# ON INFINITESIMAL CONCIRCULAR TRANSFORMATIONS

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In 1940 to 42, K. Yano [5][1] introduced the concept of concircular transformations of Riemannian manifolds and developed the theory of the concircular geometry. Recently, we have discussed concircular transformations in Riemannian manifolds, complete or compact [1]. We shall in the present paper introduce the notion of infinitesimal concircular transformations and study the properties of Riemannian monifolds, compact, complete or comformally complete, admitting an infinitesimal concircular transformotion.

## §1. Infinitesimal concircular transformations.

In a Riemannian manifold with the metric tensor $g_{\mu\lambda}$, a geodesic circle is by definition a curve $x^\kappa=x^\kappa(s)$ such that[2]

$$V^\kappa \equiv \frac{\delta^3 x^\kappa}{\delta s^3}+g_{\mu\lambda}\frac{\delta^2 x^\mu}{\delta s^2}\frac{\delta^2 x^\lambda}{\delta s^2}\frac{dx^\kappa}{ds}=0, \tag{1.1}$$

where $s$ is the arc length of the curve and $\delta/\delta s$ denotes the covariant differentiation along the curve.[3] Let $v^\kappa$ be an infinitesimal transformation, i.e. a vector field. We suppose further that the infinitesimal point-transformation $'x^\kappa=x^\kappa+\varepsilon v^\kappa(x)$ carries any geodesic circle into another one, $\varepsilon$ being an arbitrary infinitesimal constant. Then, we shall call the $v^\kappa$ an *infinitesimal concircular transformation.*

Consider a family of vectors $V^\kappa$ defined by the left-hand side of (1.1) along a curve $x^\kappa=x^\kappa(s)$, $s$ being the arc length of the curve. Denoting by $\underset{v}{\pounds}$ the Lie differentiation with respect to an infinitesimal transformation $v^\kappa$, we have[4]

$$\begin{aligned}\underset{v}{\pounds}V^\kappa=&-3F\frac{\delta^3x^\kappa}{\delta s^3}+3\left(\underset{v}{\pounds}\left\{{\kappa \atop \mu\ \lambda}\right\}\right)\frac{\delta^2x^\mu}{\delta s^2}\frac{dx^\lambda}{ds}\\ &+\nabla_\nu\left(\underset{v}{\pounds}\left\{{\kappa \atop \mu\ \lambda}\right\}\right)\frac{dx^\nu}{ds}\frac{dx^\mu}{ds}\frac{dx^\lambda}{ds}-3\frac{dF}{ds}\frac{\delta^2x^\kappa}{\delta s^2}\\ &+\left[(\underset{v}{\pounds}g_{\mu\lambda}-5Fg_{\mu\lambda})\frac{\delta^2x^\mu}{\delta s^2}\frac{\delta^2x^\lambda}{\delta s^2}\right.\\ &\left.+2g_{\nu\tau}\left(\underset{v}{\pounds}\left\{{\tau \atop \mu\ \lambda}\right\}\right)\frac{\delta^2x^\nu}{\delta s^2}\frac{dx^\mu}{ds}\frac{dx^\lambda}{ds}-\frac{d^2F}{ds^2}\right]\frac{dx^\kappa}{ds},\end{aligned} \tag{1.2}$$

Received January 3, 1960.

1) The numbers between brackets [ ] refer to the bibliography at the end of the paper.

2) $\kappa, \lambda, \mu, \nu, \omega, \tau, \xi=1, 2, \cdots n$.

3) We consider only differentiable manifolds, differentiable curves and quantities of class $C^\infty$.

4) Cf. K. Yano [6] and [7]. As to the definition of $\underset{v}{\pounds}V^\kappa$, refer to [3].

where we have put

$$F=\frac{1}{2}(\underset{v}{\pounds}\, g_{\mu\lambda})\frac{dx^\mu}{ds}\frac{dx^\lambda}{ds}.$$

By means of the definition, an infinitesimal transformation $v^\kappa$ is a concircular one, if and only if the equation $\underset{v}{\pounds}\, V^\kappa=0$ holds good along any geodesic circle $x^\kappa=x^\kappa(s)$, i.e. if and only if

$$\underset{v}{\pounds}\, V^\kappa=0$$

is valid for any curve $x^\kappa=x^\kappa(s)$ such that

$$\frac{\delta^2 x^\kappa}{\delta s^3}=-g_{\mu\lambda}\frac{\delta^2 x^\mu}{\delta s^2}\frac{\delta^2 x^\lambda}{\delta s^2}\frac{dx^\kappa}{ds}. \tag{1.3}$$

If we take account of (1.2) and substitute the relation (1.3) into $\underset{v}{\pounds}\, V^\kappa$, we have a condition among $dx^\kappa/ds$ and $\delta^2x^\kappa/\delta s^2$. This is equivalent to that the relation

$$\begin{aligned}
&3(\underset{v}{\pounds}\{{}^{\kappa}_{\mu\lambda}\})B^\mu A^\lambda-\nabla_\nu(\underset{v}{\pounds}\{{}^{\kappa}_{\mu\lambda}\})A^\nu A^\mu A^\lambda\\
&\qquad-3\left[\frac{1}{2}\nabla_\nu(\underset{v}{\pounds}\, g_{\mu\lambda})A^\nu A^\mu A^\lambda+(\underset{v}{\pounds}\, g_{\mu\lambda})B^\mu A^\lambda\right]B^\kappa\\
&+[2(\underset{v}{\pounds}\, g_{\mu\lambda}-(\underset{v}{\pounds}\, g_{\omega\tau})A^\omega A^\tau g_{\mu\lambda})B^\mu B^\lambda+2g_{\nu\tau}(\underset{v}{\pounds}\{{}^{\kappa}_{\mu\lambda}\})B^\nu A^\mu A^\lambda]\\
&-\frac{1}{2}\nabla_\omega\nabla_\nu(\underset{v}{\pounds}\, g_{\mu\lambda})A^\omega A^\nu A^\mu A^\lambda-\frac{1}{2}\nabla_\nu(\underset{v}{\pounds}\, g_{\mu\lambda})(B^\nu A^\mu A^\lambda+4A^\nu B^\mu A^\lambda)A^\kappa=0
\end{aligned} \tag{1.4}$$

is valid for any two vectors $A^\kappa$, $B^\kappa$ such that

$$g_{\mu\lambda}B^\mu A^\lambda=0,\quad g_{\mu\lambda}A^\mu A^\lambda=1.$$

We suppose now that (1.4) holds for any pair ($A^\kappa$, $B^\kappa$) of orthogonal vectors such that $g_{\mu\lambda}A^\mu A^\lambda=1$. Transvecting (1.4) with $g_{\kappa\xi}A^\xi$, if we compare the coefficients of the terms containing $B^\kappa B^\lambda$, we have

$$\underset{v}{\pounds}\, g_{\mu\lambda}=2\rho g_{\mu\lambda}\quad\text{and}\quad \rho=\frac{1}{n}g^{\mu\lambda}(\underset{v}{\pounds}\, g_{\mu\lambda}). \tag{1.5}$$

This shows that $\rho$ is a function in the manifold. Accordingly, the given $v^\kappa$ is an infinitesimal conformal transformation. We have thus

$$\underset{v}{\pounds}\left\{{}^{\ \kappa}_{\mu\ \lambda}\right\}=\delta^\kappa_\mu\rho_\lambda+\delta^\kappa_\lambda\rho_\mu-\rho^\kappa g_{\mu\lambda}, \tag{1.6}$$

where $\rho_\lambda=\partial_\lambda\rho$ and $\rho^\kappa=g^{\kappa\tau}\rho_\tau$. Substituting (1.5) and (1.6) into (1.4), we see easily that

$$(\delta^\kappa_\mu\nabla_\nu\rho_\lambda+\delta^\kappa_\lambda\nabla_\nu\rho_\mu-\nabla_\nu\rho^\kappa g_{\mu\lambda}-\delta^\kappa_\nu\nabla_\lambda\rho_\mu)A^\nu A^\mu A^\lambda=0$$

holds for any vector $A^\kappa$. This implies

$$\nabla_\mu\rho_\lambda=\phi g_{\mu\lambda}, \tag{1.7}$$

where $\phi = \nabla_\lambda \rho^\lambda / n$.

Conversely, we suppose that (1.5) and (1.7) hold. Then, it is easily seen that (1.4) is valid for a pair $(A^\kappa, B^\kappa)$ of orthogonal vectors such as $g_{\mu\lambda}A^\mu A^\lambda = 1$. This means that the given $v^\kappa$ is an infinitesimal concircular transformation. Summing up the results above obtained, we have the following

THEOREM 1. *In order that $v^\kappa$ is an infinitesimal concircular transformation, it is necessary and sufficient that*

$$\underset{v}{\pounds}\, g_{\mu\lambda} = 2\rho g_{\mu\lambda}, \qquad \nabla_\mu \rho_\lambda = \phi g_{\mu\lambda} \tag{1.8}$$

*hold, $\rho$ and $\phi$ being certain functions.*

We shall call the function $\rho$ the factor of dilatation of the infinitesimal concircular transformation $v^\kappa$.

It is easily verified that for any two infinitesimal concircular transformations their bracket product is always a concircular one. Thus, in a Riemannian manifold the set of all infinitesimal concircular transformations forms a Lie algebra.

We note here that an infinitesimal concircular transformation is a conformal one and that an infinitesimal homothetic transformation is obviously a concircular one. In the following sections, we shall confine our attention only to infinitesimal concircular transformations which is not homothetic. So, the term "concircular" will always mean "non-homothetic concircular".

## §2. The local structure.

We suppose that $v^\kappa$ is an infinitesimal concircular transformation in a Riemannian manifold[5] $M$ with the metric tensor $g_{\mu\lambda}$. By means of Theorem 1, we have then (1.8). Differentiating the both sides of the last equation of (1.8) covariantly and taking account of the Ricci formula, we have easily

$$K_{\nu\mu\lambda}{}^{\kappa}\rho_\kappa = -(\phi_\nu g_{\mu\lambda} - \phi_\mu g_{\nu\lambda}), \tag{2.1}$$

$K_{\nu\mu\lambda}{}^{\kappa}$ being the curvature tensor of the manifold, where $\phi_\lambda = \partial_\lambda \phi$.

A point of $M$ is called a *stationary point* or an *ordinary point* of the infinitesimal concircular transformation $v^\kappa$, if the gradient vector $\rho_\lambda$ vanishes or not at the point respectively. In a sufficiently small neighborhood of an ordinary point we consider the integral curves of the vector field $\rho^\kappa = g^{\kappa\tau}\rho_\tau$. By means of (1.8) we can see easily that such an integral curve is a geodesic arc. A geodesic is called a *$\rho$-curve* if it contains such an arc.

Let P be an ordinary point in $M$ and $U$ a sufficiently small coordinate neighborhood of $P$ containnig no stationary point. Then we can define in $U$ a

5) Throughout the paper we suppose that manifolds are connected and of dimension greater than 2.

family of hypersurfaces defined by the equation $\rho = \text{const.}$ There exists one and only one hypersurface $\rho = \text{const.}$ through a given ordinary point. Such a surface will be called a *$\rho$-hypersurface.* Now, keeping the notations in § **1**, we can prove the following theorem in the same way just as in a previous paper[6].

THEOREM 2. *If a Riemannian manifold $M$ admits an infinitesimal concircular transformation $v^\kappa$, then for any ordinary point of $v^\kappa$ there exists a coordinate neighborhood $U$ of the point such that we can choose in $U$ a system of coordinates $u^\kappa$ having the following properties: The function $\rho$ depends only on the $n$-th variable $u^n$ in $U$. The line element of $M$ is given by*[7]

$$ds^2 = (\rho')^2 f_{ji} du^j du^i + (du^n)^2, \tag{2.2}$$

*the functions $f_{ij}$ being independent of the variable $u^n$, where and in the following primes denote the differentiation with respect to $u^n$. The hypersurfaces defined by the equation $u^n = \text{const.}$ in $U$ are $\rho$-hypersurfaces. The curves defined by the equations $u^h = \text{const.}$ are $\rho$-curves and the variable $u^n$ indicates the arc length of the $\rho$-curves.*

A system of coordinates $u^\kappa$ having the properties given in Theorem 2 will be called a system of *adapted coordinates.*

With respect to a system of adapted coordinates, the Christoffel symbols of the line element (2.2) have the components

$$\begin{gathered}
\left\{ {h \atop i\ j} \right\} = \overline{\left\{ {h \atop j\ i} \right\}} \quad \left\{ {n \atop j\ i} \right\} = \left\{ {n \atop i\ j} \right\} = -\rho'\rho'' f_{ji} = -\frac{\rho''}{\rho'} g_{ji}, \\
\left\{ {h \atop n\ i} \right\} = \left\{ {h \atop i\ n} \right\} = \frac{\rho''}{\rho'} \delta_i^h, \quad \left\{ {n \atop n\ i} \right\} = \left\{ {n \atop i\ n} \right\} = 0, \\
\left\{ {h \atop n\ n} \right\} = 0, \quad \left\{ {n \atop n\ n} \right\} = 0,
\end{gathered} \tag{2.3}$$

where $\overline{\left\{ {h \atop j\ i} \right\}}$ denotes the Christoffel symbols constructed from $f_{ji}$.

We shall give a formula for later use. If we take account of (2.3), we can write down the equation

$$\underset{v}{\pounds}\, g_{\mu\lambda} = 2\rho g_{\mu\lambda}$$

as below:

$$\begin{gathered}
\partial_n v_n = \rho, \\
\partial_n v_i + \partial_i v_n - \frac{2\rho''}{\rho'} v_i = 0, \\
\overline{\nabla}_j v_i + \overline{\nabla}_i v_j = 2\left(\rho - \frac{\rho''}{\rho'} v_n\right) g_{ji},
\end{gathered} \tag{2.4}$$

6) See Theorem 1 in [1]; cf. also [5].

7) $h, i, j, k = 1, 2, \cdots, n-1$.

with respect to adapted coordinates $u^\kappa$, where $\overline{\nabla}_i$ denotes the convarient differentiation with respect to $\overline{\left\{ {h \atop j\ i} \right\}}$. From the first equation of (2.4) it follows

$$v_n = A + V(u^h), \tag{2.5}$$

where $A$ is a function of the variable $u^n$ such that $A' = \rho$ and the function $V$ is independent of $u^n$. Substituting this into the second equation of (2.4), and putting $V_i = \partial_i V$, we have

$$\partial_n v_i - \frac{2\rho''}{\rho'} v_i = - V_i$$

which is a linear differential equation with unknown function $v_i$ and the independent variable $u^n$. The function $V_i$ being independent of $u^n$, the function $v_i$ has the form

$$v_i = PV_i + QU_i, \tag{2.6}$$

where $U_i$ is a function independent of the variable $u^n$ and $P$, $Q$ are two functions depending only on $u^n$. Now, if we substitute (2.5) and (2.6) into the third equation of (2.4), and transvect the resulting equation with $f^{ji}$, we have

$$P(\overline{\nabla}_i V^i) + Q(\overline{\nabla}_i U^i) + (n-1)\rho''\rho' V = (n-1)\{\rho(\rho')^2 - \rho''\rho' A\}, \tag{2.7}$$

where we have put $V^i = f^{ih} V_h$ and $U^i = f^{ih} U_h$.

Denoting by $K_{\nu\mu\lambda}{}^\kappa$ and $K_{\mu\lambda} = K_{\kappa\mu\lambda}{}^\kappa$ respectively the curvature tensor and the Ricci tensor of $M$, we have easily by means of (2.3)

$$K_{nn} = -(n-1)\frac{\rho'''}{\rho'}, \quad K_{ni} = 0 \tag{2.8}$$

with respect to adapted coordinates $u^\kappa$. Therefore, any $\rho$-curve is a geodesic tangent to a Ricci direction at any regular point.

## §3. Compact manifolds.

We consider now a compact Riemannian manifold admitting an infinitesimal concircular transformation $v^\kappa$. If we take account of Theorem 2, we can prove the following lemma in the same way just as in §1 of a previous paper [1]. To formalize the lemma, we shall give some terminologies concerning the unit sphere. Denote by $S_n$ the unit sphere

$$(x^1)^2 + (x^2)^2 + \cdots + (x^{n+1})^2 = 1$$

in a Euclidean space $E_{n+1}$ of $n+1$ dimensions, $(x^1, \cdots, x^{n+1})$ being a system of rectangular coordinates in $E_{n+1}$. The points $(0, \cdots, 0, 1)$ and $(0, \cdots, 0, -1)$ of $S_n$ are called the antipodal points. The sphere defined in $S_n$ by the equation $x^{n+1} = c$ $(-1 < c < 1)$ is called a small sphere.

LEMMA 1. *Let $M$ be a compact Riemannian manifold. If $M$ admits an infinitesimal concircular transformation, then it is conformally homeomorphic to a spherical space of curvature* 1 *and there exists in $M$ exactly two stationary points* O *and* O′ *of the infinitesimal concircular transformation. The homeo-omorphism $\alpha$ of $M$ onto the unit sphere $S_n$ in $E_{n+1}$ carries these stationary points* O *and* O′ *into the antipodal points of $S_n$, any $\rho$-hypersurface into a small sphere of $S_n$ and any $\rho$-curve into a great circle passing through the antipodal points. Further, any $\rho$-hypersurface has positive constant sectional curvature.*

Keeping the same assumptions as above, we take an arbitrary $\rho$-hypersurface $S$. Then the surface $S$ is homeomorphic to a sphere of $n-1$ dimensions because of the above Lemma 1. By means of (2.7) we have

$$P\int_S \bar{\nabla}_i V^i d\sigma + Q\int_S \bar{\nabla}_i U^i d\sigma + (n-1)\rho''\rho'\int_S V d\sigma = (n-1)\,|S|\{\rho(\rho')^2 - \rho''\rho' \Lambda\},$$

where $d\sigma$ is the volume element of the surface $S$ and $|S|$ denotes the total volume of $S$. Now, we may assume $\int_S V d\sigma = 0$ without loss of generality. The above equation implies immediately

$$\rho\rho' - \rho''\Lambda = 0,$$

from which we have

$$\frac{\Lambda'''}{\Lambda''} = \frac{\Lambda'}{\Lambda}.$$

Hence we have

$$\Lambda'' + c\Lambda = 0$$

with a constant $c$. Differentiating the both sides, we find

$$\rho'' + c\rho = 0. \tag{3.1}$$

The function $\rho$ satisfies (3.1) at any ordinary point of a $\rho$-curve. Then the equation (3.1) is valid at any point of a $\rho$-curve, since there exist only isolated stationary points in any $\rho$-curve because of Lemma 1.

The constant $c$ appeared in (3.1) is necessarily positive, since the function $\rho$ is periodic along any $\rho$-curve and is not constant. We may therefore put $c = a^2$ $(a > 0)$. If we integrate (3.1), we obtain

$$\rho = A\cos(au^n + b),$$

$A$ and $b$ being certain constants. Denoting by $L$ the length of the arc of a $\rho$-curve terminated by two stationary points O and O′, we have $a = \pi/L$. If we consider the arc length $u^n$ of any $\rho$-curve such that $u^n = 0$ at O and $u^n = L$ at O′, we have $b = 0$. In fact, along any $\rho$-curve $\rho$ vanishes at the two stationary points and does not at any ordinary point. Consequently, we have along any $\rho$-curve

$$\rho = A\cos\frac{\pi s}{L}$$

and hence

$$\rho' = B\sin\frac{\pi s}{L}, \qquad B = \frac{\pi A}{L}, \tag{3.2}$$

where $s$ denotes the arc length of $\rho$-curves such that $s=0$ at the stationary point O.

If we take account of the line element (2.2), we find by means of (3.2) that the sectional curvature of the manifold is constant and positive at any ordinary point. Thus, the manifold has positive constant sectional curvature everywhere. Summing up and taking account of Lemma 1, we have the following

THEOREM 3. *If a compact Riemannian manifold admits an infinitesimal concircular transformation, then the manifold is a spherical space.*

## §4. Einstein spaces.

Let $M$ be an Einstein space and suppose that $M$ admits an infinitesimal conformal transformation $v^\kappa$, that is,

$$\underset{v}{\pounds}\, g_{\mu\lambda} = 2\rho g_{\mu\lambda},$$

$\rho$ being a function in $M$. The following formulas are well known:

$$\underset{v}{\pounds}\, K_{\nu\mu\lambda}{}^{\kappa} = \nabla_\nu\left(\underset{v}{\pounds}\left\{{\kappa \atop \mu\ \lambda}\right\}\right) - \nabla_\mu\left(\underset{v}{\pounds}\left\{{\kappa \atop \nu\ \lambda}\right\}\right).$$

Substituting (1.6) into the right-hand side, we have

$$\underset{v}{\pounds}\, K_{\nu\mu\lambda}{}^{\kappa} = \nabla_\nu\rho_\lambda\delta^\kappa_\mu - \nabla_\mu\rho_\nu\delta^\kappa_\lambda + g_{\nu\lambda}\nabla_\mu\rho^\kappa - g_{\mu\lambda}\nabla_\nu\rho^\kappa.$$

Contracting with respect to $\kappa$ and $\nu$, and putting $K_{\mu\lambda} = K_{\kappa\mu\lambda}{}^{\kappa}$, we find

$$\underset{v}{\pounds}\, K_{\mu\lambda} = -(n-2)\nabla_\mu\rho_\lambda - \nabla_\kappa\rho^\kappa g_{\mu\lambda}. \tag{4.1}$$

The manifold $M$ being Einsteinian, we have

$$K_{\mu\lambda} = \frac{K}{n} g_{\mu\lambda},$$

where $K = K_{\mu\lambda}g^{\mu\lambda}$ is a constant. Substituting this into (4.1), we obtain

$$\nabla_\mu\rho_\lambda = \phi g_{\mu\lambda},$$

where $\phi$ is a function. This shows that the infinitesimal transformation $v^\kappa$ is concircular. Thus, we have the following

LEMMA 2. *In an Einstein space any infinitesimal conformal transfor-*

*mation is concircular.*

This lemma implies together with Theorem 3 the following

THEOREM 4. *If a compact Einstein space admits a non-homothetic infinitesimal conformal transformation, then the space is a spherical space.*

## §5. Conformal circles.

We shall give for later use some remarks on conformal circles. Keeping the same notations concerning a Riemannian manifold $M$ as in §1, we define in $M$ the tensors

$$\Pi^{0}_{\mu\lambda}=-\frac{K_{\mu\lambda}}{n-1}+\frac{Kg_{\mu\lambda}}{2(n-1)(n-2)},\quad \Pi^{\kappa}_{\infty\lambda}=g^{\mu\kappa}\Pi^{0}_{\mu\lambda},\tag{5.1}$$

and consider a curve $x^{\kappa}=x^{\kappa}(s)$ satisfying the differential equations

$$\frac{\delta^{3}x^{\kappa}}{\delta s^{3}}+\frac{dx^{\kappa}}{ds}\left(g_{\mu\lambda}\frac{\delta^{2}x^{\mu}}{\delta s^{2}}\frac{\delta^{2}x^{\lambda}}{\delta s^{2}}-\Pi^{0}_{\mu\lambda}\frac{dx^{\mu}}{ds}\frac{dx^{\lambda}}{ds}\right)+\Pi^{\kappa}_{\infty\lambda}\frac{dx^{\lambda}}{ds}=0,\tag{5.2}$$

where $s$ denotes the arc length of the curve. We call such a curve a *conformal circle*. It is well known that any conformal circle is mapped into a conformal circle by every conformal transformation of the Riemannian manifold $M$ [2, 4].

Let $g$: $x^{\kappa}=x^{\kappa}(s)$ be a geodesic of $M$, $s$ being its arc length. We suppose that $g$ is tangent to a Ricci direction at any point. Then, from (5.1) we see that it holds along $g$

$$-\frac{dx^{\kappa}}{ds}\Pi^{0}_{\mu\lambda}\frac{dx^{\mu}}{ds}\frac{dx^{\lambda}}{ds}+\Pi^{\kappa}_{\infty\lambda}\frac{dx^{\lambda}}{ds}=0.$$

Taking account of this fact, we find from (5.2) that the geodesic $g$ is a conformal circle. Thus we have the following

LEMMA 3. *In a Riemannian manifold $M$, if a geodesic is tangent to a Ricci direction at any point, then it is a conformal circle.*

In a Riemannian manifold $M$, we consider a curve $C$: $x^{\kappa}=x^{\kappa}(s)$, $s$ denoting its arc length. Let $t$ be a parameter of the curve $C$ and put

$$\{t,\ s\}=\frac{d^{3}t}{ds^{3}}\Big/\frac{dt}{ds}-\frac{3}{2}\left(\frac{d^{2}t}{ds^{2}}\Big/\frac{dt}{ds}\right)^{2}.$$

A parameter $t$ of the curve $C$ is called a *projective parameter* of $C$, if it satisfies the differential equation

$$\{t,\ s\}=\frac{1}{2}g_{\mu\lambda}\frac{\delta^{2}x^{\mu}}{\delta s^{2}}\frac{\delta^{2}x^{\lambda}}{\delta s^{2}}-\Pi^{0}_{\mu\lambda}\frac{dx^{\mu}}{ds}\frac{dx^{\lambda}}{ds}\tag{5.3}$$

along $C$. It is known that, if the curve $C$ is mapped into a curve $'C$ by a conformal transformation of $M$, the parameter $'t$ induced naturally in $'C$ from the projective parameter $t$ of $C$ is also projective on $'C$ [4].

In a Riemannian manifold $M$ we consider a conformal circle $C$. For any point P of $C$ we suppose that any projective parameter $t$ of $C$ vanishing at P takes any real values along the conformal circle $C$. If any conformal circle of $M$ satisfies these conditions, we say that the Riemannian manifold $M$ is *conformally complete.*

## §6. Manifold with constant scalar curvature.

Keeping the same notations concerning a Riemannian manifold $M$ as in §1, its scalar curvature $K$ is given by $K=K_{\mu\lambda}g^{\mu\lambda}$. Let $v^\kappa$ be an infinitesimal conformal transformation in $M$. Then, by means of (4.1), we have

$$\underset{v}{\pounds} K = -2\rho K - 2(n-1)\nabla_\kappa \rho^\kappa.$$

If we now assume that the scalar curvature $K$ is constant in $M$, we find

$$\nabla_\kappa \rho^\kappa + \frac{K\rho}{n-1} = 0.$$

We suppose here that $v^\kappa$ is an infinitesimal concircular transformation. Then, from the second equation of (1.8) it follows

$$\nabla_\mu \rho_\lambda + \frac{K\rho}{n-1} g_{\mu\lambda} = 0. \tag{6.1}$$

Let $u^\kappa$ be a system of adapted coordinates of the infinitesimal concircular transformation $v^\kappa$ in a regular neighborhood $U$. From (6.1) it follows that the function $\rho$ satisfies

$$\rho'' + \frac{K\rho}{n(n-1)} = 0 \tag{6.2}$$

along $\rho$-curve, where $\rho'' = d^2\rho/(du^n)^2$.

As has been noted at the end of §2, any $\rho$-curve is a geodesic and tangent to a Ricci direction at any regular point. On the other hand, since the function $\rho$ is a solution of the differential equation (6.2) along any $\rho$-curve, any singular point lying on a $\rho$-curve, if it exists, is isolated. Hence, the $\rho$-curve is tangent to a Ricci direction at any point. Consequently, by virtue of Lemma 3 we see that any $\rho$-curve is a conformal circle.

Consider a $\rho$-curve and let $t$ be its projective parameter. Denoting by $s$ the arc length of the $\rho$-curve $x^\kappa = x^\kappa(s)$ and taking account of $\delta^2 x^\kappa/\delta s^2 = 0$, we find from (5.3)

$$\{t,\, s\} = -\overset{0}{\Pi}_{\mu\lambda} \frac{dx^\mu}{ds}\frac{dx^\lambda}{ds}.$$

Using of adapted coordinates $u^\kappa$ and taking account of (2.8) and (5.1), we have from the above equation

$$\{t,\ s\} = -\frac{n-1}{n-2}\frac{\rho'''}{\rho'} - \frac{K}{2(n-1)(n-2)}. \tag{6.3}$$

Now, differentiating the both sides of (6.2), we find

$$\frac{\rho'''}{\rho'} = -\frac{K}{n(n-1)}.$$

If we substitute this into (6.3), we obtain

$$\{t,\ s\} = \frac{K}{2n(n-1)}. \tag{6.4}$$

We suppose moreover that the Riemannian manifold $M$ is conformally complete. We consider now the three possible cases: (i) $K=0$, (ii) $K<0$ and (iii) $K>0$.

In the case (i) $K=0$, we consider a $\rho$-curve and its arc length $s$ vanishing at a point P. For a constant $s_0 \neq 0$, we have a solution of (6.4)

$$t = \frac{s}{s-s_0}.$$

The projective parameter $t$ of the $\rho$-curve vanishes at the point P, but it does not, however, take the value 1. This contradicts the fact that the manifold $M$ is conformally complete. Therefore, the case (i) does not occure.

In the case (ii) $K<0$, we consider a $\rho$-curve and its arc length $s$ vanishing at a point P. Then, the function

$$t = \frac{2}{k}\tanh\frac{ks}{2}, \quad k = \sqrt{\frac{-K}{n(n-1)}},$$

is a solution of (6.4). The projective parameter $t$ of the $\rho$-curve vanishes at the point P, but it takes only the values in the interval $(-1, 1)$. This contradicts the fact that the manifold $M$ is conformally complete. Therefore, the case (ii) does not occur.

Finally, we consider the case (iii) $K>0$. We take a $\rho$-curve and its arc length $s$ vanishing at a point P. Then, the function

$$t = \frac{2}{k}\tan\frac{ks}{2}, \quad k = \sqrt{\frac{K}{n(n-1)}},$$

is a solution of (6.4). The projective parameter $t$ of the $\rho$-curve vanishes at the point P. Since the manifold $M$ is conformally complete, the projective parameter $t$ takes any real value. The arc length $s$ has to take hence any value contained in the interval $I_0 = (-\pi/k,\ \pi/k)$. Consequently, the $\rho$-curve must contain an arc $C$ consisting of all points corresponding to $s \in I_0$. The length of the arc $C$ is obviously equal to $2\pi/k$. The point P being taken arbitrarily in the $\rho$-curve, we see that for any point of the $\rho$-curve there exists in the $\rho$-curve an arc of length $2\pi/k$ having the point as its middle point. Therefore, we can say that in any $\rho$-curve its arc length $s$ takes any real value.

Taking the arc length $s$ of a $\rho$-curve suitably, and integrating (6.2), we have along the $\rho$-curve

$$\rho = A \sin ks,$$

where $A$ is a constant. Since the arc length $s$ takes any real value, there exist at least two isolated singular points in the $\rho$-curve. Consequently, in the same way just as in §3, we can prove the following

THEOREM 5. *Let $M$ be a conformally complete Riemannian manifold having constant scalar curvature. If $M$ admits an infinitesimal concircular transformation, then $M$ is a spherical space.*

In a similar way, we can prove the following

THEOREM 6. *Let $M$ be a conformally complete Einstein space. If $M$ admits a non-homothetic infinitesimal conformal transformation, then $M$ is a spherical space.*

## §7. Holonomy groups.

If we take account of Theorem 2 we can prove the following theorems in the same way just as in a previous paper [1].

THEOREM 7. *If a complete non-flat Riemannian manifold admits an infinitesimal concircular transformation, then its local homogeneous holonomy group at any ordinary point is the special orthogonal group $SO(n)$.*

THEOREM 8. *If a non-flat conformally flat Riemannian manifold admits an infinitesimal concircular transformation, then the local homogeneous holonomy group at any ordinary point is the group $SO(n)$.*

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