ON CONFORMAL MAPPING OF A RIEMANN SURFACE ONTO A CANONICAL COVERING SURFACE

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§1. Introduction

The problem to map conformally a multiply-connected planar domain onto one of various canonical domains has been discussed by several authors.

In the present paper we will discuss a problem to map conformally a finite Riemann surface each boundary component of which is a continuum onto a certain canonical covering surface. With respect to this problem Ahlfors has shown that a finite Riemann surface each boundary component of which is a continuum can be mapped conformally onto a many-sheeted disk and further has discussed a certain extremal problem for these mapping functions (cf. [2]). Kusunoki has discussed the mapping of such a Riemann surface onto a covering surface cut along parallel slits as an application of the theory of Abelian integrals (cf. [5]).

We concerned ourselves in [7] with the problem to map conformally a planar domain onto a covering surface of annular type cut along concentric circular slits by a certain extremal method. Further in [8] we concerned ourselves with the similar problem in the case where the image covering surface is of the circular type. It seems, however, to me that the problem constructing such a canonical mapping function is more significant when the basic set is a Riemann surface. In the present paper we shall concern ourselves with the problem to map conformally a finite Riemann surface each boundary component of which is a continuum onto such a canonical covering surface.

First we shall prove by a rather elementary method that a finite Riemann surface each boundary component of which is a continuum can be mapped conformally onto a many-sheeted disk (§3, 1). This fact is known as the Bieberbach-Grunsky's theorem when the basic domain is planar and has been discussed in detail by Ahlfors. Next we shall concern ourselves with the problem to map conformally a finite Riemann surface onto a covering surface of annular type cut along concentric circular slits and shall show that for the finite Riemann surface of non-vanishing genus there does not exist necessarily a covering surface of annular type cut along concentric circular slits onto which the other is conformally mapped (§3, 2). This is remarkably different from the fact that when the basic domain is planar it can always be mapped conformally onto such a canonical covering surface preassigning arbitrarily the rotation number of the image of each boundary component about the origin

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(cf. [7]). Finally we shall concern ourselves with the case of circular type and shall show that a finite Riemann surface can be mapped conformally onto a covering surface of circular type cut along concentric circular slits so that the radius of the image (a circle or a circular slit) of each boundary component takes an arbitrarily preassigned value (§3, 3).

§2. Preliminaries

1. Let R be a finite Riemann surface each boundary component of which is a continuum. It is always possible to show that such a Riemann surface R is embedded conformally in a compact Riemann surface so that each boundary component of the former constitutes of a simple closed analytic curve on the latter. Thus we may assume that R is so taken in advance. We assume that the genus of R is g and its boundary α consists of r simple closed analytic curves $\alpha_1, \cdots, \alpha_r$ $(r \geqq 1)$. Let $\alpha_{r+1}, \cdots, \alpha_N$ $(N = r + 2g)$ be a canonical homology basis of R such that α_{r+2j-1} and α_{r+2j} $(j = 1, \cdots, g)$ are conjugate each other.

2. *Canonical basis of normalized harmonic differentials.* Let ω_j $(j = 1, \cdots, r)$ be the harmonic measure of the boundary component α_j with respect to R, respectively.

Let ω_j $(j = r+1, \cdots, N)$ be a normalized potential of the first kind uniquely determined by the following conditions, respectively:

(i) ω_j is one-valued and harmonic on R cut along α_j;

(ii) ω_j has a jump of the value 1 through α_j;

(iii) $\omega_j = 0$ on α.

We call the system of the differentials $d\omega_1, \cdots, d\omega_N$ that are constructed by this procedure a *canonical basis of normalized harmonic differentials.*

3. Let F be a covering surface over the unit disk which covers each point of the unit disk exactly n times. Then we call F an *n-sheeted unit disk.* Let G be a covering surface over the w-plane each boundary component of which has the projection on the w-plane consisting of a circle or a circular slit centred at the origin and further let there be no inner point of G on 0 or ∞. Then we call G a *covering surface of annular type cut along concentric circular slits centred at the origin.*[1] Let H be a bounded covering surface over the w-plane each boundary component of which has a projection on the w-plane consisting of a circle or a circular slit centred at the origin and let further there exist a point of H on 0. Then we call H a *covering surface of circular type cut along concentric circular slits centred at the origin.*[1]

4. Let γ_k $(k = 2, \cdots, N)$ be a simple analytic curve on R starting from a point q_k^0 on α_1 and ending at a point q_k^1 on α_k (for $2 \leqq k \leqq r$) or on α_1 (for $r+1 \leqq k \leqq N$) and further satisfy a condition

$$\gamma_k^1 \times \alpha_j = \delta_j^k \qquad (j = r+1, \cdots, N),$$

1) It is permitted that there is no concentric circular slit.

respectively, where $\gamma_k^1 \times \alpha_j$ denotes the intersection number between γ_k^1 and α_j, and δ_j^k the Kronecker's symbol. Let

$$\gamma_k^1: \qquad p_k(t) \qquad (0 \leqq t \leqq 1;\ p_k(0) = q_k^0,\ p_k(1) = q_k^1)$$

be a parameter representation of the curve γ_k^1 $(k = 2, \cdots, N)$. The representation

$$\gamma_k^\tau: \qquad p_k(t) \qquad (0 \leqq t \leqq \tau,\ \tau \leqq 1)$$

denotes a subarc of γ_k^1 starting from q_k^0 and ending to $p_k(\tau)$. Now let

$$u_j^k(t) = \int_{\gamma_k^t} d\omega_j \qquad (j,\ k = 2, \cdots, N).$$

Obviously

$$(1) \qquad u_j^k(0) = 0, \quad u_j^k(1) = \delta_j^k \qquad (j,\ k = 2, \cdots, N).$$

Consider $N-1$ functions of $N-1$ variables

$$(2) \qquad u_j = \sum_{k=2}^{N} u_j^k(t_k) \qquad (0 \leqq t_k \leqq 1;\ j,\ k = 2, \cdots, N).$$

They define a continuous mapping φ of a closed unit cube

$$E_{N-1} = \{0 \leqq t_k \leqq 1;\ k = 2, \cdots, N\}$$

of the $(N-1)$-dimensional euclidean space $\mathfrak{E}_{N-1}$ into the $(N-1)$-dimensional euclidean space $\mathfrak{E}_{N-1}^*$.

The following lemma will play a fundamental role in the present paper.

LEMMA 1. *The mapping*

$$u_j = u_j(t_2, \cdots, t_N) \equiv \sum_{k=2}^{N} u_j^k(t_k) \qquad (\text{mod } 1;\ j = 2, \cdots, N)$$

takes in E_{N-1} *all values of the closed unit cube*

$$E_{N-1}^* = \{0 \leqq u_j \leqq 1;\ j = 2, \cdots, N\}$$

of $\mathfrak{E}_{N-1}^*$.

Proof. Though this lemma can be proved immediately, we will make here use of the following Brouwer's fundamental theorem on the mapping degree in topology (cf. [6]):

The mapping degree ρ *of the continuous mapping* f^* *of the* $(N-1)$*-sphere* $\mathfrak{S}_{N-1}$ *on another* $\mathfrak{S}_{N-1}^*$ *depends only upon* f^* *and it remains constant for any mapping homotopic to* f^*.

Noting to (1), by means of defining

$$u_j = f_j(t_2, \cdots, t_N) = [t_j] + \sum_{k=2}^{N} u_j^k(\tau_k) \quad (-\infty < t_k < +\infty;\ j,\ k = 2, \cdots, N)$$

from the functions of (2), we see that φ are extensible to a continuous function f of the whole space $\mathfrak{E}_{N-1}$ into $\mathfrak{E}^*_{N-1}$, where

$$\tau_k = t_k - [t_k] \qquad (k = 2, \cdots, N),$$

and [ ] denotes the Gauss' symbol. Further if we take their compactification to the $(N-1)$-spheres $\mathfrak{S}_{N-1}$ and $\mathfrak{S}^*_{N-1}$ which are obtained from $\mathfrak{E}_{N-1}$ and $\mathfrak{E}^*_{N-1}$ by means of the adjunction of points P_0 and P_0^* corresponding to the infinities, respectively, then f are extensible to the continuous function f^* of $\mathfrak{S}_{N-1}$ into $\mathfrak{S}^*_{N-1}$ by means of $f^* = f$ in $\mathfrak{E}_{N-1}$ and $P_0^* = f^*(P_0)$. Then, according to the Brouwer's fundamental theorem, we conclude that f^* takes all values of $\mathfrak{S}^*_{N-1}$ (obviously $\rho = 1$ in the present case). According to the above reasoning, the assertion of the lemma will be almost obvious. q. e. d.

Now, if $t_\kappa = 1$ for some κ $(2 \leqq \kappa \leqq N)$, by taking $t_\kappa = 0$ in place of it, we see that

$$\sum_{k=2}^{N} u_j^k(t_k) \qquad (j = 2, \cdots, N)$$

vary only for integral values and thus the values of

$$(3) \qquad u_j(t_2, \cdots, t_N) \qquad (j = 2, \cdots, N)$$

are invariant. Thus the proposition will remain valid also taking

$$E^0_{N-1} = \{0 \leqq t_k < 1;\ k = 2, \cdots, N\}$$

as a basic region in place of E_{N-1}. Further if $t_\kappa = 0$ for some κ $(2 \leqq \kappa \leqq N)$, the values of the functions of (3) remain unchanged by deleting the term corresponding to $k = \kappa$ in

$$\sum_{k=2}^{N} u_j^k(t_k) \qquad (j = 2, \cdots, N).$$

According to the above remark, we obtain the following lemma readily from LEMMA 1.

LEMMA 2. *For arbitrarily preassigned point* $(u_2^*, \cdots, u_N^*) \in E^*_{N-1}$, *we have*

$$\sum_{k=1}^{n} \int_{\gamma_k} d\omega_j \equiv u_j^* \qquad (\text{mod } 1;\ j = 2, \cdots, N)$$

by taking at most $N-1$ *points* $p_1, \cdots, p_n$ $(0 \leqq n \leqq N-1)$ *suitably on the Riemann surface* R, *where* γ_k $(k = 1, \cdots, n)$ *are arbitrary analytic paths on* R *from points on* α_1 *to* p_k, *respectively. Some of* $p_1, \cdots, p_n$ *may possibly be repeated. Further, we always have* $n \geqq 1$ *unless* $(u_2^*, \cdots, u_N^*)$ *is a vertex of* E^*_{N-1}.

§ 3. Conformal mapping onto a canonical covering surface.

1. *Conformal mapping onto a many-sheeted disk.* We retain notations introduced in § 2, unless otherwise stated.

LEMMA 3. *Let $p_1, \cdots, p_n$ $(n \geqq r)$ be n points of the Riemann surface R. Then, in order that there exists a function $w = \Phi(p)$*[2] *which maps conformally R onto an n-sheeted unit disk F over the w-plane such that the image points on F of $p_1, \cdots, p_n$ and only these have the same projection $w = 0$ and the rotation number about $w = 0$ of the image β_j of each boundary component α_j $(j = 1, \cdots, r)$ is equal to ν_j $(\nu_j \geqq 1;\ \sum_{j=1}^{r} \nu_j = n)$, it is necessary and sufficient that the $N-1$ equations*

$$\text{(A)} \qquad \begin{cases} \displaystyle\sum_{k=1}^{n} \int_{\gamma_k} d\omega_j = \nu_j & (j = 2, \cdots, r), \\ \displaystyle\sum_{k=1}^{n} \int_{\gamma_k} d\omega_j \equiv 0 & (\text{mod } 1;\ j = r+1, \cdots, N) \end{cases}$$

are satisfied for these points, where γ_k $(k = 1, \cdots, n)$ are arbitrary analytic paths from points on α_1 to p_k. Some of $p_1, \cdots, p_n$ may possibly be repeated in case where Φ has multiple zeros. In the present case, a mapping function Φ is given by

$$(4) \qquad w = \Phi(p) \equiv \exp\left\{-\sum_{k=1}^{n}\left(G(p, p_k) + i\int_{p_0}^{p} d\tilde{G}_k\right)\right\},$$

where $G(p, p_k)$ are the Green's functions of R with poles at p_k, $d\tilde{G}_k$ are the conjugate differentials of the Green differentials $dG_k \equiv dG(p, p_k)$ and p_0 is an arbitrary point on R.

Proof. If there exists the desired mapping function $w = \Phi(p)$, the following conditions will be satisfied:

(i) Φ has zeros at p_k $(k = 1, \cdots, n)$ according to multiplicity and is a one-valued regular function on R which has no zeros other than these n zeros;

(ii) $|\Phi| \equiv 1$ on α;

$$\text{(iii)} \qquad \begin{cases} \displaystyle\int_{\alpha_j} d\mathfrak{J} \lg \Phi = 2\pi\nu_j & (j = 1, \cdots, r), \\ \displaystyle\int_{\alpha_j} d\mathfrak{J} \lg \Phi \equiv 0 & (\text{mod } 2\pi;\ j = r+1, \cdots, N). \end{cases}$$

Conversely, if these conditions are satisfied for an analytic function $w = \Phi(p)$, it is the desired mapping function for which the conditions in the lemma are

2) Though Φ is a mapping of R onto F, we regard that Φ assumes values projected onto the w-plane from F so far as a confusion does not arise. For preciseness we should denote it as $w = w \circ \Phi(p)$ where $w = w(q)$ is the projection map of F onto the w-plane.

satisfied.

According to the conditions (i) and (ii)

$$\lg |\Phi| = -\sum_{k=1}^{n} G(p, p_k).$$

Thus

$$d\mathfrak{J} \lg \Phi = -\sum_{k=1}^{n} d\tilde{G}_k.$$

Then, the condition (iii) can be expressed in the following form:

$$(5) \qquad \begin{cases} -\sum_{k=1}^{n} \int_{\alpha_j} d\tilde{G}_k = 2\pi\nu_j & (j=1, \cdots, r), \\ -\sum_{k=1}^{n} \int_{\alpha_j} d\tilde{G}_k \equiv 0 & (\text{mod } 2\pi;\ j=r+1, \cdots, N). \end{cases}$$

Consequently, in order that there exists the desired mapping function Φ, it is necessary and sufficient that the condition (5) is satisfied.

Now, let du be any harmonic differential and let dG be the Green differential, then the mixed Dirichlet integral of them vanishes. For, by means of the Green's formula, we have

$$(dG, du) = \iint_R dG\, d\tilde{u} = \int_\alpha G\, d\tilde{u} = 0,$$

where (dG, du) is the mixed Dirichlet integral of dG and du, and $d\tilde{u}$ is the conjugate differential of du. According to this result, again using the Green's formula we obtain

$$\begin{cases} (d\omega_1, dG_k) = \int_{\alpha_1} d\tilde{G}_k + 2\pi\left(1 + \int_{\gamma_k} d\omega_1\right) = 0 & (k=1, \cdots, n), \\ (d\omega_j, dG_k) = \int_{\alpha_j} d\tilde{G}_k + 2\pi\int_{\gamma_k} d\omega_j = 0 & (j=2, \cdots, N;\ k=1, \cdots, n), \end{cases}$$

or

$$\begin{cases} -\int_{\alpha_1} d\tilde{G}_k = 2\pi\left(1 + \int_{\gamma_k} d\omega_1\right) & (k=1, \cdots, n), \\ -\int_{\alpha_j} d\tilde{G}_k = 2\pi\int_{\gamma_k} d\omega_j & (j=2, \cdots, N;\ k=1, \cdots, n). \end{cases}$$

Inserting these relations to (5) we have

$$(6) \qquad \begin{cases} \sum_{k=1}^{n} \left(1 + \int_{\gamma_k} d\omega_1\right) = \nu_1, & \\ \sum_{k=1}^{n} \int_{\gamma_k} d\omega_j = \nu_j & (j=2, \cdots, r), \\ \sum_{k=1}^{n} \int_{\gamma_k} d\omega_j \equiv 0 & (\text{mod } 1;\ j=r+1, \cdots, N). \end{cases}$$

However, the first equation in (6) necessarily follows from the $r-1$ equations subsequent to it, since

$$\sum_{k=1}^{n}\left(1+\int_{\gamma_k} d\omega_1\right)=n-\sum_{k=1}^{n}\int_{\gamma_k}\sum_{j=2}^{r} d\omega_j=n-\sum_{j=2}^{r}\sum_{k=1}^{n}\int_{\gamma_k} d\omega_j$$
$$=n-\sum_{j=2}^{r}\nu_j=\nu_1,$$

by reason that

$$\sum_{j=1}^{r} d\omega_j=0.$$

So the first half of the lemma has been verified.

The latter half of the lemma will be obvious from the procedure of the above proof. q. e. d.

The following theorem which is easily verified by using LEMMAS 2 and 3 is known as the Bieberbach-Grunsky's theorem in the case where the basic domain is planar and has been discussed in detail by Ahlfors (cf. [1], [2], [3], [4], [9], [10]).

THEOREM 1. *Let an arbitrary point p_1 be given on the Riemann surface R. Then, by means of taking further at most $N-1$ points $p_2, \cdots, p_n$ $(r\leqq n \leqq N)$ suitably on R, we can always find a function $w=\Phi(p)$ which maps conformally R onto the n-sheeted unit disk F over the w-plane such that the image points on F of $p_1, \cdots, p_n$ and only these have the same projection $w=0$. Some of $p_1, \cdots, p_n$ may possibly be repeated in case where Φ has multiple zeros.*

Proof. Let γ_1 be an arbitrary analytic path on R from a point on α_1 to p_1 and put

$$\int_{\gamma_1} d\omega_j \equiv u_j^* \qquad (\text{mod } 1;\ 0\leqq u_j^*<1;\ j=2, \cdots, N). \tag{7}$$

By LEMMA 2, it is possible to take at most $N-1$ points $p_2, \cdots, p_n$ on R such that

$$\sum_{k=2}^{n}\int_{\gamma_k} d\omega_j \equiv 1-u_j^* \qquad (\text{mod } 1;\ j=2, \cdots, N), \tag{8}$$

where γ_k $(k=2, \cdots, n)$ are arbitrary analytic paths on R from points on α_1 to p_k, respectively. Then by (7) and (8) we have

$$\sum_{k=1}^{n}\int_{\gamma_k} d\omega_j \equiv 0 \qquad (\text{mod } 1;\ j=2, \cdots, N).$$

Thus, according to LEMMA 3, (4) gives a desired mapping function. q. e. d.

2. *Conformal mapping onto a covering surface of annular type.*

LEMMA 4. *In order that there exists a function $w=\Psi(p)$ which maps conformally the Riemann surface R onto a covering surface of annular type G cut along concentric circular slits centred at $w=0$ over the w-plane such that the rotation number about $w=0$ of the image β_j of each boundary component α_j ($j=1, \cdots, r$; $r \geqq 2$) is equal to ν_j ($\sum_{j=1}^{r} \nu_j = 0$ and $\nu_j \neq 0$ for some j) and the radius of a circle or a circular slit being the projection of β_j is equal to C_j, respectively, it is necessary and sufficient that the following $N-1$ equations*

$$\text{(B)} \quad \begin{cases} \sum_{k=2}^{r} a_k \int_{\alpha_j} d\tilde{\omega}_k = \nu_j & (j=2, \cdots, r), \\ \sum_{k=2}^{r} a_k \int_{\alpha_j} d\tilde{\omega}_k \equiv 0 & (\text{mod } 1;\ j=r+1, \cdots, N) \end{cases}$$

are satisfied, where $d\tilde{\omega}_k$ are the conjugate differentials of $d\omega_k$, respectively and

$$a_k = \frac{\lg C_k - \lg C_1}{2\pi} \qquad (k=2, \cdots, r).$$

In the present case, a mapping function Ψ is given by

$$w = \Psi(p) \equiv C_1 \exp\left\{2\pi \sum_{k=2}^{r} a_k \left(\omega_k(p) + i \int_{p_0}^{p} d\tilde{\omega}_k\right)\right\},$$

where p_0 is an arbitrary point on R.

Proof. If there exists the desired mapping function $w=\Psi(p)$, the following conditions will be satisfied:

(i) Ψ is a one-valued regular function which has no zeros on R;

(ii) $|\Psi| \equiv C_k$ on α_k ($k=1, \cdots, r$);

$$\text{(iii)} \quad \begin{cases} \int_{\alpha_j} d\Im \lg \Psi = 2\pi\nu_j & (j=1, \cdots, r), \\ \int_{\alpha_j} d\Im \lg \Psi \equiv 0 & (\text{mod } 2\pi;\ j=r+1, \cdots, N). \end{cases}$$

Conversely, if these conditions are satisfied for an analytic function $w=\Psi(p)$, it is the desired mapping function for which the conditions in the lemma are satisfied.

According to the conditions (i) and (ii)

$$\lg |\Psi| = \sum_{k=1}^{r} c_k \omega_k = \sum_{k=2}^{r} (c_k - c_1)\omega_k + c_1 = 2\pi \sum_{k=2}^{r} a_k \omega_k + c_1,$$

where

$$c_k = \lg C_k \qquad (k=2, \cdots, r).$$

Thus

$$d\mathfrak{J}\lg\Psi = 2\pi\sum_{k=2}^{r} a_k d\tilde{\omega}_k .$$

Then, the condition (iii) can be expressed in the following form:

$$(9)\qquad \begin{cases} \sum_{k=2}^{r} a_k \int_{\alpha_j} d\tilde{\omega}_k = \nu_j & (j=1,\cdots,r), \\ \sum_{k=2}^{r} a_k \int_{\alpha_j} d\tilde{\omega}_k \equiv 0 & (\text{mod } 2\pi;\ j=r+1,\cdots,N). \end{cases}$$

Consequently, in order that there exists the desired mapping function Ψ, it is necessary and sufficient that the condition (9) is satisfied. However, the equation for $j=1$ in (9) necessarily follows from the equations for $j=2,\cdots,r$, since

$$\sum_{k=2}^{r} a_k \int_{\alpha_1} d\tilde{\omega}_k = \sum_{k=2}^{r} a_k\left(-\sum_{j=2}^{r}\int_{\alpha_j} d\tilde{\omega}_k\right) = -\sum_{j=2}^{r}\sum_{k=2}^{r} a_k \int_{\alpha_j} d\tilde{\omega}_k = -\sum_{j=2}^{r}\nu_j = \nu_1 ,$$

by reason that

$$\int_{\alpha} d\tilde{\omega}_k = 0 \qquad (k=2,\cdots,r).$$

So the first half of the lemma has been verified.

The latter half of the lemma will be obvious from the procedure of the above proof. q. e. d.

In the $(N-1)$-dimensional euclidean space $\mathfrak{E}^*_{N-1}$, let $\mathfrak{H}^*_{r-1}$ be an $(r-1)$-dimensional subspace of $\mathfrak{E}^*_{N-1}$ which is spanned by $r-1$ independent vectors

$$\mathfrak{V}_k = \left\{\int_{\alpha_2} d\tilde{\omega}_k,\ \cdots,\ \int_{\alpha_N} d\tilde{\omega}_k\right\} \qquad (k=2,\cdots,r).$$

By LEMMA 4 we obtain immediately the following theorem.

THEOREM 2. *In order that there exists a covering surface of annular type G cut along concentric circular slits centred at the origin onto which the Riemann surface R having at least two boundary components can be conformally mapped, it is necessary and sufficient that the subspace* $\mathfrak{H}^*_{r-1}$ *contains an integral point not coincident with the origin.*

Let $\mathfrak{F}(r, g)$ be the class of the Riemann surfaces R of genus g and having r $(r \geqq 2)$ boundary components such that there exists a covering surface of annular type cut along concentric circular slit onto which R can be conformally mapped. If $R \in \mathfrak{F}(r, g)$ $(g \geqq 1)$, then by THEOREM 2 it seems plausible that we obtain a Riemann surface R^* which does not belong to $\mathfrak{F}(r, g)$ by only a little varying of moduli of R.

Especially, if $g=0$ (therefore $r=N$), $\mathfrak{H}^*_{r-1}$ is identical to $\mathfrak{E}^*_{N-1}$. Thus we obtain the following corollary (cf. [7]).

COROLLARY. *A multiply-connected planar domain of finite connectivity each boundary component of which is a continuum can be mapped conformally onto a covering surface of annular type cut along concentric circular slits centred at the origin. Further we can preassign the rotation number about the origin of the image of each boundary component arbitrarily under the condition that the sum of the rotation numbers is equal to zero but the rotation numbers of all boundary components are not equal to zero.*

3. *Conformal mapping onto a covering surface of circular type.*

LEMMA 5. *Let $p_1, \cdots, p_n$ $(n \geqq 1)$ be n points on the Riemann surface R. Then, in order that there exists a function $w=\chi(p)$ which maps conformally R onto a covering surface of circular type H cut along concentric circular slits centred at $w=0$ over the w-plane such that the image points on H of $p_1, \cdots, p_n$ and only these have the same projection $w=0$, the rotation number about $w=0$ of the image β_j of each boundary component α_j $(j=1, \cdots, r)$ is equal to ν_j $(\sum_{j=1}^{r} \nu_j = n)$ and the radius of a circle or a circular slit being the projection of β_j is equal to C_j, respectively, it is necessary and sufficient that the following $N-1$ eqations*

$$\text{(C)} \qquad \begin{cases} \displaystyle\sum_{k=1}^{n} \int_{\gamma_k} d\omega_j + \sum_{k=2}^{r} a_k \int_{\alpha_j} d\tilde{\omega}_k = \nu_j & (j=2, \cdots, r), \\ \displaystyle\sum_{k=1}^{n} \int_{\gamma_k} d\omega_j + \sum_{k=2}^{r} a_k \int_{\alpha_j} d\tilde{\omega}_k \equiv 0 & (\text{mod } 1;\ j=r+1, \cdots, N) \end{cases}$$

are satisfied for these points $p_1, \cdots, p_n$, where γ_k $(k=1, \cdots, n)$ are arbitrary analytic paths from points on α_1 to p_k, respectively and

$$a_k = \frac{\lg C_k - \lg C_1}{2\pi} \qquad (k=2, \cdots, r).$$

Some of $p_1, \cdots, p_n$ may possibly be repeated in case where χ has multiple zeros. In the present case, a mapping function χ is given by

$$(10) \quad w = \chi(p) \equiv C_1 \exp\left\{-\sum_{k=1}^{n}\left(G(p, p_k) + i\int_{p_0}^{p} d\tilde{G}_k\right) + 2\pi \sum_{k=2}^{r} a_k \left(\omega_k(p) + i\int_{p_0}^{p} d\tilde{\omega}_k\right)\right\},$$

where p_0 is an arbitrary point on R.

Proof. If there exists the desired mapping function $w=\chi(p)$, the following conditions will be satisfied:

(i) χ has the zeros p_k $(k=1, \cdots, n)$ according to multiplicity and is a one-valued regular function on R which has no zeros other than n zeros;

(ii) $|\chi| \equiv C_k$ on α_k $(k=1, \cdots, r)$;

$$\text{(iii)} \qquad \begin{cases} \displaystyle\int_{\alpha_j} d\Im \lg \chi = 2\pi\nu_j & (j=1, \cdots, r), \\ \displaystyle\int_{\alpha} d\Im \lg \chi \equiv 0 & (\text{mod } 2\pi;\ j=r+1, \cdots, N). \end{cases}$$

Conversely, if these conditions are satisfied for an analytic function $w=\chi(p)$, it is the desired mapping function for which the conditions in the lemma are satisfied.

According to the conditions (i) and (ii)

$$\begin{aligned}\lg|\chi| &= -\sum_{k=1}^{n} G(p,\ p_k) + \sum_{k=1}^{r} c_k\omega_k \\ &= -\sum_{k=1}^{n} G(p,\ p_k) + 2\pi\sum_{k=2}^{r} a_k\omega_k + c_1,\end{aligned}$$

where $c_k = \lg C_k$ $(k=1, \cdots, r)$. Thus

$$d\Im \lg \chi = -\sum_{k=1}^{n} d\tilde{G}_k + 2\pi\sum_{k=2}^{r} a_k d\tilde{\omega}_k.$$

Then, the condition (iii) can be expressed in the following form:

$$\begin{cases} -\sum_{k=1}^{n}\int_{\alpha_j} d\tilde{G}_k + 2\pi\sum_{k=2}^{r} a_k\int_{\alpha_j} d\tilde{\omega}_k = 2\pi\nu_j & (j=1, \cdots, r), \\ -\sum_{k=1}^{n}\int_{\alpha_j} d\tilde{G}_k + 2\pi\sum_{k=2}^{r} a_k\int_{\alpha_j} d\tilde{\omega}_k \equiv 0 & (\text{mod } 2\pi;\ j=r+1, \cdots, N). \end{cases}$$

Thus, by using of the argument similar to the proofs of LEMMAS 3 and 4 which we omit here, we can prove the present lemma. q. e. d.

THEOREM 3. *Let C_j $(j=1, \cdots, r)$ be arbitrary r positive real numbers, where some of them may coincide with each other. Then there exists a function χ which maps conformally the Riemann surface R onto a covering surface of circular type H cut along concentric circular slits centred at $w=0$ over the w-plane such that the radius of a circle or a circular slit being the projection of the image β_j of each boundary component α_j $(j=1, \cdots, r)$ is equal to C_j, respectively and the covering degree at $w=0$ is equal to at most N.*

Proof. We put

$$\sum_{k=2}^{r} a_k\int_{\alpha_j} d\tilde{\omega}_k \equiv u_j^* \qquad (\text{mod } 1;\ 0 \leqq u_j^* < 1;\ j=2, \cdots, N), \tag{11}$$

where

$$a_k = \frac{\lg C_k - \lg C_1}{2\pi} \qquad (k=2, \cdots, r).$$

First, we assume that

$$u_j^* \neq 0$$

for some j $(2 \leqq j \leqq N)$. Then, by LEMMA 2 we have

$$\sum_{k=1}^{n}\int_{\gamma_k} d\omega_j \equiv 1 - u_j^* \qquad (\text{mod } 1;\ j=2, \cdots, N), \tag{12}$$

when we take at most $N-1$ suitable points $p_1, \cdots, p_n$ $(1 \leqq n \leqq N-1)$ on R, where γ_k $(k=1, \cdots, n)$ are arbitrary analytic paths on R from points on α_1 to p_k, respectively. Thus, by (11) and (12) we have

$$\sum_{k=1}^{n}\int_{\gamma_k} d\omega_j + \sum_{k=2}^{r} a_k \int_{\alpha_j} d\tilde{\omega}_k \equiv 0 \qquad (\text{mod } 1;\ j=2, \cdots, N).$$

Then, by LEMMA 5, (10) gives the desired mapping function. In this case the covering degree at $w=0$ may be equal to at most $N-1$.

Next, we assume that

$$u_j^* = 0$$

for all j $(2 \leqq j \leqq N)$. Then, the only following two cases may arise:

(i) $C_1 = \cdots = C_r$;

(ii) there exists a function $w = \Psi(p)$ which maps conformally R onto a covering surface of annular type G such that the radius of a circle or a circular slit, the projection of the image of each boundary component α_j $(j=1, \cdots, r)$, is equal to C_j, respectively. The case (i) has been already solved in THEOREM 1. In the case (ii) we have

$$\sum_{k=2}^{r} a_k \int_{\alpha_j} d\tilde{\omega}_k \equiv 0 \qquad (\text{mod } 1;\ j=2, \cdots, N).$$

Otherwise, by THEOREM 1 we have

$$\sum_{k=1}^{n}\int_{\gamma_k} d\omega_j \equiv 0 \qquad (\text{mod } 1;\ j=2, \cdots, N),$$

when we preassign arbitrarily a point p_1 on R and further select at most $N-1$ suitable points $p_2, \cdots, p_n$ $(r \leqq n \leqq N)$ on R. Thus we obtain

$$\sum_{k=1}^{n}\int_{\gamma_k} d\omega_j + \sum_{k=2}^{r} a_k \int_{\alpha_j} d\tilde{\omega}_k \equiv 0 \qquad (\text{mod } 1;\ j=2, \cdots, N).$$

Then by LEMMA 5, (10) gives a desired mapping function.[3] q. e. d.

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3) It would be noted that our proof contains more than the statement of the theorem.

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