A NOTE ON A RENEWAL THEOREM

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1. Let X_i $(i=1, 2, \cdots)$ be independent random variables, having the finite mean values $E(X_i)=m_i>0$ $(i=1, 2, \cdots)$. When

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}m_i=m$$

exists, then it holds that

$$\lim_{T\to\infty}\frac{1}{T}\int_{-\infty}^{T}dt\sum_{n=1}^{\infty}\Pr(t<S_n\leqq t+h)=\frac{h}{m} \tag{1.1}$$

where

$$S_n=\sum_{i=1}^{n}X_i,$$

with some restrictions. This problem is a renewal theorem in a wide sense and have been treated by Kawata [2] and the author [1] under somewhat different conditions. In the following, we shall extend this theorem in a sense under the assumptions which have been considered in [2].

2. In the first place, we shall prepare the following lemmas.

Lemma 1. *Let $F(t)$ be a non-decreasing function,*

$$\int_{-\infty}^{0}e^{-s_0t}dF(t)<+\infty \qquad \textit{for some } s_0>0 \tag{2.1}$$

and

$$\int_{-\infty}^{\infty}e^{-st}dF(t)\sim\frac{A}{s^{\gamma}} \qquad \textit{as } s\downarrow 0 \textit{ for some } \gamma>0. \tag{2.2}$$

Then

$$F(t)\sim\frac{At^{\gamma}}{\Gamma(\gamma+1)} \qquad \textit{as } t\to\infty.$$

Proof. Since

Received September 27, 1959.

$$0 \leqq \int_{-\infty}^{0} e^{-st} dF(t) \leqq \int_{-\infty}^{0} e^{-s_0 t} dF(t) < +\infty \qquad \text{for } 0 \leqq s \leqq s_0$$

by (2.1), we have

$$\int_{0}^{\infty} e^{-st} dF(t) \sim \frac{A}{s^{\gamma}} \qquad \text{for } s \downarrow 0.$$

Hence by a classical Tauberian theorem, it results that

$$F(t) \sim \frac{At^{\gamma}}{\Gamma(\gamma+1)} \qquad \text{as } t \to \infty.$$

LEMMA 2. *Let X_i $(i = 1, 2, \cdots)$ be independent random variables such that $E(X_i) = m_i > 0$. Suppose that the distribution function $F_n(x)$ of X_n satisfies*

$$\int_{-\infty}^{0} e^{-s_0 t} dF_n(x) < +\infty \qquad \textit{for some } s_0 > 0 \tag{2.3}$$

and further that both of

$$\lim_{A \to \infty} \int_{A}^{\infty} x\, dF_n(x) = 0 \tag{2.4}$$

and

$$\lim_{A \to \infty} \int_{-\infty}^{-A} e^{-s_0 x} dF_n(x) = 0 \tag{2.5}$$

hold uniformly with respect to n. If

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} m_i = m > 0, \tag{2.6}$$

then

$$\lim_{s \downarrow 0} s^{\alpha+1} \sum_{n=1}^{\infty} n^{\alpha} \varphi_n(s) = \frac{\alpha!}{m^{\alpha+1}} \qquad \textit{for } \alpha = 0, 1, 2, \cdots, \tag{2.7}$$

where

$$\varphi_n(s) = \int_{-\infty}^{\infty} e^{-sx} d\sigma_n(x),$$

$\sigma_n(x)$ being the distribution function of S_n.

Proof. Let ε be any given positive number. Under the conditions of this lemma, there exist an N and an $s_2 < s_0$ such that

(2.8) $$e^{-ns(m+2\varepsilon)} \leqq \varphi_n(s) \leqq e^{-ns(m-2\varepsilon)} \quad \text{for } n > N \text{ and } 0 \leqq s \leqq s_2,$$

which has been given in the course of the proof of Lemma 2 in [2]. Hence we have

$$\begin{aligned} & s^{\alpha+1}\sum_{n=1}^{\infty}(n+\alpha)(n+\alpha-1)\cdots(n+1)\varphi_n(s) \\ & < s^{\alpha+1}\sum_{n=1}^{N}(n+\alpha)(n+\alpha-1)\cdots(n+1)C \\ & \quad + s^{\alpha+1}\sum_{n=0}^{\infty}(n+\alpha)(n+\alpha-1)\cdots(n+1)e^{-ns(m-2\varepsilon)} \\ & = s^{\alpha+1}\sum_{n=1}^{N}(n+\alpha)(n+\alpha-1)\cdots(n+1)C + \frac{s^{\alpha+1}\cdot\alpha!}{(1-e^{-s(m-2\varepsilon)})^{\alpha+1}}, \quad \text{for } 0 < s \leqq s_2 \end{aligned}$$

where

$$C = \sup_{\substack{0<s\leqq s_2 \\ n=1,2,\cdots,N}} \varphi_n(s) \leqq 1 + \max_{n=1,2,\cdots,N}\int_{-\infty}^{0} e^{-s_2 x}d\sigma_n(x) < +\infty.$$

Thus

$$\overline{\lim_{s\downarrow 0}}\, s^{\alpha+1}\sum_{n=1}^{\infty}(n+\alpha)(n+\alpha-1)\cdots(n+1)\varphi_n(s) \leqq \frac{\alpha!}{(m-2\varepsilon)^{\alpha+1}}$$

and since ε is arbitrary, we get

(2.9) $$\overline{\lim_{s\downarrow 0}}\, s^{\alpha+1}\sum_{n=1}^{\infty}(n+\alpha)(n+\alpha-1)\cdots(n+1)\varphi_n(s) \leqq \frac{\alpha!}{m^{\alpha+1}}.$$

On the other hand

$$\begin{aligned} & s^{\alpha+1}\sum_{n=1}^{\infty}(n+\alpha)(n+\alpha-1)\cdots(n+1)\varphi_n(s) \\ & \geqq s^{\alpha+1}\sum_{n=0}^{\infty}(n+\alpha)(n+\alpha-1)\cdots(n+1)e^{-ns(m+2\varepsilon)} \\ & \quad - s^{\alpha+1}\sum_{n=0}^{N}(n+\alpha)(n+\alpha-1)\cdots(n+1) \\ & = \frac{s^{\alpha+1}\cdot\alpha!}{(1-e^{-s(m+2\varepsilon)})^{\alpha+1}} - s^{\alpha+1}\sum_{n=0}^{N}(n+\alpha)(n+\alpha-1)\cdots(n+1) \quad \text{for } 0 < s \leqq s_2. \end{aligned}$$

Hence

$$\underline{\lim_{s\downarrow 0}}\, s^{\alpha+1}\sum_{n=1}^{\infty}(n+\alpha)(n+\alpha-1)\cdots(n+1)\varphi_n(s) \geqq \frac{\alpha!}{(m+2\varepsilon)^{\alpha+1}}$$

from which it results that

$$\varliminf_{s \downarrow 0} s^{\alpha+1} \sum_{n=1}^{\infty} (n+\alpha)(n+\alpha-1)\cdots(n+1)\varphi_n(s) \geqq \frac{\alpha!}{m^{\alpha+1}}. \tag{2.10}$$

(2.9) and (2.10) show that

$$\lim_{s \downarrow 0} s^{\alpha+1} \sum_{n=1}^{\infty} (n+\alpha)(n+\alpha-1)\cdots(n+1)\varphi_n(s) = \frac{\alpha!}{m^{\alpha+1}}. \tag{2.11}$$

From (2.11) with $\alpha=0, 1, 2, \cdots$, we get (2.7).

LEMMA 3. *Let $\{a_n\}$ be a sequence of real numbers. If, in addition to the conditions of Lemma 2, we assume that*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} a_i = a$$

exists, then we have

$$\lim_{s \downarrow 0} s^{\alpha+1} \sum_{n=1}^{\infty} a_n n^\alpha \varphi_n(s) = \frac{a \cdot \alpha!}{m^{\alpha+1}} \qquad \textit{for } \alpha = 0, 1, 2, \cdots. \tag{2.12}$$

Proof. Since, describing

$$\frac{1}{n} \sum_{i=1}^{n} a_i = a + \varepsilon_n,$$

we have

$$a_n = a + n\varepsilon_n - (n-1)\varepsilon_{n-1}, \qquad \varepsilon_n \to 0 \quad (n \to \infty)$$

and

$$\sum_{n=1}^{\infty} a_n n^\alpha \varphi_n(s) = a \sum_{n=1}^{\infty} n^\alpha \varphi_n(s) + \sum_{n=1}^{\infty} n^\alpha \varphi_n(s)[n\varepsilon_n - (n-1)\varepsilon_{n-1}],$$

it suffices to prove

$$\lim_{s \downarrow 0} s^{\alpha+1} \sum_{n=1}^{\infty} n^\alpha \varphi_n(s)[n\varepsilon_n - (n-1)\varepsilon_{n-1}] = 0. \tag{2.13}$$

Noticing

$$\sum_{n=1}^{\infty} |n^\alpha \varphi_n(s) \cdot n\varepsilon_n| \leqq \sup_{n=1,2,\cdots} |\varepsilon_n| \cdot \sum_{n=1}^{\infty} n^{\alpha+1} \varphi_n(s) < +\infty,$$

we get

$$\begin{aligned} &\sum_{n=1}^{\infty} n^\alpha \varphi_n(s)[n\varepsilon_n - (n-1)\varepsilon_{n-1}] \\ &= \sum_{n=1}^{\infty} n\varepsilon_n [n^\alpha \varphi_n(s) - (n+1)^\alpha \varphi_{n+1}(s)] \end{aligned}$$

$$= \sum_{n=1}^{\infty} n\varepsilon_n \varphi_n(s)[n^\alpha - (n+1)^\alpha f_{n+1}(s)]$$

$$= \sum_{n=1}^{\infty} n\varepsilon_n \varphi_n(s) \cdot n^\alpha (1 - f_{n+1}(s)) + \sum_{n=1}^{\infty} n\varepsilon_n \varphi_n(s)(n^\alpha - (n+1)^\alpha) f_{n+1}(s)$$

$$= I_1 + I_2,$$

say, where

$$f_n(s) = \int_{-\infty}^{\infty} e^{-sx} dF_n(s), \qquad 0 \leqq s \leqq s_0.$$

Now, for any given positive number ε, we choose an integer N such that

$$|\varepsilon_n| < \varepsilon \qquad \text{for } n > N.$$

Since there are positive constants C_1 and s_2 such that $m_n < C_1$ for $n = 1, 2, \cdots$ *and*

$$f_n(s) = 1 - sm_n + s\eta_n,$$

where

$$|\eta_n| < \varepsilon(C_1 + 2) \equiv C_2 \qquad \text{for} \quad 0 \leqq s \leqq s_2,$$

uniformly with respect to n, which have been proved in the course of the proof of Lemma 2 in [2], we have

$$|s^{\alpha+1} \cdot I_1| < s^{\alpha+2} \cdot (C_1 + C_2) \cdot C \sum_{n=1}^{N} n^{\alpha+1} |\varepsilon_n| + \varepsilon(C_1 + C_2) \cdot s^{\alpha+2} \sum_{n=N+1}^{\infty} n^{\alpha+1} \varphi_n(s)$$

$$\textit{for } 0 < s \leqq s_2,$$

where

$$C = \sup_{\substack{0 < s \leqq s_2 \\ n = 1, 2, \cdots, N}} \varphi_n(s) < +\infty,$$

and hence

$$\overline{\lim_{s \downarrow 0}} |s^{\alpha+1} \cdot I_1| \leqq \varepsilon (C_1 + C_2) \frac{(\alpha+1)!}{m^{\alpha+2}}$$

and since ε is arbitrary, we get

$$\lim_{s \downarrow 0} s^{\alpha+1} \cdot I_1 = 0. \tag{2.14}$$

On the other hand,

$$|s^{\alpha+1} \cdot I_2| < s^{\alpha+1}[1 + (C_1 + C_2)s] \cdot C \sum_{n=1}^{N} \alpha(n+1)^\alpha \cdot |\varepsilon_n|$$

$$+[1+(C_1+C_2)s]\varepsilon\cdot s^{\alpha+1}\sum_{n=N+1}^{\infty}\left(\alpha n^{\alpha}+\binom{\alpha}{2}n^{\alpha-1}+\cdots+n\right)\varphi_n(s)$$

for $0<s\leqq s_2$.

Hence

$$\overline{\lim_{s\downarrow 0}}|s^{\alpha+1}\cdot I_2|\leqq\alpha\cdot\varepsilon\frac{\alpha!}{m^{\alpha+1}},$$

from which it results that

(2.15) $$\lim_{s\downarrow 0}s^{\alpha+1}\cdot I_2=0.$$

(2.14) and (2.15) give (2.12).

3. THEOREM 1. *If, in addition to the conditions of Lemma* 3, *we assume that* $a_n\geqq 0$ $(n=1,2,\cdots)$, *then we have*

(3.1) $$\lim_{t\to\infty}\frac{1}{t^{\alpha+1}}\sum_{n=1}^{\infty}a_n\cdot n^{\alpha}\Pr(S_n\leqq t)=\frac{a}{(\alpha+1)m^{\alpha+1}}\qquad \text{for } \alpha=0,1,2,\cdots.$$

Proof. Describing

$$H_N(t)=\sum_{n=1}^{N}a_nn^{\alpha}\Pr(S_n\leqq t),$$

we have

$$\int_{-\infty}^{\infty}e^{-st}dH_N(t)=\sum_{n=1}^{N}a_n\cdot n^{\alpha}\int_{-\infty}^{\infty}e^{-st}d\sigma_n(t)=\sum_{n=1}^{N}a_nn^{\alpha}\varphi_n(s)$$

and so

(3.2) $$\lim_{N\to\infty}\int_{-\infty}^{\infty}e^{-st}dH_N(t)$$

exists and is equal to

$$\sum_{n=1}^{\infty}a_nn^{\alpha}\varphi_n(s)\qquad \text{for } 0<s\leqq s_2.$$

Since we know by (2.8) that there exists a constant C_3 such that

$$\varphi_n(s)\leqq C_3\qquad \text{for } n=1,2,\cdots \text{ and } 0<s\leqq s_2,$$

we have

$$\int_{-\infty}^{\infty}e^{-st}dH_N(t)\leqq C_3\sum_{n=1}^{N}a_nn^{\alpha}$$

and

$$\int_{-\infty}^{-A} e^{-st} dH_N(t) \leqq C_3 \sum_{n=1}^{N} a_n n^\alpha \qquad \text{for any positive } A.$$

Taking σ less than s_2 and $s = s_2$,

$$\begin{aligned} C_3 \sum_{n=1}^{N} a_n \cdot n^\alpha &\geqq \int_{-\infty}^{-A} e^{-(s_2-\sigma)t} \cdot e^{-\sigma t} dH_N(t) \\ &\geqq e^{(s_2-\sigma)A} \int_{-\infty}^{-A} e^{-\sigma t} dH_N(t) \\ &\geqq e^{(s_2-\sigma)A} \int_{-B}^{-A} e^{-\sigma t} dH_N(t), \end{aligned}$$

which gives

$$\int_{-B}^{-A} e^{-\sigma t} dH_N(t) \leqq e^{-(s_2-\sigma)A} C_3 \cdot \sum_{n=1}^{N} a_n n^\alpha.$$

Letting $\sigma \to 0$ and $B \to \infty$, we get

$$H_N(-A) \leqq e^{-s_2 A} C_3 \cdot \sum_{n=1}^{N} a_n n^\alpha.$$

Therefore if $0 < s \leqq s_3 < s_2$,

$$\lim_{t \to -\infty} e^{-st} H_N(t) = 0.$$

Thus we get by partial integration that for $0 < s \leqq s_3$

$$\int_{-\infty}^{\infty} e^{-st} dH_N(t) = s \int_{-\infty}^{\infty} e^{-st} H_N(t)\, dt. \tag{3.3}$$

Since $H_N(t)$ increases as $N \to \infty$ and tends to a non-decreasing function $H(t)$, the existence of the limit (3.2) and (3.3) show that

$$\lim_{N \to \infty} \int_{-\infty}^{\infty} e^{-st} H_N(t)\, dt = \int_{-\infty}^{\infty} e^{-st} H(t)\, dt \tag{3.4}$$

exists for $0 < s \leqq s_3$. $H(t)$ is equal to

$$\sum_{n=1}^{\infty} a_n n^\alpha \Pr(S_n \leqq t).$$

The existence of the right side integral shows that for $0 < s \leqq s_4 \leqq s_3$

$$H(t) = o(e^{st}) \qquad \text{for } |t| \to \infty.$$

Hence

$$s\int_{-\infty}^{\infty} e^{-st}H(t)\,dt = \int_{-\infty}^{\infty} e^{-st}dH(t)$$

exists and is equal to

$$\sum_{n=1}^{\infty} a_n n^{\alpha}\varphi_n(s) \qquad \text{for } 0 < s \leqq s_4$$

by (3.2), (3.3) and (3.4), so that we have by Lemma 3

$$\int_{-\infty}^{\infty} e^{-st}dH(t) \sim \frac{a\cdot\alpha!}{m^{\alpha+1}s^{\alpha+1}} \qquad \text{as } s\downarrow 0.$$

Then, by Lemma 1, we get

$$\lim_{t\to\infty}\frac{1}{t^{\alpha+1}}H(t) = \lim_{t\to\infty}\frac{1}{t^{\alpha+1}}\sum_{n=1}^{\infty} a_n n^{\alpha}\Pr(S_n \leqq t) = \frac{a}{(\alpha+1)m^{\alpha+1}}.$$

COROLLARY 1. *If, in addition to the conditions of Lemma 3, we assume that* a_n $(n=1,2,\cdots)$ *are bounded from below, we have*

$$\lim_{t\to\infty}\frac{1}{t^{\alpha+1}}\sum_{n=1}^{\infty} a_n n^{\alpha}\Pr(S_n \leqq t) = \frac{a}{(\alpha+1)m^{\alpha+1}} \qquad \textit{for } \alpha=0,1,2,\cdots.$$

Proof. Since we can take a constant $c \geqq 0$ such that

$$a_i + c \geqq 0 \qquad \text{for } i=1,2,\cdots,$$

we have

$$\frac{1}{t^{\alpha+1}}\sum_{n=1}^{\infty} a_n n^{\alpha}\Pr(S_n \leqq t) = \frac{1}{t^{\alpha+1}}\sum_{n=1}^{\infty}(a_n+c)n^{\alpha}\Pr(S_n \leqq t) - \frac{c}{t^{\alpha+1}}\sum_{n=1}^{\infty} n^{\alpha}\Pr(S_n \leqq t)$$

which converges to

$$\frac{a+c}{(\alpha+1)m^{\alpha+1}} - \frac{c}{(\alpha+1)m^{\alpha+1}} = \frac{a}{(\alpha+1)m^{\alpha+1}} \qquad (t\to\infty).$$

COROLLARY 2. *If, in addition to the conditions Lemma 2, we assume that* $\{a_n\}$ *is a sequence of real numbers and* a_n *can be expressed as the difference of* b_n *and* c_n *for* $n=1,2,\cdots,$ *where* b_n *and* c_n *are bounded from below and have the properties that*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} b_i = b \quad \textit{and} \quad \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} c_i = c$$

exist, we have

$$\lim_{t\to\infty}\frac{1}{t^{\alpha+1}}\sum_{n=1}^{\infty}a_n n^{\alpha}\Pr(S_n\leqq t)=\frac{a}{(\alpha+1)m^{\alpha+1}}\qquad \textit{for } \alpha=0,1,2,\cdots,$$

where $a=b-c$.

4. Taking $\sigma_n(t)$ as $a_n\Pr(S_n\leqq t)$ in the course of the proof of Theorem 2 in [1], we get the following

THEOREM 2. *Under the conditions of Theorem* 1, *we have*

$$\lim_{T\to\infty}\frac{1}{T^{\alpha+1}}\int_{-\infty}^{T}dt\sum_{n=1}^{\infty}a_n n^{\alpha}\Pr(t<S_n\leqq t+h)=\frac{ah}{(\alpha+1)m^{\alpha+1}} \tag{4.1}$$

for $\alpha=0,1,2,\cdots$.

COROLLARY 3. *Under the conditions of Corollary* 2, *we have*

$$\lim_{T\to\infty}\frac{1}{T^{\alpha+1}}\int_{-\infty}^{T}dt\sum_{n=1}^{\infty}a_n n^{\alpha}\Pr(t<S_n\leqq t+h)=\frac{ah}{(\alpha+1)m^{\alpha+1}}$$

for $\alpha=0,1,2,\cdots$.

The proof follows easily from Theorem 2.

REMARK. If $\alpha\geqq 0$ may be not necessarily an integer, we have in place of (3.1) and (4.1) that

$$\lim_{t\to\infty}\frac{1}{T^{\alpha+1}}\sum_{n=1}^{\infty}a_n\cdot\frac{\Gamma(n+\alpha+1)}{n!}\Pr(S_n\leqq t)=\frac{a}{(\alpha+1)m^{\alpha+1}},$$

and

$$\lim_{T\to\infty}\frac{1}{T^{\alpha+1}}\int_{-\infty}^{T}dt\sum_{n=1}^{\infty}a_n\cdot\frac{\Gamma(n+\alpha+1)}{n!}\Pr(t<S_n\leqq t+h)=\frac{ah}{(\alpha+1)m^{\alpha+1}}.$$

COROLLARY 4. *Let* A_i $(i=1,2,\cdots)$ *be independent identically distributed random variables, having finite variances. Then, under the conditions of Lemma* 2, *we have*

$$\lim_{T\to\infty}\frac{1}{T^{\alpha+1}}\int_{-\infty}^{T}dt\sum_{n=1}^{\infty}A_n n^{\alpha}\Pr(t<S_n\leqq t+h)=\frac{ah}{(\alpha+1)m^{\alpha+1}} \tag{4.2}$$

with probability 1, *where* $a=E\{X_i\}$.

Proof. Since the sequences $\{\max(A_i,0)\}$ and $\{\max(-A_i,0)\}$ of random variables obey respectively the the strong law of large numbers and $A_i=\max(A_i,0)-\max(-A_i,0)$, this theorem follows from Corollary 3.

In conclusion, the author expresses his sincerest thanks to Professors T. Kawata and K. Kunisawa who have given valuable advices.

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