THE PROJECTIVE TRANSFORMATION ON A SPACE WITH PARALLEL RICCI TENSOR

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Introduction.

Recently we have proved [2] that a complete connected Riemannian space M, $2<\dim M$, with parallel Ricci tensor does not admit a non-isometric conformal transformation, unless M is isometric either to the Euclidean space or to the sphere. An analogous fact is true for a projective transformation, as the following main theorem of this paper shows.

THEOREM 1. *Let g and $\hat{g}$ be two complete Riemannian metrics on a connected manifold M with dimension >1 whose Ricci tensors R and R' are parallel. If g and $\hat{g}$ are projectively related, then* 1) *Levi-Civita connections of g and $\hat{g}$ coincide, or* 2) *g and $\hat{g}$ are of positive constant curvature.*

On the other hand Tanaka [5] studied projective transformations of affine connections. To describe his theorem we explain some terminologies. Two affine connections without torsion L and $\hat{L}$ (on the same manifold) are said *projectively relaled* when there exists a 1-form ϕ satisfying

$$\hat{L}^i_{jk}=L^i_{jk}+\delta^i_j\phi_k+\delta^i_k\phi_j, \tag{0.1}$$

where δ is Kronecker's delta. ϕ is then called *the associated form.* Two Riemannian metrics on the same manifold are said projectively related when their Levi-Civita connections are projectively related. B denoting the Ricci tensor of L, the *symmetrized Ricci tensor* R shall have the components $R_{ij}=(B_{ij}+B_{ji})/2$. Now Tanaka's theorem states:

THEOREM T. *Let L and $\hat{L}$ be two complete and torsion-free affine connections (on a connected manifold M with* $\dim M>1$) *whose Ricci tensors are parallel. Assume that they are projectively related.*

1) *If the symmetrized Ricci tensors R and $\hat{R}$ are both positive semi-definite, then, for any point x in M any vector X at x, $R_{ij}(x)X^i=0$ is equivalent to $\hat{R}'_{ij}(x)X^i=0$ and implies $\phi_i(x)X^i=0$, ϕ being the associated form*

2) *In the other case, L and $\hat{L}$ coincide.*

By Theorem T we have only to prove Theorem 1 in the two cases I) and II); I) R and $\hat{R}$ are non-zero, degenerate and positive semi-definite, II) R and $\hat{R}$

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are positive definite. But we shall give a complete proof.

From Theorem 1 follows easily Theorem 2 which is not covered by Theorem T.

THEOREM 2. *Under the hypothesis of Theorem T, if R and $\hat{R}$ are positive definite then* 1) *L and $\hat{L}$ coincide, or* 2) *L and $\hat{L}$ are the Levi-Civita connections of Riemannian metrics of positive constant curvature.*

To close Introduction we must pay attention to Tashiro's results [6]: if one of two complete Riemannian metrics which are projectively related is (locally) reducible then their Levi-Civita connections coincide. Ishihara, Sumitomo and Yano-Nagano obtained some results concerning projective transformations, which are covered by the above theorems.

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1. Construction of M'.

Let M be an n-dimensional differentiable (i.e. C^∞-differentiable) manifold, $1<n$, and E the one-dimensional Euclidean space. We consider the direct product $M\times E$ of these differentiable manifolds, which will be denoted by M'. M' is covered by the coordinate systems (x^0, x^i) which are pairs of a fixed cartesian coordinate (x^0) on E and arbitrary coordinate systems (x^i) on M. Given an affine connection L on M without torsion, we define two affine connection L'' and L' on M' in terms of these coordinates by

$$L''^{\lambda}_{\mu\nu}=0 \quad \text{if} \quad \lambda\mu\nu=0, \quad \text{and} \quad =L^{\lambda}_{\mu\nu} \quad \text{if} \quad \lambda\mu\nu\neq 0, \tag{1.1}$$

$$L'^{\lambda}_{\mu\nu}=L''^{\lambda}_{\mu\nu}+\delta^{\lambda}_{\mu}\delta^{0}_{\nu}+\delta^{\lambda}_{\nu}\delta_{\mu}-g''_{\mu\nu}w^{\lambda}, \tag{1.2}$$

where w is the vector field on M' with the components $w^\lambda=\delta^\lambda_0$, and g'' is the tensor field defined at $x'\in M'$ by

$$g''_{\mu\nu}X^\mu X^\nu=(X^0)^2+X^iX^jR_{ij}/(n-1) \tag{1.3}$$

for any vector X at x', R being the symmetrized Ricci tensor of L. We have here adopted the conventions: Greek indices run over $0, 1, \cdots, n$ and Latin ones run over $1, \cdots, n$. The affine connection L' will be called the Thomas connection of L or of a Riemannian metric g when L is the Levi-Civita connection of g [7].

(1.4) *Any geodesic (=path) of the Thomas connection is mapped to that of L by the natural projection of M' onto M.*

PROPOSITION 1. *Let L and $\hat{L}$ be affine connections without torsion on a differentiable manifold. Assume that the Ricci tensors of L and $\hat{L}$ are symmetric and that M is simply connected. If these affine connections are projectively related, then there exists a transformation α of M' (i.e. a dif-*

feomorphism of M' onto itself) which transforms the Thomas connection L' of L to that of $\hat{L}$. (See [8].)

Proof. The associated one-form ϕ is exact; i.e. there exists a function ρ on M with $d\rho = \phi$, because the Ricci tensors are symmetric and M is simply connected (see [4] for example). Now α is defined by $\alpha(x^0, x^i) = (x^0 + \rho, x^i)$, and satisfies the required condition as is easily seen.

REMARK. When $\hat{L}$ and L are Levi-Civita connections, simple-connectedness of M is a redundant condition, as we have $\phi = d[\log(\hat{G}/G)]/2(n+1)$ where G and $\hat{G}$ are the determinants of the metric tensors.

(1.5) *If R is parallel with respect to L then the tensor field* $\exp(2x^0)\cdot g''$ *is parallel with respect to the Thomas connection L', as is seen by means of some straightforward calculation.*

2. Tanaka's method.

Based on Tanaka's idea [5], but using no projective connection, we shall prove his Theorem T under some restrictions:

(2.1) *Let L and $\hat{L}$ be complete affine connections on a manifold M such that the Ricci tensors are symmetric and parallel. Assume that these connections are projectively related.* 1) *In case both of the Ricci tensors are positive semi-definite, we have $R_{ij}X^j = 0$ if and only if $\hat{R}_{ij}X^j = 0$ for any vector X, and this implies $\phi_i X^i = 0$, ϕ being the associated form.* 2) *Otherwise we have $L = \hat{L}$.*

Proof. We can suppose that M is simply connected. For the moment we consider L only and put $\hat{L}$ aside. Given a geodesic γ' of L' on M' with an affine parameter t, the equation of γ' is written as

$$DDx^\lambda + L'^\lambda_{\mu\nu} Dx^\mu Dx^\nu = 0, \qquad D \text{ denoting } d/dt. \tag{2.2}$$

Let f be the function $\exp(2x^0)$ on γ'. By (1.5) we get the first integral of (2.2).

$$f\cdot g''_{\mu\nu} Dx^\mu Dx^\nu = a \qquad (a = \text{const.}). \tag{2.3}$$

Solving (2.2) for $\lambda = 0$, we obtain

$$f = at^2 + 2bt + c \qquad (b,\ c = \text{const.}), \tag{2.4}$$

and

$$b/c = X^0, \tag{2.5}$$

where X^0 is the first component of the initial tangent vector $X' = Dx(0)$. We note that c is strictly positive.

Let s denote an affine parameter of the image geodesic $\pi\gamma'$ (see (1.4)) of

γ'. s is a function of t. Then fDs is a non-zero constant k: $fDs = k$. Since L is complete, the range of s is $(-\infty, \infty)$. By Cauchy's theorem applied to the differential equation $fDs = k$, we infer that the domain of t is the interval containing 0 given by $0 < f$. Owing to (2.3), (2.4) and (2.5) this implies

$$0 < g''_{\mu\nu}(x)X^\mu X^\nu + 2X^0 t + 1, \qquad x = \gamma'(0).$$

In other words, given a direction X' at $x' \in M'$, the geodesic γ' with an initial vector Y' in that direction X' is defined exactly for the interval $0 \leqq u < 1$ of the affine parameter u, provided that Y' satisfies

$$\begin{aligned} 0 &= g''_{\mu\nu}(x)Y^\mu Y^\nu + 2Y^0 + 1 \\ &= (Y^0 + 1)^2 + Y^i Y^j R_{ij}(x)/(n-1). \end{aligned} \tag{2.6}$$

γ' is defined for $0 \leqq u < \infty$ if no vector Y' in that direction satisfies (2.6).

Given an affine connection L mentioned in (2.1), we assign to each point $x' \in M'$ a quadric $Q(x')$ on the tangent space at x' defined by (2.6). To $\hat{L}$ in (2.1) corresponds $\hat{Q}(x')$ in the same way. By Proposition 1 these two figures must coincide, or precisely $\delta\alpha Q(x') = \hat{Q}(\alpha(x'))$, where $\delta\alpha$ is the differential of α. Since $\delta\alpha(Y')$ has the components $(Y^0 + Y^a \nabla_a \rho, Y^i)$, this gives that (2.6) implies

$$(Y^0 + Y^a \nabla_a \rho + 1)^2 + Y^i Y^j \hat{R}_{ij}/(n-1) = 0. \tag{2.7}$$

Now assume that some vector $Y = (Y^i)$ at a point $x \in M$ satisfies

$$R_{ij} Y^i Y^j < 0. \tag{2.8}$$

Then there exists a number $Y^0 \neq -1$ such that the vector $Y' = (Y^0, Y^i)$ at any point $x' \in \pi^{-1}(x) \subset M'$ satisfies (2.6). We have (2.7). Since the vector $Y'' = (Y^0, -Y^i)$ satisfies (2.6) too, it follows

$$(Y^0 + 1) Y^a \nabla_a \rho = 0, \quad \text{and so } Y^a \nabla_a \rho = 0.$$

The last equation is satisfied by every vector Z sufficiently near to Y; $Z^a \nabla_a \rho = 0$. This shows $\phi = d\rho = 0$ at x. Since R is parallel, at *any* point in M there exists a vector Y satisfying (2.8) under the above assumption and we have $\phi = 0$ on M. The second half 2) of (2.1) is thus proved.

Next we assume that a vector Y satisfies

$$R_{ij} Y^i Y^j = 0.$$

Then, putting $Y^0 = -1$, the vector $Y' = (Y^0, Y^i)$ tangent to M' satisfies (2.6), and (2.7) reads

$$(Y^a \nabla_a \rho) + Y^i Y^j R_{ij}/(n-1) = 0. \tag{2.9}$$

It follows

$$Y^a \nabla_a \rho = 0, \tag{2.10}$$

because otherwise we should have $Y^i Y^j \hat{R}_{ij} < 0$ and so by the above arguments $Y^a \nabla_a \rho = 0$. (2.9) and (2.10) give $Y^i Y^j \hat{R}_{ij} = 0$. When R is symmetric and positive semi-definite, $R_{ij}Y^iY^j=0$ is equivalent to $R_{ij}Y^j=0$. Thus the first half 1) of (2.1) is also proved.

3. The positive definite case.

This section is devoted to the proof of Theorem 2 in the introduction. The hypothesis of Theorem 2 and the notations in Section 1 will be preserved.

Let f be the function $\exp(2x^0)$ on M'. Then $g'=fg''$ (see (1.3)) and $\hat{g}'=f\hat{g}''$ define Riemannian metrics on M' whose Levi-Civita connections are L' and $\hat{L}'$ respectively by (1.5).

PROPOSITION 2. *Under the above hypothesis, M' with g' is either irreducible or locally flat. In the latter case L is a Levi-Civita connection of positive constant curvature.*

Proof. Assume that M' with g' is reducible; i.e. the homogeneous holonomy group H' of g' is reducible. Then there exists a parallel tensor field P of type (1.1) on M' such that, for each point x' in M', $P(x')$ is an orthogonal projection of the tangent space at x' onto a non-trivial subspace invariant under H. We identify P with the distribution assigning this subspace to x'. Let Q denote $I-P$; i.e. $Q^\lambda_\mu=\delta^\lambda_\mu-P^\lambda_\mu$. Proposition 2 will be proved after several lemmas.

(3.1) *Given any real number c the subset $\{x'\in M';\ 0\leqq x^0\}$ of M' is complete with respect to the metric on M' defined from g'.*

(3.2) *The vector field w defined in Section 1 is concurrent: $\nabla'_\mu w^\lambda=\delta^\lambda_\mu$. Pw is concurrent on any integral manifold of P, where Pw is the vector field with $(Pw)^\lambda=P^\lambda_\alpha w^\alpha$.*

(3.3)
$$\nabla'(Qw)=Q.$$

(3.4) *The length of Qw is constant on a connected integral manifold of P.*

In fact from (3.3) follows

$$P^\alpha_\nu\nabla'_\alpha((Qw)_\beta(Qw)^\beta)=2P^\alpha_\nu(Qw)_\beta Q^\beta_\alpha=0.$$

(3.5) *The union U of integral manifolds of P to which w is tangent at each point is nowhere dense.*

Proof. Let V be an open subset contained in U. We have $Qw=0$ on V, whence $\nabla'(Qw)=0$, contrarily to (3.3). Thus V is vacuous.

(3.6) *A connected integral manifold N of P is locally flat, if w is not tangent to N (at a point).*

Proof. Then w is not tangent to N at any point by (3.4). It suffices to verify (3.6) in case N is a maximal connected integral manifold. Let z be an arbitrary point of N. Assume that $Pw=0$ at z. By (3.2) the curvature tensor S of N is invariant by Pw; $\mathcal{L}_{Pw}S=0$, $\mathcal{L}$ denoting the Lie derivative [9]. By (2.2) and the equality $Pw(z)=0$, we find that $S(z)=0$. Next suppose that $Pw \neq 0$ at z. Consider the trajectory γ of Pw issuing from z in such a direction that the length λ of Pw is a decreasing function of the arc length s of γ. By (3.2), $2\lambda+s$ is constant on Y. By (3.4) and the assumption of (3.6) the length $\|w\|$ of w is bounded below on γ. This shows that the first coordinate $x^0=\log\|w\|$ is bounded below. From (3.1) it follows that s can attain the value s_0 such that $2\lambda+s_0=0$; i.e. there exists a point y on γ at which $Pw=0$. We have $S(y)=0$ as shown before. On the other hand $\|S\|\lambda^2$ is constant on γ [10] where

$$\|S\|^2=S_{\alpha\beta\gamma\delta}S^{\alpha\beta\gamma\delta}.$$

Therefore S must vanish on γ. In particular we have $S(z)=0$, and (3.6) is proved.

By (3.5) and (3.6) any integral manifold of P is locally flat. The analogue holds good for Q too. Thus M' is locally flat, and the first half of Proposition 2 is established. Since the symmetrized Ricci tensor R of an affine connection L without torsion is parallel and positive definite, L is the Levi-Civita connection of the Riemannian metric R, and R coincides with the Ricci tensor of L. In particular M with the metric tensor R is an Einstein space. If M' with $g'=fg''$ is locally flat, then g'' is locally conformally flat. Since M' with g'' is the Riemann product of the Euclidean space E and the Einstein space M with R, it follows that M with R is locally conformally flat. Thus M with R is a space of constant curvature. This completes the proof of Proposition 2.

Proof of Theorem 2. By Proposition 1, M' with g'' is irreducible if and only if M' with $\hat{g}''$ is irreducible. Then α is a homothetic transformation ([1], [3]). Owing to the definition (1.3) of g'' and $\hat{g}''$ α is then an isometry. Hence R coincides with $\hat{R}$. Hence the Levi-Civita connection L of the metric tensor R coincides with $\hat{L}$. If M' is reducible, Theorem 2 follows from Proposition 2 immediately.

4. The non-definite case.

Eventually we have to survey the case that the Ricci tensors of g and $\hat{g}$ are positive semi-definite but not definite in order to complete the proof of Theorem 1. In this case applies Tanaka's theorem mentioned in the introduction, since both g and $\hat{g}$ are then reducible. We shall however give an independent proof. M can be assumed to be simply connected. M with g (or $\hat{g}$) is then a Riemann product of a space N with the vanishing Ricci tensor and a space S (or $\hat{S}$) with the parallel positive definite Ricci tensor. N is common to g and $\hat{g}$ because of 1) in Theorem T. Let D be the distribution on M which is parallel with respect to g and whose maximal connected integral submanifolds are isometric to S. The distribution $\hat{D}$ is defined analogously from $\hat{g}$.

If D coincides with $\hat{D}$, then the associated form ϕ vanishes on M, as follows immediately from (0.1). In this case Theorem 1 is thus proved.

Now assume $D \neq \hat{D}$. Then there exists a point x in M such that the maximal connected integral submanifold $S(x)$ of D which contains x is different from $\hat{S}(x)$. For the sake of brevity we write S for $S(x)$, $\hat{S}$ for $\hat{S}(x)$ and N for the (totally geodesic) submanifold containing x isometric to N whose tangent space at x is orthocomplement of that of S at x with respect to g.

Let μ be the orthogonal projection of M with g onto S, and ν that of M with g onto N. $\hat{\mu}$ and $\hat{\nu}$ are analogously defined from $\hat{g}$; $\hat{\nu}(M)=\nu(M)=N$.

S with g and $\hat{S}$ with $\hat{g}$ are isometric to the sphere. Hence S with $\hat{g}$ and $\hat{S}$ with g are projectively flat and so spaces of constant curvature. Being compact and simply connected, they are isometric to the sphere. Restricted to $\hat{S}$ with $\hat{g}$, μ is a mapping onto S and sends any geodesic to a geodesic with the affine parameters preserved. Restricted to some neighborhood U of x in $\hat{S}$ with g, μ is a diffeomorphism and so an affine transformation. Since U with g is irreducible, it is a homothetic transformation. It follows that μ, restricted to $\hat{S}$ with g, is a homothetic transformation onto S; in particular it is a diffeomorphism of $\hat{S}$ onto S. Therefore ν, restricted to $\hat{S}$ with g, is a homothetic taansformation of $\hat{S}$ onto $\nu(\hat{S}) \subset N$.

Consider the submanifolds B and $\hat{B}$ of M such that $B=\{p \in M;\ \nu(p) \in \nu(\hat{S}),\ \mu(p) \in S\}$ and $\hat{B}=\{p \in M;\ \hat{\nu}(p) \in \nu(\hat{S})$ and $\hat{\mu}(p) \in \hat{S}\}$. B with g and $\hat{B}$ with $\hat{g}$ are both isometric to the Riemannian product $S \times S$. Let λ be the map of $\hat{B}$ into B defined by the conditions: $\hat{\nu}=\nu\lambda$ and $\mu\hat{\mu}=\mu\lambda$ on $\hat{B}$. Then λ is a projectiue transformation of $\hat{B}$ with $\hat{g}$ onto B with g. By theorem 2, λ is an affine transformation. Restricted to $\hat{S}$, λ coincides with μ. Hence g and $\hat{g}$ on $\hat{S}$ has the same Levi-Civita connection. By (0.1) and 2) in Theorem T we conclude that ϕ vanishes on $\hat{S}$ and so on M.

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