SOME THEOREMS IN AN EXTENDED RENEWAL THEORY, I

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1. Prof. J. L. Doob [1] has treated the renewal theory in terms of the theory of probability as follows: Let $X_1, X_2, \cdots$ be non-negative mutually independent random variables. It is supposed that $X_2, X_3, \cdots$ have a common distribution function. Let $N(t)$ be the number of sums $X_1, X_1+X_2, \cdots$ which are less than t. In other words $N(t)$ is the random variable such that

$$\sum_{\nu=1}^{N(t)} X_\nu < t \leqq \sum_{\nu=1}^{N(t)+1} X_\nu. \tag{1.1}$$

The process $X(t)$ is defined as follows:

$$\begin{aligned} X(t) &= t - \sum_{\nu=1}^{N(t)} X_\nu \qquad \text{if} \quad N(t) \geqq 1, \\ &= X_0 + t \qquad\qquad \text{if} \quad N(t) = 0, \end{aligned} \tag{1.2}$$

where X_0 is a random variable. Then renewal theory may be said to be a theory on $N(t)$ and $X(t)$, and Doob has investigated many properties on $N(t)$ and $X(t)$.

Recently Prof. T. Kawata [2] has found in an application that it has been necessary to weaken the assumption that $X_2, X_3, \cdots$ have a common distribution function and discussed an asymptotic property of $E\{N(t+h) - N(t)\}$ at $t \to \infty$ with the condition that there exists

$$\lim_{n\to\infty} \frac{1}{n}(a_1 + \cdots + a_n) = a$$

where $a_\nu = E\{X_\nu\}$ $(\nu = 1, 2, \cdots)$. In this paper we shall give some results on $N(t)$ and $X(t)$ by the methods of Doob [1] in the case where the existence of

$$\lim_{n\to\infty} \frac{a_2 + \cdots + a_n}{n-1} = a$$

and some other conditions are assumed.

2. Through this section we set the following assumptions:

(2.1) $X_1, X_2, \cdots$ are non-negative and mutually independent random variables.

(2.2) $X_2, X_3, \cdots$ have finite means $a_2, a_3, \cdots$ respectively and there exists a positive constant L such that $a_\nu \geqq L$ for $\nu = 2, 3, \cdots$.

(2.3) $X_2, X_3, \cdots$ have finite variances $\sigma_2^2, \sigma_3^2, \cdots$ respectively and there exists a positive constant K such that $\sigma_\nu^2 \leqq K < +\infty$ for $\nu = 2, 3, \cdots$.

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(2.4) There exists

$$\lim_{n\to\infty}\frac{1}{n-1}(a_2+\cdots+a_n)=a$$

where a may be $+\infty$.

In the following $N(t)$ and $X(t)$ are the random variables defined by (1.1) and (1.2) respectively and a^{-1} is interpreted as zero if $a=+\infty$.

LEMMA 1. *Let $\tilde{X}_1, \tilde{X}_2, \cdots$ be mutually independent random variables such that*

$$P\{\tilde{X}_1=0\}=1,\quad P\{\tilde{X}_\nu=1\}=p_\nu,\quad P\{\tilde{X}_\nu=0\}=1-p_\nu \quad \text{for } \nu=2, 3\cdots.$$

If there exists a positive constant p such that $p_\nu \geqq p$ for $\nu=2, 3, \cdots$, then we have

$$E\{\tilde{N}(t)^\alpha\}<+\infty \tag{2.5}$$

and

$$\varlimsup_{t\to\infty} E\left\{\left(\frac{\tilde{N}(t)}{t}\right)^\alpha\right\}<+\infty \qquad \text{for} \quad \alpha\geqq 1, \tag{2.6}$$

where $\tilde{N}(t)$ is the number of sums $\tilde{X}_1, \tilde{X}_1+\tilde{X}_2, \cdots$ which are less than t.

Proof. It is evident under the conditions of this lemma that $\tilde{N}(t)$ can be finite with probability 1. If $\nu\geqq 1$, then we have by the definition of $\tilde{N}(t)$ that $\tilde{X}_1+\cdots+\tilde{X}_{\tilde{N}(\nu)}=\nu-1$, $\tilde{X}_{\tilde{N}(\nu)+1}=1$, $\tilde{X}_{\tilde{N}(\nu)+2}=\cdots=\tilde{X}_{\tilde{N}(\nu+1)}=0$ and $\tilde{X}_{\tilde{N}(\nu+1)+1}=1$. Since the event $\tilde{N}(\nu)=j$ is determined by $\tilde{X}_1, \tilde{X}_2, \cdots, \tilde{X}_{j+1}$ only, we have

$$\begin{aligned} P\{\tilde{N}(\nu+1)-\tilde{N}(\nu)=k\} &= \sum_{j=0}^{\infty} P\{\tilde{N}(\nu)=j\} \\ &\qquad \times P\{\tilde{X}_{j+2}=0\}\cdots P\{\tilde{X}_{j+k}=0\}P\{\tilde{X}_{j+k+1}=1\} \\ &= \sum_{j=0}^{\infty} P\{\tilde{N}(\nu)=j\}(1-p_{j+2})\cdots(1-p_{j+k})p_{j+k+1} \\ &\leqq (1-p)^{k-1}\sum_{j=0}^{\infty}P\{\tilde{N}(\nu)=j\}=(1-p)^{k-1}. \end{aligned}$$

The same evaluation is obtained for $\nu=0$, where $\tilde{N}(0)\equiv 0$. Therefore we have

$$E\{[\tilde{N}(\nu+1)-\tilde{N}(\nu)]^\alpha\}\leqq\sum_{k=1}^{\infty}k^\alpha(1-p)^{k-1}\equiv A_\alpha<+\infty,$$

so that, using the well-known property of the convex function x^α, we have

$$\begin{aligned} E\{N(t)^\alpha\} &= E\left\{\left[\sum_{\nu=0}^{[t]}(\tilde{N}(\nu+1)-\tilde{N}(\nu))\right]^\alpha\right\} \\ &\leqq E\left\{([t]+1)^{\alpha-1}\sum_{\nu=0}^{[t]}[\tilde{N}(\nu+1)-\tilde{N}(\nu)]^\alpha\right\}\leqq([t]+1)^\alpha\cdot A_\alpha<+\infty, \end{aligned}$$

which implies (2.5). Consequently we get

$$E\left\{\left(\frac{\tilde{N}(t)}{t}\right)^\alpha\right\}\leqq\frac{([t]+1)^\alpha}{t^\alpha}A_\alpha,$$

and hence

$$\overline{\lim_{t\to\infty}} E\left\{\left(\frac{\tilde{N}(t)}{t}\right)^{\alpha}\right\} \leqq A_{\alpha} < +\infty,$$

which give (2.6).

Using this lemma we shall prove the Theorem 1. (2.7) in this theorem shall be made more precise in theorem 4 with some other assumptions.

THEOREM 1. *If* $X_1, X_2, \cdots$ *are the random variables satisfying the conditions* (2.1) – (2.4), *then we have*

$$P\left\{\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{a}\right\} = 1, \tag{2.7}$$

$$E\{N(t)^{\alpha}\} < \infty, \tag{2.8}$$

and

$$\lim_{t\to\infty} \frac{E\{N(t)^{\alpha}\}}{t^{\alpha}} = \frac{1}{a^{\alpha}} \qquad \textit{for} \quad \alpha > 0. \tag{2.9}$$

Proof. Suppose $a < +\infty$. According to the strong law of large numbers which is found to be true by (2.3), and using the fact that $\lim_{n\to\infty} X_1/n = 0$ (a.s.), we have that

$$\lim_{n\to\infty} \frac{X_1 + \cdots + X_n}{n} = \lim_{n\to\infty} \frac{a_2 + \cdots + a_n}{n-1} = a \quad \text{(a.s.)}. \tag{2.10}$$

Since $a \geqq L > 0$ by (2.2), $N(t)$ is finite by (2.10) with probability 1, and

$$\lim_{t\to\infty} N(t) = \infty \qquad \text{(a.s.)}. \tag{2.11}$$

Let E_1 be the subset of Ω on which either of (2.10) and (2.11) does not hold where Ω is the probability space. If $\omega \in \Omega - E_1$, there exists a suitable number $n_\varepsilon = n_\varepsilon(\omega)$ for an arbitrary small $\varepsilon > 0$ such that

$$-\varepsilon n < X_1 + \cdots + X_n - an < \varepsilon n \qquad \text{for} \quad n > n_\varepsilon. \tag{2.12}$$

Since $N(t) > n_\varepsilon$ for large t by (2.11), we get by (1.1) and (2.12) that

$$-\varepsilon N(t) < X_1 + \cdots + X_{N(t)} - aN(t) < t - aN(t) \tag{2.13}$$

so that

$$-\varepsilon < \frac{t}{N(t)} - a.$$

When $t \to \infty$, we have that

$$-\varepsilon + a \leqq \underline{\lim_{t\to\infty}} \frac{t}{N(t)}. \tag{2.14}$$

Since we get from (2.12) by the similar way that

$$t - a(N(t)+1) \leqq X_1 + \cdots + X_{N(t)+1} - a(N(t)+1) < \varepsilon(N(t)+1),$$

it is known that

$$\overline{\lim_{t\to\infty}}\frac{t}{N(t)}\leqq a+\varepsilon \tag{2.15}$$

which gives with (2.14) that

$$\lim_{t\to\infty}\frac{t}{N(t)}=a \qquad \text{for all} \quad \omega\in\Omega-E_1.$$

Therefore we have

$$\lim_{t\to\infty}\frac{N(t)}{t}=\frac{1}{a} \qquad \text{(a.s.)} \tag{2.16}$$

which is equivalent to (2.7). Now using Tchebyshev's inequality, we know by (2.2) and (2.3) that there exist positive constants λ and p such that

$$P\{X_\nu\geqq\lambda\}\equiv p_\nu\geqq p>0 \qquad \text{for} \quad \nu=2, 3, \cdots.$$

Defining $\tilde{X}_1, \tilde{X}_2, \cdots$ by

$$\begin{aligned} &\tilde{X}_1\equiv 0, \\ &\tilde{X}_\nu=0 \quad \text{if} \quad X_\nu<\lambda, \\ &\quad\;\, =1 \quad \text{if} \quad X_\nu\geqq\lambda, \end{aligned} \qquad \text{for} \quad \nu=2, 3, \cdots,$$

they satisfy the all assumptions in Lemma 1. From the fact that

$$X_1+\cdots+X_{N(t)}<t \quad \text{and} \quad \tilde{X}_\nu\leqq\frac{X_\nu}{\lambda} \qquad \text{for} \quad \nu=1, 2, \cdots,$$

we get

$$\tilde{X}_1+\cdots+\tilde{X}_{N(t)}<\frac{t}{\lambda},$$

so that

$$N(t)\leqq\tilde{N}\Big(\frac{t}{\lambda}\Big)$$

which implies (2.8) with (2.5). Since we have by (2.6) that, taking $\beta>1$ so that $\alpha\beta>1$,

$$\overline{\lim_{t\to\infty}}E\Big\{\Big(\frac{N(t)}{t}\Big)^{\alpha\beta}\Big\}\leqq\frac{1}{\lambda^{\alpha\beta}}\overline{\lim_{t\to\infty}}E\Big\{\Big(\frac{\tilde{N}(t)}{t}\Big)^{\alpha\beta}\Big\}<+\infty,$$

it will be known from (2.16) that

$$\lim_{t\to\infty}E\Big\{\Big(\frac{N(t)}{t}\Big)^{\alpha}\Big\}=E\Big\{\lim_{t\to\infty}\Big(\frac{N(t)}{t}\Big)^{\alpha}\Big\}=\frac{1}{a^\alpha}$$

which concludes the proof of our theorem. The proof of the case $a=\infty$ will similarly be done with slight modifications.

REMARK 1. If we want to get (2.7) only, we may assume the weaker condition instead (2.3) that

(2.3′) $X_2, X_3, \cdots$ obey the strong law of large numbers.

The following two theorems can be proved in our case by the similar way as in Theorem 2 and Theorem 9 of Doob's paper [1].

THEOREM 2. *If* $a < +\infty$ *with the assumptions in Theorem* 1, *we have*

$$P\left\{\lim_{t\to\infty} \frac{X(t)}{t} = 0\right\} = 1, \tag{2.17}$$

and if in addition $E\{X_0\} < +\infty$, *it follows that*

$$\lim_{t\to\infty} \frac{E\{X(t)\}}{t} = 0. \tag{2.18}$$

THEOREM 3. *If, in addition of the conditions of Theorem* 1, *we assume that*

$$b = \lim_{n\to\infty} \frac{E\{X_2^2\} + \cdots + E\{X_n^2\}}{n-1} \tag{2.19}$$

exists and

$$\sum_{\nu=2}^{\infty} \nu^{-2} \operatorname{Var}\{X_\nu^2\} < +\infty, \tag{2.20}$$

then we have that

$$P\left\{\lim_{t\to\infty} \frac{1}{t}\int_0^t X(s)\,ds = \frac{b}{2a}\right\} = 1. \tag{2.21}$$

3. We have used the strong law of large numbers in the course of the proofs in the preceding section. If we can assume the law of iterated logarithm on $X_2, X_3, \cdots$, we can make Theorem 1 and Theorem 2 more complete.

We substitue for (2.3) and (2.4) the conditions that

$$a < +\infty \quad \text{and} \quad \frac{a_2 + \cdots + a_n}{n-1} = a + o\left(\sqrt{\frac{\log\log n}{n}}\right) \qquad (n \to \infty), \tag{3.1}$$

$$B_n \equiv \sum_{\nu=2}^{n} \sigma_\nu^2 \to \infty \qquad (n\to\infty) \tag{3.2}$$

and

(3.3) $X_2 - a_2, X_3 - a_3, \cdots$ obey the law of iterated logarithm.

Then we have that

$$\overline{\lim_{n\to\infty}} \frac{|X_1 + \cdots + X_n - (a_2 + \cdots + a_n)|}{\sqrt{2B_n \log\log B_n}} = 1 \qquad \text{(a.s.)}. \tag{3.4}$$

THEOREM 4. *Under the conditions* (2.1), (2.2), (3.1), (3.2) *and* (3.3), *we have that*

$$P\left\{\overline{\lim_{n\to\infty}} \frac{|N(t) - t/a|}{\sqrt{2t\log\log t}} \leqq \frac{\sqrt{B}}{(\sqrt{a})^3}\right\} = 1 \tag{3.5}$$

where

$$\overline{\lim_{n\to\infty}} \frac{B_n}{n-1} = \overline{\lim_{n\to\infty}} \frac{\sigma_2^2 + \cdots + \sigma_n^2}{n-1} = B. \tag{3.6}$$

Proof. Let E_2 be the subset of Ω on which (3.4) does not be satisfied. If $\omega \in \Omega - E_1 - E_2$, there exists a number $m_\varepsilon = m_\varepsilon(\omega)$ for an arbitrary small ε such that

$$-(1+\varepsilon) < \frac{X_1 + \cdots + X_n - (a_2 + \cdots + a_n)}{\sqrt{2B_n \log\log B_n}} < 1 + \varepsilon \quad \text{for } n > m_\varepsilon. \tag{3.7}$$

If we make t so large that $N(t) > m_\varepsilon$, then it follows by (1.1) and (3.1) that

$$\begin{aligned} -(1+\varepsilon)\sqrt{2B_{N(t)} \log\log B_{N(t)}} &< X_1 + \cdots + X_{N(t)} - (a_2 + \cdots + a_{N(t)}) \\ &< t - aN(t) + o(\sqrt{N(t) \log\log N(t)}). \end{aligned} \tag{3.8}$$

Since, noting Remark 1, we have that

$$\left(\frac{1}{a} - \varepsilon'\right) t < N(t) < \left(\frac{1}{a} + \varepsilon'\right) t$$

for sufficient large t where ε' is an arbitrary small number, we get that

$$\begin{aligned} \left(\frac{1}{a} - \varepsilon'\right) t \log\log \left(\frac{1}{a} - \varepsilon'\right) t &< N(t) \log\log N(t) \\ &< \left(\frac{1}{a} + \varepsilon'\right) t \log\log \left(\frac{1}{a} + \varepsilon'\right) t, \end{aligned}$$

so that

$$\lim_{t\to\infty} \frac{N(t) \log\log N(t)}{t \log\log t} = \frac{1}{a}. \tag{3.9}$$

Therefore it follows that

$$\begin{aligned} &\overline{\lim_{t\to\infty}} \frac{B_{N(t)} \log\log B_{N(t)}}{t \log\log t} \\ &= \overline{\lim_{t\to\infty}} \frac{B_{N(t)} \log\log B_{N(t)}}{N(t) \log\log N(t)} \cdot \lim_{t\to\infty} \frac{N(t) \log\log N(t)}{t \log\log t} \\ &= \frac{1}{a} \overline{\lim_{t\to\infty}} \frac{B_{N(t)}}{N(t)-1} = \frac{B}{a} \end{aligned} \tag{3.10}$$

From (3.8), (3.9) and (3.10), we have

$$-\sqrt{\frac{B}{a}} \leqq \underline{\lim_{t\to\infty}} \frac{t - aN(t)}{\sqrt{2t \log\log t}} \tag{3.11}$$

because ε can be taken as an arbitrary small number. By the same way, if we take $n = N(t) + 1$ in the right inequality in (3.7), we get that

$$\overline{\lim_{t\to\infty}} \frac{t - aN(t)}{\sqrt{2t \log\log t}} \leqq \sqrt{\frac{B}{a}} \tag{3.12}$$

which gives (3.5) with (3.11).

REMARK 2. Without the assumption (3.1), we have that

$$P\left\{\frac{\underline{C}-\sqrt{B}}{(\sqrt{a})^3} \leqq \varliminf_{t\to\infty} \frac{t/a-N(t)}{\sqrt{2t\log\log t}} \leqq \varlimsup_{t\to\infty} \frac{t/a-N(t)}{\sqrt{2t\log\log t}} \leqq \frac{\bar{C}+\sqrt{B}}{(\sqrt{a})^3}\right\}=1, \tag{3.13}$$

where

$$\varlimsup_{n\to\infty}\left(\frac{a_2+\cdots+a_n}{n-1}-a\right)\sqrt{\frac{n}{2\log\log n}}=\bar{C}$$

and

$$\varliminf_{n\to\infty}\left(\frac{a_2+\cdots+a_n}{n-1}-a\right)\sqrt{\frac{n}{2\log\log n}}=\underline{C}.$$

This remark is available in the following Theorem 5.

THEOREM 5. *Under the conditions in Theorem 4, we have that*

$$P\left\{\varlimsup_{t\to\infty} \frac{|X(t)/a+(N(t)-t/a)|}{\sqrt{2t\log\log t}} \leqq \frac{\sqrt{B}}{(\sqrt{a})^3}\right\}=1. \tag{3.14}$$

Proof. If $\omega\in\Omega-E_1-E_2$ we have by (3.7) and (1.2) that

$$-(1+\varepsilon)\sqrt{2B_{N(t)}\log\log B_{N(t)}} < t-X(t)-aN(t)+o(\sqrt{N(t)\log\log N(t)})$$

so that

$$-\sqrt{\frac{B}{a}} \leqq \varliminf_{t\to\infty} \frac{t-aN(t)-X(t)}{\sqrt{2t\log\log t}} \tag{3.15}$$

By the same way, we get that

$$\varlimsup_{t\to\infty} \frac{t-aN(t)-X(t)}{\sqrt{2t\log\log t}} \leqq \sqrt{\frac{B}{a}} \tag{3.16}$$

which gives (3.14) with (3.15).

Combining Theorems 4 and 5 we get the following corollary which will be a more complete form of Theorem 2.

COROLLARY. *Under the conditions in Theorem 4, we have that*

$$P\left\{\varlimsup_{t\to\infty} \frac{X(t)}{\sqrt{2t\log\log t}} \leqq 2\cdot\sqrt{\frac{B}{a}}\right\}=1 \tag{3.17}$$

4. Let $X_1, X_2, \cdots; Y_1, Y_2, \cdots; Z_1, Z_2, \cdots$ be non-negative and mutually independent random variables with finite means $a_1, a_2, \cdots; b_1, b_2, \cdots; c_1, c_2, \cdots$ respectively. We assume the conditions analogous to (2.1)–(2.4) for every above sequence and that $a\geqq b\geqq\cdots\geqq c$. Then defining $M(t)$ as the random variable such that

$$\max\left\{\sum_{\nu=1}^{M(t)} X_\nu, \sum_{\nu=1}^{M(t)} Y_\nu, \cdots, \sum_{\nu=1}^{M(t)} Z_\nu\right\} < t \leqq \max\left\{\sum_{\nu=1}^{M(t)+1} X_\nu, \sum_{\nu=1}^{M(t)+1} Y_\nu, \cdots, \sum_{\nu=1}^{M(t)+1} Z_\nu\right\},$$

we know easily that

$$P\left\{\lim_{t\to\infty}\frac{M(t)}{t}=\frac{1}{a}\right\}=1 \quad \text{and} \quad M(t)\leqq\tilde{N}\left(\frac{t}{\lambda}\right),$$

so that

$$\lim_{t\to\infty}\frac{E\{M(t)^{\alpha}\}}{t^{\alpha}}=\frac{1}{a^{\alpha}} \qquad \text{for} \quad \alpha>0.$$

In this case, we have also a conclusion analoguous to Theorem 4.

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References

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