\int_{\partial D\frown R}\frac{\partial}{\partial n}G^{(m)}(z,\zeta)f(\zeta)\,ds.$$

Letting m tend to infinity, we get

$$E_f{}^D(z)=\frac{1}{2\pi}\int_{\partial D\frown R}\frac{\partial}{\partial n}G_D(z,\zeta)f(\zeta)\,ds. \qquad \text{Q.E.D.}$$

Definition of the operator l_A. For $A\in N(R^*)$ and $f\in S(R)$,

$$(3.6) \qquad (l_Af)(z)=\begin{cases} f(z) & \text{on } A, \\ H_f{}^{R-A}(z) & \text{on } R-A. \end{cases}$$

It is easily seen that l_Af belongs to $S(R)$. Hence l_A is an operator of $S(R)$ into $S(R)$. We have

LEMMA 3.5. *The operator l_A possesses the following properties*:
(i) $0\leqq l_Af\leqq f$;
(ii) $f\leqq g$ *implies* $l_Af\leqq l_Ag$;
(iii) $l_A(cf+g)=cl_Af+l_Ag$, *where c is non-negative number*;
(iv) $l_{A\smile B}f\leqq l_Af+l_Bf$;
(v) $A\supset B$ *implies* $l_Af\geqq l_Bf$ and $l_A(l_Bf)=l_B(l_Af)=l_Bf$;
(vi) $\lim_n f_n\in S(R)$ *implies*

$$l_A(\lim_n f_n)\leqq \underline{\lim}_n\, l_Af_n$$

and moreover the existence of $g\in S(R)$ such that $g\geqq f_n$ $(n=1, 2, \cdots)$ implies

$$l_A(\lim_n f_n)=\lim_n l_Af_n;$$

(vii) $l_AK(z,\zeta)$ *is lower semicontinuous on ∂R with respect to ζ*;
(viii) $R\frown A_n\nearrow R\frown A$, *i.e.* $R\frown A=R\frown(\cup_n A_n)$ *implies*

$$\lim_n l_{A_n}f=l_Af.$$

Proof. The properties (i)-(v) is clear from the definition of l_A. By using the representation $l_Af=G_f{}^{R-A}$ on $R-A$, (vi) follows at once from Fatou's theorem and Lebesgue's theorem. (vii) is a direct consequence of (vi).

To prove (viii), put $f_n=l_{A_n}f$. As $\{f_n\}$ is an increasing sequence dominated by l_Af, so we may set $g=\lim_n f_n$ on R. In any point of A, $g=f$. In $R-A$, g is a solution of (1.1) from monotone compactness of $\{f_n\}$, which is a sequence of (1.1) in $R-A$. In each point ζ in $\partial A\frown R$, $\lim_{R-A\ni z\to\zeta}f_n(z)=f(\zeta)$ for some n. Thus $\lim_{R-A\ni z\to\zeta}g(z)=f(\zeta)$ at each point ζ in ∂A. Thus $l_Af=H_f{}^{R-A}\leqq g\leqq l_Af$ in $R-A$. Hence $l_Af=g$ on R. Q.E.D.

LEMMA 3.6. *For any A in $N(R^*)$ there exists a measure λ on R^* such that $S_\lambda\subset A$ and*

$$(l_Af)(z)=K^\lambda(z)$$

holds on R.

Proof. Let $\{R_n\}$ be an exhaustion of R and $A_n=A\frown R_n$. By the continuity of l_A with respect to A,

$$(l_A f)(z) = \lim_n (l_{A_n} f)(z)$$

on R. By Theorem 2.5, there exists a measure μ_n on R with $S_{\mu_n} \subset A_n$ such that

$$(l_{A_n} f)(z) = U^{\mu_n}(z)$$

on A_n except an F_σ-set E of ∂A_n with $C(E)=0$ and $(l_{A_n} f)(z) \geqq U^{\mu_n}(z)$ on R. Let $U^\nu(z)$ be Evans' potential of E on R and ε be an arbitrary positive number. Then

$$\varliminf_{R-\bar{A} \ni z \to \zeta} (U^{\mu_n}(z) + \varepsilon U^\nu(z)) \geqq f(\zeta)$$

on ∂A_n and by $l_{A_n} f = H_f^{R-A_n}$ on $R - A_n$, we get $U^{\mu_n} + \varepsilon U^\nu \geqq l_{A_n} f$ on $R - A_n$. Letting $\varepsilon \searrow 0$, we have $U^{\mu_n} \geqq l_{A_n} f$ on $R - A_n$. Hence

$$U^{\mu_n}(z) = (l_{A_n} f)(z)$$

on $R - E$ and by Lemma 2.6 this holds on the whole space R.

Now put

$$d\lambda_n(\zeta) = G(z_0, \zeta)\, d\mu_n(\zeta).$$

Then

$$(l_{A_n} f)(z) = K^{\lambda_n}(z).$$

As $\lambda_n(R^*) = K^{\lambda_n}(z_0) \leqq (l_A f)(z_0)$, so we may assume that λ_n converges to a measure λ vaguely. Now let D be a nice compact subdomain of R and μ_n' be the restriction of μ_n on $\bar{D}$ and λ_n' be the restriction of λ_n on $R^* - \bar{D}$. Clearly $\{\mu_n'(R^*)\}$ and $\{\lambda_n'(R^*)\}$ are bounded and so we may assume that μ_n' and λ_n' converge to μ' and λ' vaguely, respectively. Clearly $\lambda = \lambda' + \mu'$ and $K^\lambda = K^{\lambda'} + U^{\mu'}$.

If z lies in D, then $K(z, \zeta)$ is continuous on $R^* - \bar{D}$ as a function of ζ and so $\lim_n K^{\lambda_n'}(z) = K^{\lambda'}(z)$. By Lemma 2.8, $U^{\mu'}(z) = \varliminf_n U^{\mu_n'}(z)$ on R except a set of capacity zero. As $(l_{A_n} f)(z) = K^{\lambda_n'}(z) + U^{\mu_n'}(z)$, so

$$(l_A f)(z) = K^{\lambda'}(z) + U^{\mu'}(z) = K^\lambda(z)$$

on D except a set of capacity zero. The difference $(l_A f)(z) - K^{\lambda'}(z)$ is continuous on D and equals to $U^{\mu'}(z)$ except a set of capacity zero and so by Lemma 2.6 $(l_A f)(z) - K^{\lambda'}(z) = U^{\mu'}(z)$ on D or $(l_A f)(z) = K^\lambda(z)$ on D. Since D is arbitrary $(l_A f)(z) = K^\lambda(z)$ on R. Q.E.D.

3.4. Operator L_X. We denote by $\mathfrak{P}(R)$ the totality of non-negative solutions of (1.1) on R. This is a subset of $S(R)$. Let F be a compact set contained in ∂R. Let $N(F)$ be the totality of A in $N(R^*)$ such that the open kernel of A contains F. We easily see that $N(F)$ constitutes a fundamental base of neighborhood system of F. Needless to say $N(F)$ contains a countable subbase.

For closed set F in ∂R we define a function $L_F u$ on R by

$$(L_F u)(z) = \inf\{(l_A u)(z);\ A \in N(F)\} \tag{3.7}$$

for each u in $\mathfrak{P}(R)$. We now show that $L_F u \in \mathfrak{P}(R)$. In fact, we can choose a decreasing sequence $A_n \in N(F)$ such that $\cap A_n = F$ and $(L_F u)(z_0) = \lim_n (l_{A_n} u)(z_0)$. By monotone compactness, $\lim_n l_{A_n} u = v$ belongs to $\mathfrak{P}(R)$. Next choose an arbitrary point z in R. We can also choose a decreasing sequense $B_n \in N(F)$ such that $\cap B_n = F$ and $(L_F u)(z) = \lim_n (l_{B_n} u)(z)$. By considering $B_n \cap A_n$, we may assume $B_n \subset A_n$. By monotone compactness, $\lim_n l_{B_n} u = w$ belongs to $\mathfrak{P}(R)$ and $v \geqq w$. As $v(z_0) = w(z_0)$, so by (II) in § 1.2, $v = w$. This shows that $v(z) = (L_F u)(z)$ at every point z in R. Thus $L_F u \in \mathfrak{P}(R)$. Thus L_F defines an operator of $\mathfrak{P}(R)$ into $\mathfrak{P}(R)$.

Incidently we have also proved that there exists a sequence $\{A_n\}$ in $N(F)$ such that $A_n \searrow F$ and

(3.8) $$L_F u = \lim_n l_{A_n} u.$$

LEMMA 3.7. *The operator L_F possesses the following properties*:
(i) $0 \leqq L_F u \leqq u$;
(ii) $u \leqq v$ *implies* $L_F u \leqq L_F v$;
(iii) $L_F(cu + v) = cL_F u + L_F v$, *where c is a non-negative constant*;
(iv) $L_{F \cup K} u \leqq L_F u + L_K u$;
(v) $L_{\partial R} u = u$;
(vi) $F \supset K$ *implies* $L_F u \geqq L_K u$ *and* $L_F(L_K u) = L_K u$;
(vii) $F_n \searrow F$, *i.e.* $\cap F_n = F$ *implies* $L_{F_n} u \searrow L_F u$.

Proof. These can be verified easily from (3.8) and Lemma 3.5. Q.E.D.

Nextly let G be a relatively open subset of ∂R. We put

(3.9) $$(L_G u)(z) = \sup \{(L_F u)(z);\ F \text{ is compact in } \partial R \text{ and } F \subset G\}$$

for each u in $\mathfrak{P}(R)$. By the similar method by which we proved $L_F u \in \mathfrak{P}(R)$, we can prove $L_G u \in \mathfrak{P}(R)$. From Lemma 3.7, we easily get

LEMMA 3.8. *The operator L_G possesses the following properties*:
(i) $0 \leqq L_G u \leqq u$;
(ii) $u \leqq v$ *implies* $L_G u \leqq L_G v$;
(iii) $L_G(cu + v) = cL_G u + L_G v$;
(iv) $L_{\bigcup_{n=1}^{\infty} G_n} u \leqq \sum_{n=1}^{\infty} L_{G_n} u$;
(v) $G \supset U$ *implies* $L_G u \geqq L_U u$ *and* $L_G(L_U u) = L_U u$;
(vi) $G_n \nearrow G$, *i.e.* $\cup G_n = G$ *implies* $L_{G_n} u \nearrow L_G u$.

Finally for arbitrary set X in ∂R, we set

(3.10) $$(L_X^* u)(z) = \inf \{(L_G u)(z),\ G \text{ is relatively open set in } \partial R \text{ and } G \supset X\}$$

for each u in $\mathfrak{P}(R)$. Similarly as before, it can be shown that $L_X^* u \in \mathfrak{P}(R)$. From Lemma 3.8 and (vii) in Lemma 3.7, it follows the following

LEMMA 3.9. *The operator L_X^* possesses the following properties*:
(i) $0 \leqq L_X^* u \leqq u$;
(ii) $u \leqq v$ *implies* $L_X^* u \leqq L_Y^* v$;

(iii) $L_X^*(cu+v) = cL_X^*u + L_X^*v$, *where c is a non-negative constant;*

(iv) $L^*_{\bigcup_{n=1}^{\infty} X_n} u \leqq \sum_{n=1}^{\infty} L_{X_n}^* u$;

(v) $X \supset Y$ *implies* $L_X^* u \geqq L_Y^* u$ *and* $L_Y^*(L_Y^* u) = L_Y^* u$;

(vi) *for any relatively open subset G of* ∂R, $L_G^* u = L_G u$;

(vii) *for any compact set F of* ∂R, $L_F^* u = L_F u$.

Hence we may use the notation L_X instead of L_X^*.

LEMMA 3.10. *For any compact set F in* ∂R, *there exists a measure* μ *with* $S_\mu \subset \partial R$ *and*

$$(L_F u)(z) = K^\mu(z) = \int_F K(z, \zeta)\, d\mu(\zeta).$$

Proof. We choose $A_n \in N(F)$ such that $A_n \searrow F$ and $l_{A_n} u \searrow L_F u$. By Lemma 3.6, there exists a measure μ_n with $S_{\mu_n} \subset A_n$ and $l_{A_n} u = K^{\mu_n}$. As $\mu_n(A_n) = K^{\mu_n}(z_0) \leqq u(z_0)$, so we may assume that $\{\mu_n\}$ converges to a measure μ vaguely. Hence $K^{\mu_n}(z) \to K^\mu(z)$ for each point z in R, since $K(z, \zeta)$ is continuous on $R^* - z$ with respect to ζ. The fact that $S_{\mu_n} \subset A_m$ $(n \geqq m)$ implies $S_\mu \subset A_m$ and so $S_\mu \subset F$. Q.E.D.

LEMMA 3.11. *For any u in* $\mathfrak{P}(R)$ *there exists a measure* μ *with* $S_\mu \subset \partial R$ *such that*

$$u(z) = \int_{\partial R} K(z, \zeta)\, d\mu(\zeta).$$

Conversely, for any measure μ *with* $S_\mu \subset \partial R$ *the function*

$$\int_{\partial R} K(z, \zeta)\, d\mu(\zeta)$$

belongs to $\mathfrak{P}(R)$.

Proof. The first assertion follows from Lemma 3.8, since $u = L_{\partial R} u$. The second assertion is obvious. Q.E.D.

We introduce a Fréchet norm $|||u|||$ in $\mathfrak{P}(R)$ by

$$|||\tilde{u}||| = \sum_{n=1}^{\infty} 2^{-n} \sup_{R_n} |\tilde{u}(z)|\,(1 + |\tilde{u}(z)|)^{-1}, \qquad \tilde{u} \in \mathfrak{P}(R) \ominus \mathfrak{P}(R),$$

where $\{R_n\}$ is an exhaustion of R. Then clearly $\mathfrak{P}(R)$ is complete with respect to $|||u|||$. Now for each subset X of ∂R let $\mathfrak{L}_X$ be the totality of u with the form

$$u(z) = \sum_{i=1}^{\infty} c_i K(z, \zeta_i),$$

where c_i are non-negative constants and $\zeta_i \in X$.

Then Lemma 3.10 and 11 are equivalent to the following: $\mathfrak{L}_F$ *is dense in* $L_F \mathfrak{P}(R)$ *with respect to* $|||u|||$ *for each compact set F in* ∂R.

3.5. Minimal solutions. We say that a solution of (1.1) is minimal if $u \in \mathfrak{P}(R)$, $u \neq 0$ and $u \geqq v$ for some v in $\mathfrak{P}(R)$ implies the existence of a constant c_v such that $v = c_v u$. We denote by $\mathfrak{P}^M(R)$ the totality of minimal solutions u of (1.1).

Let u be in $\mathfrak{P}^M(R)$ and K be a compact set in ∂R. We can find a constant c such that

$$L_K u = cu.$$

By (vi) in Lemma 3.7, $L_K(L_K u) = L_K u$. Hence $L_K u = cL_K u$ or $c^2 = c$. Thus

$$L_K u = 0 \qquad \text{or} \qquad L_K u = u.$$

Assume that the latter is the case. Let $K = \cup_{i=1}^n F_i$, where F_i is compact and the diameter $d(F_i) < 2^{-1}$. By (iii) in Lemma 3.7,

$$\sum_{i=1}^n L_{F_i} u \geqq L_K u = u.$$

Thus one of $L_{F_i} u$ satisfies $L_{F_i} u = cu$ $(c > 0)$. Applying L_{F_i}, we see that $c = 1$. Hence we showed the existence of a compact set $K_1 \subset K$ such that $d(K_1) < 2^{-1}$ and $L_{K_1} u = u$. Repeating this process, we get a sequence $\{K_n\}$ of compact sets such that $K \supset K_1 \supset K_2 \supset \cdots \supset K_n \supset \cdots$ with $d(K_n) < 2^{-n}$ and $L_{K_n} u = u$. Then $\cap K_n$ is a one point ζ in K and by (vii) in Lemma 3.7,

$$L_\zeta u = u.$$

Thus by Lemma 3.10

$$u = u(z_0) K(z, \zeta), \qquad \zeta \in \partial R.$$

Finally suppose that $L_\zeta K(z, \zeta) \neq K(z, \zeta)$. By Lemma 3.10, we get a non-negative number $c = L_\zeta K(z_0, \zeta) < 1$ such that

$$L_\zeta K(z, \zeta) = cK(z, \zeta).$$

By applying again L_ζ, $c = c^2$ or $c = 0$. So $L_\zeta K(z, \zeta) = 0$.

From the above consideration, we obtain

LEMMA 3.12. (i) *Any minimal solution of* (1.1) *is a multiple of a* $K(z, \zeta)$ *with* ζ *in* ∂R.

(ii) $L_\zeta K(z, \zeta) = K(z, \zeta)$ *or* $= 0$ *on* R *and* $K(z, \zeta)$ $(\zeta \in \partial R)$ *is minimal if and only if* $L_\zeta K(z, \zeta) = K(z, \zeta)$ *on* R.

(iii) *Assume* $K(z, \zeta)$ *is minimal. Then* $L_F K(z, \zeta) = K(z, \zeta)$ *or* $= 0$ *on* R *according to* $\zeta \in F$ *or* $\zeta \notin F$, *where* F *is a compact set in* ∂R.

Now we put

$$M = \{\zeta \in \partial R;\ L_\zeta K(z, \zeta) = K(z, \zeta) \text{ on } R\}$$

and

$$E = \{\zeta \in \partial R;\ L_\zeta K(z, \zeta) = 0 \text{ on } R\}.$$

Clearly $M \frown E$ is empty and $M \smile E = \partial R$.

Now fix a constant a with $0 < a < 1$. Assume that for a point ζ in ∂R we can find an integer $n \geqq 1$ such that for any $A \in N(\zeta)$ with $d(A) < 1/n$ implies $l_A K(z_0, \zeta) \leqq a$. Then we can find a smallest integer $n(\zeta)$ with the above prescrived property. If for a point ζ in ∂R we cannot find such an integer, then we put $n(\zeta) = \infty$. It is clear from the definition of L_ζ that $n(\zeta) < \infty$ for any ζ in E and $n(\zeta) = \infty$ for any ζ in M. Hence we may write

$$M=\{\zeta\in\partial R;\ n(\zeta)=\infty\} \quad\text{and}\quad E=\{\zeta\in\partial R;\ n(\zeta)<\infty\}.$$

We put

$$E_n=\{\zeta\in\partial R;\ n(\zeta)<n\}.$$

From (vii) in Lemma 3.5, E_n is compact. Clearly $E_n\nearrow E$, i.e. $E=\cup E_n$.

Thus we have shown that M is a G_δ-set with respect to ∂R and E is an F_σ-set.

LEMMA 3.13. *Let $X\subset E$, Y be an arbitrary set in ∂R and $u\in\mathfrak{P}(R)$. Then*

(i) $L_X u=0$;

(ii) $L_Y u=L_{Y\frown M}u$.

Proof. First let K be compact in E_n with $d(K)<1/n$. We can find $A\in N(K)$ with $d(A)<1/n$. Let $v_m\in\mathfrak{L}_K$ such that $|||v_m-L_K u|||\to 0$. Let $v_m=\sum c_i K(z,\zeta_i)$ where $\zeta_i\in K$. Then

$$l_A v_m(z_0)=\sum c_i l_A K(z_0,\zeta_i)\leqq a\sum c_i K(z_0,\zeta_i)=a v_m(z_0).$$

From (vi) in Lemma 3.5,

$$l_A L_K u(z_0)\leqq \underline{\lim}_m l_A v_m(z_0)\leqq \lim_m a v_m(z_0)=a L_K u(z_0).$$

Since $l_A(L_K u)\searrow L_K(L_K u)=L_K u$, $L_K u(z_0)\leqq aL_K u(z_0)$. Thus $L_K u(z_0)=0$.

Next we can write $E_n=\cup_{i=1}^N K_i$ with $d(K_i)<1/n$. Then

$$L_{E_n}u\leqq \sum_i L_{K_i}u=0.$$

Finally by (iv) in Lemma 3.9,

$$L_X u\leqq L_E u\leqq \sum_n L_{E_n}u=0.$$

Thus we have proved (i).

Now we prove (ii). As $L_{Y\frown E}u=0$, so

$$L_Y u=L_{(Y\frown E)\smile(Y\frown M)}u\leqq L_{(Y\frown E)}u+L_{(Y\frown M)}u=L_{Y\frown M}u\leqq L_Y u,$$

which implies $L_Y u=L_{Y\frown M}u$. Q.E.D.

LEMMA 3.14. *Let $u\in\mathfrak{P}(R)$. Then there exists a measure μ on ∂R such that $\mu(E)=0$ and*

$$u(z)=\int_M K(z,\zeta)\,d\mu(\zeta).$$

Proof. There exists a decreasing sequence $\{u_n\}_{n=0}^\infty\subseteq\mathfrak{P}(R)$ with $u_0=u$ and a sequence $\{K_n\}_{n=1}^\infty$ of compact set in M such that

$$u_{n-1}=L_{K_n}u_{n-1}+u_n \quad\text{and}\quad u_n(z_0)\leqq 1/n.$$

This can be shown by induction as follows. First put $u_0=u$. As $L_E u_0=0$, so there exists a relatively open subset G_1 of ∂R such that $G_1\supset E$ and $L_{G_1}u_0(z_0)<1$. Put $K_1=\partial R-G_1$. Then K_1 is a compact subset of M. Let $u_1=u_0-L_{K_1}u_0$. Then $0\leqq u_1\leqq u_0$ and $u_0=L_{K_1}u_0+u_1$. As $u_0=L_{\partial R}u_0\leqq L_{G_1}u_0+L_{K_1}u_0$, so $u_1=u_0-L_{K_1}u_0\leqq L_{G_1}u_0$ and hence $u_1(z_0)\leqq L_{G_1}u_0(z_0)\leqq 1$. Thus u_0 and K_1 are required one.

Assume that u_i $(0 \leqq i < n)$ and K_i $(1 \leqq i < n)$ are required ones. As $L_E u_{n-1} = 0$, so there exists a relatively open subset G_n of ∂R such that $G_n \supset E$ and $L_{G_n} u_{n-1}(z_0) < 1/n$. Put $K_n = \partial R - G_n$. Then K_n is a compact subset of M. Let $u_n = u_{n-1} - L_{K_n} u_{n-1}$. Then $0 \leqq u_n \leqq u_{n-1}$ and $u_{n-1} = u_n + L_{K_n} u_{n-1}$. As u_{n-1} $L_{\partial R} u_{n-1} \leqq L_{G_n} u_{n-1} + L_{K_n} u_{n-1}$, so $u_n = u_{n-1} - L_{K_n} u_{n-1} \leqq L_{G_n} u_{n-1}$ and hence $u_n(z_0) \leqq L_{G_n} u_{n-1}(z_0) < 1/n$. Thus our induction is completed.

Next we get

$$u_0 = \sum_{i=1}^{n} L_{K_i} u_{i-1} + u_n.$$

Clearly $u_n \searrow 0$, and so we get

$$u = \sum_{n=1}^{\infty} L_{K_n} u_{n-1}.$$

By Lemma 3.10, we have a measure μ_n with $S_{\mu_n} \subset K_n$ such that $L_{K_n} u_{n-1}(z) = K^{\mu_n}(z)$ on R. As

$$\sum_{i=1}^{n} \mu_i(\partial R) = \sum_{i=1}^{n} K^{\mu_i}(z_0) \leqq u(z_0) \quad \text{and} \quad \sum_{i=1}^{n} \mu_i(E) = 0,$$

so

$$\mu = \sum_{n=1}^{\infty} \mu_n$$

defines a measure on ∂R with $\mu(E) = 0$ and $u = \sum_{n=1}^{\infty} K^{\mu_n} = K^{\mu}$. Q.E.D.

LEMMA 3.15. *Let μ be a measure on ∂R with $\mu(E) = 0$ and*

$$u(z) = \int_M K(z, \zeta)\, d\mu(\zeta).$$

Then $L_G u(z) = \int_{G \cap M} K(z, \zeta)\, d\mu(\zeta)$ for any open set G in ∂R. In particular

$$\mu(G) = L_G u(z_0).$$

Thus the measure μ in Lemma 3.14 is unique.

Proof. Take a sequence $\{K_n\}$ of compact sets in M such that $K_n \nearrow$ and $\mu(M - K_n) \searrow 0$. We put

$$u_n(z) = \int_{K_n} K(z, \zeta)\, d\mu(\zeta).$$

Then $u_n \nearrow u$. As $0 \leqq L_G u - L_G u_n = L_G(u - u_n) \leqq u - u_n \searrow 0$, so we have $L_G u_n \nearrow L_G u$. If we can prove

$$L_G u_n(z) = \int_{G \cap K_n} K(z, \zeta)\, d\mu(\zeta),$$

then we get

$$L_G u(z) = \int_{G \cap M} K(z, \zeta)\, d\mu(\zeta).$$

Thus we may assume that $K = S_\mu \subset M$ and so

$$u(z)=\int_K K(z,\ \zeta)\,d\mu(\zeta).$$

We put $F=K-G$, which is a compact set. Let H be a compact set in G. If we can prove

$$L_H u(z)=\int_{K\frown H} K(z,\ \zeta)\,d\mu(\zeta), \tag{3.11}$$

then taking the supremum with respect to $H\subset G$ we get $L_G u(z)=\int_{K\frown G} K(z,\zeta)\,d\mu(\zeta)$. Hence we have only to prove (3.11).

Let $A_n\in N(H)$ and $A_n\searrow H$. First we prove that $l_{A_n}K(z_0,\ \zeta)\searrow 0$ uniformly for $\zeta\in F$. Contrary to the assertion, assume the existence of positive number η and points ζ_n in F such that

$$l_{A_n}K(z_0,\ \zeta_n)\geqq\eta.$$

Taking a subsequence, if necessary, we may assume $\zeta_n\to\zeta_0\in F$. By Lemma 3.10, there exists a measure ν_n on R^* with $S_{\nu_n}\subset A_n$ such that

$$l_{A_n}K(z,\ \zeta_n)=K^{\nu_n}(z)$$

on R. As $\nu_n(R^*)=K^{\nu_n}(z_0)=l_{A_n}K(z_0,\ \zeta_n)$, so $\eta\leqq\nu_n(R^*)\leqq 1$. Taking again a subsequence, if necessary, we may assume that $\{\nu_n\}$ converges to ν vaguely. Clearly $S_\nu\subset H$ and $1\geqq\nu(H)\geqq\eta$. Since

$$K(z,\ \zeta_n)\geqq l_{A_n}K(z,\ \zeta_n)=\int_{A_n} K(z,\ \zeta)\,d\nu_n(\zeta),$$

so

$$K(z,\ \zeta_0)\geqq\int_H K(z,\ \zeta)\,d\nu(\zeta)>0.$$

Easily, we can select a sequence $\{H_m\}$ of compact sets such that

$$H\supset H_1\supset H_2\supset\cdots\supset H_m\cdots$$

and the diameter of H_m is less than $1/m$ and $\nu(H_m)>0$. As $K(z,\ \zeta_0)$ is minimal and

$$K(z,\ \zeta_0)\geqq\int_{H_m} K(z,\ \zeta)\,d\nu(\zeta)>0,$$

so we get a sequence $\{a_m\}$ of positive numbers such that

$$K(z,\ \zeta_0)=a_m\int_{H_m} K(z,\ \zeta)\,d\nu(\zeta).$$

Putting

$$\lambda_m(X)=a_m\nu(X\frown H_m),$$

we have

$$K(z,\ \zeta_0)=K^{\lambda_m}(z)$$

on R and putting $z=z_0$ we see that $\lambda_m(H_m)=1$. Selecting a subsequence, if necessary, we may assume that λ_m converges to a measure λ vaguely and clearly

$$S_\lambda=\bigcap_{m=1}^{\infty}H_m,$$

which must consist of only one point ξ in H and $\lambda(\xi)=1$. Thus we get $K(z, \zeta_0)=K(z, \xi)$, which shows $\zeta_0=\xi\in H$. This contradicts the fact that $\zeta_0\in F=K-G$ and $H\subset G$.

Now

$$u(z)=\int_{K\frown G}K(z, \zeta)\,d\mu(\zeta)+\int_F K(z, \zeta)\,d\mu(\zeta).$$

As $\int_F K(z, \zeta)\,d\mu(\zeta)$ is the limit of a sequence $\{v_m\}$ such that v_m is the form

$$v_m(z)=\sum_{\nu=1}^{N_m}c_\nu K(z, \zeta_\nu),$$

where $\zeta_\nu\in F$, so from the above we can find a sequence $a_n\searrow 0$ such that

$$l_{A_n}v_m(z_0)=\sum_\nu c_\nu l_{A_n}K(z_0, \zeta_\nu)\leqq\sum_\nu c_\nu a_n\leqq a_n\sum_\nu c_\nu K(z_0, \zeta_\nu)=a_n v_m(z_0).$$

Hence by (vi) in Lemma 3.5,

$$L_H\int_F K(z_0, \zeta)\,d\mu(\zeta)\leqq L_{A_n}\int_F K(z_0, \zeta)\,d\mu(\zeta)\leqq \underline{\lim}_m\, L_{A_n}v_m(z_0)\leqq a_n L_H\int_F K(z_0, \zeta)\,d\mu(\zeta).$$

This shows that

$$L_H u(z)=L_H\int_{K\frown G}K(z, \zeta)\,d\mu(\zeta). \tag{3.12}$$

By monotone continuity of L_H, making $G\searrow H$ we have

$$L_H u(z)=L_H\int_{K\frown H}K(z, \zeta)\,d\mu(\zeta).$$

Next let $H_n=\{\zeta\in\partial R;\ d(\zeta, \zeta')\geqq 1/n,\ \zeta'\in K\frown H\}$. Then H_n is compact and $H_n\frown(K\frown H)$ is empty, so by considering $(K\frown H, H_n)$ as (F, H) in the preceding argument, we see $L_{H_n}\int_{K\frown H}K(z, \zeta)\,d\mu(\zeta)=0$ and

$$L_{H\frown H_n}\int_{K\frown H}K(z, \zeta)\,d\mu(\zeta)\leqq L_H\int_{K\frown H}K(z, \zeta)\,d\mu(\zeta)\leqq\int_{K\frown H}K(z, \zeta)\,d\mu(\zeta).$$

making $n\nearrow\infty$ and so $H\smile H_n\nearrow\partial R$, we get

$$\int_{K\frown H}K(z, \zeta)\,d\mu(\zeta)=L_H\int_{K\frown H}K(z, \zeta)\,d\mu(\zeta).$$

Combining this with (3.12), we get (3.11). Q.E.D.

From Lemmas 3.14 and 15, we finally obtain

THEOREM 3.1. *For any u in $\mathfrak{P}(R)$, there exists a unique measure μ such that $\mu(\partial R-M)=0$ and*

$$u(z)=\int_M K(z, \zeta)\,d\mu(\zeta),$$

where M is the totality of points ζ in ∂R such that $K(z, \zeta)$ is minimal in $\mathfrak{P}(R)$.

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