# ON THE NON-SEPARABILITY OF SINGULAR REPRESENTATION OF OPERATOR ALGEBRA

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In [3], Feldman and Fell have raised the question whether any separable representation of a $W^*$-algebra without the direct summand of finite type I is always $\sigma$-weakly continuous or not and they have shown that this is almost affirmative, but the case of type $\mathrm{II}_1$ (that is, finite and of type II) remains open. The purpose of the present note is to settle this remainning problem in its positive sense.

We have investigated, in [7], the conjugate space of operator algebra and have given an alternative proof of some parts of their above results. We shall use the notation and the result in [7].

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In the proof of our theorem we shall also use the following lemma which played an essential role in [3].

LEMMA. *Let $S$ be the set of all sequences of integers $J=\{j_1, j_2, \cdots\}$ such that $1 \leqq j_n \leqq 2^n$ for each $n$. Then there exists a subset $S_0$ of the power of the continuum, such that, for any two distinct sequences $J$, $J'$ in $S_0$, the set of all $n$ for which $j_n = j_n'$ is finite.*

Now let $\boldsymbol{M}$ be a $C^*$-algebra and $\varphi$ a positive linear functional on $\boldsymbol{M}$. Putting

$$\mathfrak{m}_\varphi = \{x \in M;\ \langle x^*x, \varphi\rangle = 0\}$$

which is called the left kernel of $\varphi$, the quotient space $\boldsymbol{M}/\mathfrak{m}_\varphi$ becomes the pre-Hilbert space in the usual way canonically induced inner product by $\varphi$. We denote the element of $\boldsymbol{M}/\mathfrak{m}_\varphi$ corresponding to $x \in \boldsymbol{M}$ by $\eta_\varphi(x)$. Then we get a Hilbert space $H_\varphi$ as the completion of $\boldsymbol{M}/\mathfrak{m}_\varphi$ and a representasion $\pi_\varphi$ of $\boldsymbol{M}$, as the left multiplication operators on $H_\varphi$, where $\pi_\varphi$ is called cyclic representation or $\varphi$-representation. When $H_\varphi$ is separable, we shall call $\varphi$ separable positive linear functional. If a positive linear functional $\psi$ is majorized by a scalar multiple of $\varphi$, $H_\psi$ is considered as a closed subspace of $H_\varphi$ by [6]. Hence any positive linear functional majorized by separable one is separable too, which

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implies that the singular part of every separaple positive linear functional is so.

Now we state our main theorem in the following form:

THEOREM. *Let $M$ be a $W^*$-algebra of finite type II, then any separable positive linear functional on $M$ is necessarily $\sigma$-weakly continuous.*

Proving this theorem, we can show that the unsettled part of the conjecture in [3] is affirmative. That is,

COROLLARY. *Any separable representation of a $W^*$-algebra of finite type II is necessarily $\sigma$-weakly continuous.*

*Proof of Corollary.* Let $H$ be a separable Hilbert space, $M$ a $W^*$-algebra of finite type II and $\pi$ a representation of $M$ onto $H$. For a unit vector $\xi \in H$, put $<x, \varphi> = (\pi(x)\xi, \xi)$ for each $x \in M$. Then we have

$$<y^*x, \varphi> = (\pi(x)\xi, \pi(y)\xi),$$

so that $H_\varphi \cong [\pi(M)\xi]$. It follows that $\varphi$ is a separable state on $M$. Hence our above theorem implies the $\sigma$-weak continuity of $\varphi$, so that $\pi$ is $\sigma$-weakly continuous.

Thus, combinning this result with that of Feldman and Fell, it is shown that *any separable representation of a $W^*$-algebra without the direct summand of finite type* I *is necessarily $\sigma$-weakly continuous.*

*Proof of theorem.* From the above discussion, it suffices to prove that there exists no singular separable state on $W^*$-algebra of finite type II. Suppose that there exists a separable singular state $\varphi$ on $M$. We shall show that this is impossible.

By [8], the separability and the singularity of $\pi_\varphi$ imply that $\pi_\varphi$ cannot be faithful on $eMe$ for any non-zero projection $e$ of $M$, so that the kernel $\pi_\varphi^{-1}(O)$ of $\pi_\varphi$ is $\sigma$-weakly dense in $M$. Hence the unit $I$ of $M$ is the $\sigma$-weak limit of some directed sequence $\{x_\alpha\}$ in $\pi_\varphi^{-1}(O)$, so that we have

$$I = \text{w-lim}_\alpha\, x_\alpha^{\natural},$$

where $\natural$ means the $\natural$-application of $x_\alpha$ in $M$. Since $x_\alpha^{\natural}$ is uniformly approximated by some element of the form $\sum_{i=1}^n \lambda_i u_i^{-1} x_\alpha u_i$ for $\lambda_i \geqq 0$, $\sum_{i=1}^n \lambda_i = 1$ and unitary $u_i \in M$, $x_\alpha^{\natural}$ belongs to the intersection of the center $Z$ and $\pi_\varphi^{-1}(O)$. Then it is easily seen that $Z \cap \pi_\varphi^{-1}(O)$ is $\sigma$-weakly dense in $Z$. Besides, since the left kernel of the restriction of $\varphi$ onto $Z$ is $\pi_\varphi^{-1}(O) \cap Z$, the restriction of $\varphi$ onto $Z$ is singular. Now define the trace $\tau$ by $<x, \tau> = <x^{\natural}, \varphi>$, which is easily seen to be singular.

Next, let $A$ be any fixed maximal abelian subalgebra of $M$ and $\mu_\varphi$ and $\mu_\tau$ the Radon-measures on the spectrum space $\Omega$ of $A$ induced by $\varphi$ and $\tau$, respectively. Furthermore let $\mu$ be the Radon-measure on the spectrum space $\Gamma$ of $Z$ induced by $\varphi = \tau$.

We shall divide into two cases according to the relation between the measures $\mu_\varphi$ and $\mu_\tau$.

Case I. $\mu_\varphi$ is absolutely continuous with respect to $\mu_\tau$: In this case, there exists a $\mu_\tau$-integrable non-negative function $f(\omega)$ on $\Omega$ such that

$$\mu_\varphi(E)=\int_E f(\omega)\,d\mu_\tau(\omega)$$

for each Borel subset $E$ of $\Omega$ by Radon-Nikodym's Theorem. Since $\mu_\varphi(\Omega)=1$, $f$ is strictly positive on some compact set $K$ with positive mass relative to $\mu_\tau$, so that the restrictions of $\mu_\varphi$ and $\mu_\tau$ on $K$ are equivalent each other.

Because Radon-measure $\mu_\tau$ is a regular measure, there exists a sequence of open sets $G_n$ such that

$$G_n\supset G_{n+1}\supset K \qquad \text{and} \qquad \lim_n \mu_\tau(G_n)=\mu_\tau(K)>0.$$

Since $G_n{}^c$ and $K$ are disjoint compact sets, there exist two disjoint open sets $G_n{}'$ and $G_n{}''$ such that

$$G_n{}'\supset K,\ G_n{}''\supset G_n{}^c \qquad \text{and} \qquad G_n{}'\cap G_n{}''=\phi.$$

Besides $G_n{}''^c$ is compact and $G_n{}''^c\supset G_n{}'$, hence $G_n{}''^c$ contains the closure $\overline{G_n{}'}$ of $G_n{}'$ which is open and closed. Considering $\overline{G_n{}'}$ in place of $G_n$, we can say that there exists a sequence of open and compact sets $E_n{}'$ such that

$$E_n{}'\supset E_{n+1}{}'\supset K \qquad \text{and} \qquad \lim_n \mu_\tau(E_n{}')=\mu_\tau(E_n{}')=\mu_\tau(K).$$

Now let $e_n{}'$ be the projection of $\boldsymbol{A}$ corresponding to $E_n{}'$. Since $\tau$ is a singular state over $\boldsymbol{Z}$, there exists a monotone decreasin sequence of central projections $z_n$ such that $\langle z_n, \tau\rangle=1$ and $\sigma$-weakly convergent to zero by [8].

Put $e_n=e_n{}'z_n$ and $E_n$ to be open and closed set in $\Omega$ corresponding to $e_n$. Then $\{e_n\}$ is a monotone decreasing sequence and $\sigma$-weakly convergent to zero, and we have easily

$$\lim_n \mu_\tau(E_n)=\mu_\tau\Big(\bigcap_{n=1}^\infty E_n\Big)=\mu_\tau\Big(\bigcap_{n=1}^\infty E_n\cap K\Big)=\mu_\tau(K)$$

and

$$e_n=\sum_{k\geqq n}(e_k-e_{k+1}).$$

Hence we have

$$\lim_{n\to\infty}\int g(\omega)e_n(\omega)\,d\mu_\tau(\omega)=\lim_{n\to\infty}\int_{E_n} g(\omega)\,d\mu_\tau(\omega)$$
$$=\int_K g(\omega)\,d\mu_\tau(\omega)$$

for each $\mu_\tau$-integrable function $g$ on $\Omega$.

Next, we construct a partition of $e_n-e_{n+1}$ for each positive integer $j=0$, 1, 2, $\cdots$, consisting of $2^j$ equivalent orthogonal projection $p^j_{n,k}$ $(k=1,\cdots,2^j)$ of $\boldsymbol{A}$ such that

$$e_n - e_{n+1} = \sum_{k=1}^{2^j} p_{n,k}^j, \qquad p_{n,i}^j \sim p_{n,k}^j$$

and

$$p_{n,k}^j = p_{n,2k-1}^{j+1} + p_{n,2k}^{j+1}.$$

Put

$$u_n^j = \sum_{k=1}^{2^j} (-1)^k p_{n,k}^j$$

for each $n$ and $j$ then $u_n^j u_n^{j'}$ is represented, for distinct $j$ and $j'$, as the difference of two orthogonal equivalent projections with sum $e_n - e_{n+1}$. Let us write

$$u(J) = \sum_{n=1}^{\infty} u_n^{j_n}$$

for each sequence $J$ of $S_0$ in Lemma. Then we have

$$u(J)^* u(J) = \sum_{n=1}^{\infty} (e_n - e_{n+1}) = e_1.$$

On the other hand, if $j_n \neq j_n'$ for $n \geqq n_0$,

$$e_n u(J)^* u(J') e_n = p_n - q_n$$

for $n \geqq n_0$, where $p_n$ and $q_n$ are orthagonal equivalent projections and $p_n + q_n = e_n$. Hence it follows that

$$\begin{aligned}\int_K |u(J)(\omega)|^2 d\mu_\tau(\omega) &= \lim_n \langle e_n u(J)^* u(J) e_n, \tau\rangle \\ &= \lim_n \langle e_n, \tau\rangle = \mu_\tau(K) > 0\end{aligned}$$

and

$$\begin{aligned}\int_K \overline{u(J)(\omega)} u(J')(\omega) d\mu_\tau(\omega) &= \lim_n \langle e_n u(J)^* u(J') e_n, \tau\rangle \\ &= \lim_n \langle p_n - q_n, \tau\rangle = 0.\end{aligned}$$

Therefore $\{u(J)\}$ is a family of non-zero mutually orthogonal functions in $L^2(K, \mu_\tau)$ and has the power of continuum by Lemma, which implies the non-separability of $L^2(K, \mu_\tau)$. Consequently, $L^2(K, \mu_\varphi)$ is non-separable by the equivalence of $\mu_\varphi$ and $\mu_\tau$ on $K$. Now $L^2(K, \mu_\varphi)$ is a subspace of $L(\Omega, \mu_\varphi)$ and $L^2(\Omega, \mu_\varphi)$ is a subspace of $H_\varphi$, which implies the non-separability of $H_\varphi$. This contradicts the separability of $\varphi$.

Case II. $\mu_\varphi$ is not absolutely continuous with respect to $\mu_\tau$: In this case, there exists a compace subset $K$ of $\Omega$ such that $\mu_\varphi(K) > 0$ and $\mu_\tau(K) = 0$. From the similar arguments as in the previous case, there exists a sequence $\{E_n\}$ of open and closed sets in $\Omega$ such that

$$E_n \supset E_{n+1} \supset K \quad \text{and} \quad \lim_n \mu_\tau(E_n) = 0.$$

Let $e_n$ be the projection of $A$ corresponding to $E_n$, then we have

$$e_n{}^\natural \geqq e^\natural{}_{n+1} \geqq 0$$

and

$$\lim_{n\to\infty}\int_{\Gamma} e_n^{\natural}(\gamma)\,d\mu(\gamma) = \lim_n \langle e_n, \tau\rangle = \lim_n \mu_\tau(E_n) = 0.$$

It follows that the sequence $e_n^{\natural}(\gamma)$ of functions on $\Gamma$ is convergent to zero $\mu$-almost everywhere. Hence, by Egoroff's Theorem, $e_n^{\natural}(\gamma)$ is uniformly convergent to zero on some compact subset $F$ of $\Gamma$ with $\mu(F) > 1 - \varepsilon$ for any $\varepsilon > 0$. Therefore, considering a subsequence of $\{e_n\}$, we may assume $e_n^{\natural}(\gamma) < 1/4^{n+2}$ for all $\gamma \in F$. Put

$$G_n = \{\gamma \in \Gamma;\ e_n^{\natural}(\gamma) < 1/4^{n+2}\}.$$

Then $G_n$ is open and contains $F$. We have $e_n^{\natural}(\gamma) \leqq 1/4^{n+2}$ on the closure $\bar{G}_n$ of of $G_n$ which is open and closed. Consider the projection $g_n$ of $Z$ corresponding to open and closed set $\bar{G}_1 \cap \cdots \cap \bar{G}_n$, and put $f_n = e_n g_n$, then we have

$$g_n \geqq g_{n+1},\ f_n \geqq f_{n+1} \quad\text{and}\quad f_n^{\natural} \leqq 1/4^{n+2},$$

so that $f_n$ converges to zero $\sigma$-weakly. Let $U_n$ be the open and closed subset of $\Omega$ corresponding to $g_n$ and $U = \cap_{n=1}^{\infty} U_n$, we get

$$\mu_\varphi(U) = \lim_{n\to\infty} \mu_\varphi(U_n) = \lim_{n\to\infty} \langle g_n, \varphi\rangle$$
$$= \mu\left(\bigcap_{n=1}^{\infty} \bar{G}_n\right) \geqq \mu(F) > 1 - \varepsilon,$$

which implies

$$\mu_\varphi(U \cap K) = \mu_\varphi(U) + \mu_\varphi(K) - \mu_\varphi(U \cup K)$$
$$> 1 - \varepsilon + \mu_\varphi(K) - \mu_\varphi(U \cup K)$$
$$> \mu_\varphi(K) - \varepsilon > 0$$

for sufficiently small $\varepsilon > 0$.

Now if we consider the space $H_\varphi$, then we have

$$\pi_\varphi(f_n) \geqq \pi_\varphi(f_{n+1}) \quad\text{and}\quad \|\pi_\varphi(f_n)\eta_\varphi(I)\|^2 = \langle f_n, \varphi\rangle$$
$$= \langle e_n g_n, \varphi\rangle = \mu_\varphi(U_n \cap E_n) \geqq \mu_\varphi(U \cap K) > 0$$

for all $n$. It follows that $\pi_\varphi(f_n)\eta_\varphi(I)$ converges to the non-zero vector $\xi$ of $H_\varphi$ which belongs to $\cap_{n=1}^{\infty} \pi_\varphi(f_n) H_\varphi$.

Put $h_n = f_n - f_{n+1}$, $p_{1,1} = h_1$ and suppose that orthogonal projections $\{p_{k,j}\}$ are constructed for $k = 1, \cdots, n-1$ and $1 \leqq j \leqq 2^k$ such as

$$h_k = p_{k.1} \sim p_{k,j} \qquad \text{for}\quad j = 1, 2, \cdots, 2^k$$

and $f_n$ is orthogonal to $p_{k,j}$. Let us put

$$p_n = \sum_{k=1}^{n-1} \sum_{j=1}^{2^k} p_{k,j} + f_n,$$

then we have

$$p_n{}^{\natural} = \sum_{k=1}^{n-1} \sum_{j=1}^{2^k} p_{k,j}{}^{\natural} + f_n{}^{\natural} \leqq \sum_{k=1}^{n-1} 1/4^{k+2} + 1/4^{n+2}$$
$$= \sum_{k=1}^{n-1} (1/16)(1/2^k) + 1/4^{n+2} < 1/8.$$

We get $(I - p_n)^{\natural} \geqq 7/8$, so that there exist orthogonal equivalent projections $p_{n,j}$ $1 \leqq j \leqq 2^n$ such that

$$h_n = p_{n,1} \sim p_{n,j} \leqq I - p_n \qquad \text{for} \quad 2 \leqq j \leqq 2^n.$$

Therefore, by the mathematical induction, there exists a family of orthogonal projections $\{p_{n,j}\}$ such as above.

Considering partial isometries $u_{n,j}$ as

$$u_{n,j}{}^* u_{n,j} = p_{n,1} = h_n \qquad \text{and} \qquad u_{n,j} u_{n,j}{}^* = p_{n,j},$$

we have

$$u_{n,j}{}^* u_{n,j'} = u_{n,j}{}^* p_{n,j} p_{n,j'} u_{n,j'} = 0$$

for $j \neq j'$. Hence, if we put

$$u(J) = \sum_{n=1}^{\infty} u_{n,j_n}$$

for each sequence $J$ of $S_0$, we have

$$u(J)^* u(J) = \sum_{n=1}^{\infty} h_n = f_1$$

and

$$f_{n_0} u(J)^* u(J') f_{n_0} = 0$$

if $j \neq j_n'$ for $n \geqq n_0$. It follows that

$$(\pi_\varphi[u(J)]\xi,\ \pi_\varphi[u(J)]\xi) = (\pi_\varphi[u(J)^* u(J)]\xi,\ \xi)$$
$$= (\pi_\varphi(f_1)\xi,\ \xi) = \|\xi\|^2 > 0$$

and

$$(\pi_\varphi[u(J)]\xi,\ \pi_\varphi[u(J')]\xi) = (\pi_\varphi[u(J')^* u(J)]\xi,\ \xi)$$
$$= \lim_n (\pi_\varphi(f_n)\pi_\varphi[u(J')^* u(J)]\pi_\varphi(f_n)\xi,\ \xi) = 0.$$

Therefore $\{\pi_\varphi[u(J)]\xi\}$ is an ortogonal system in $H_\varphi$, so that $H_\varphi$ is non-separable by Lemma. This contradicts the seprability of $\varphi$, which completes the proof.

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