ON COEFFICIENT PROBLEMS FOR SOME PARTICULAR CLASSES OF ANALYTIC FUNCTIONS

BY YÛSAKU KOMATU

1. Introduction.

Recently, Hummel [3] has established a variational formula on the class of univalent functions starlike in the unit circle. The variation employed consists in displacing every boundary point of the image domain in the radial direction with respect to the center of starlikeness. He then has applied the variational formula thus obtained to a coefficient problem of a general nature for the class under consideration. His result states:

Let $F(a_2, a_3, \cdots, a_n)$ *be any function of the* $n-1$ *complex variables* a_2, $a_3, \cdots, a_n$ *having a continuous derivative in each variable. Then any function* $f(z)=z+\sum_{\nu=2}^{\infty}a_\nu z^\nu$ *which maximizes* $\Re F(a_2, a_3, \cdots, a_n)$ *within the class starlike in* $|z|<1$ *with respect to the origin must be of the form*

$$f^*(z)=z\prod_{\mu=1}^{m}(1-\kappa_\mu z)^{-\sigma_\mu}$$

where $|\kappa_\mu|=1$, $\sigma_\mu>0$ *for all* μ $(1\leqq\mu\leqq m)$, $\sum_{\mu=1}^{m}\sigma_\mu=2$, *and* $m\leqq n-1$.

Hummel's variational formula is itself of considerable interest. His procedure of obtaining the above-mentioned result would seem also an interesting attempt to apply the variational method. The proof described in his paper depends on a linearization process, namely on the reduction of the problem of maximizing $\Re F(a_2, a_3, \cdots, a_n)$ to that of maximizing $\Re\sum_{\nu=2}^{n}\lambda_\nu a_\nu$, λ_ν being the value of $\partial F/\partial a_\nu$ corresponding to a maximizing function. Accordingly, it must be indeed supposed, even if not explicitly stated, that at least one among the λ's does not vanish, what seems, however, a priori not self-evident. Nevertheless, Hummel's result remains valid. It will be further shown that the result can be derived more simply without making use of the variational formula. Namely, we shall give in the present paper a new proof of the Hummel's theorem. For that purpose, we consider previously an analogous coefficient problem for a related class of functions.

2. The class $\Re$.

Let $\Re$ be the class which consists of analytic functions $\Phi(z)$ with positive real part in $|z|<1$ and normalized by $\Phi(0)=1$. Let the Taylor expansion of $\Phi(z)$ be

Received February 13, 1959.

$$\Phi(z)=1+\sum_{\nu=1}^{\infty}C_\nu z^\nu.$$

Now, suppose further that the functional to be maximized within this class may depend on, besides an assigned number of the beginning Taylor coefficients, also their complex conjugates. For the sake of brevity, put

$$(\Re C;\Im C)_1^n=(\Re C_1,\Im C_1,\cdots,\Re C_n,\Im C_n),$$

n being an arbitrarily assigned positive integer. Then, so far as we are interested in such a functional as defined by the real part of a function analytic in $C_1,\cdots,C_n$, we may consider more generally a real-valued functional which is merely suposed to be harmonic in the $2n$ variables $(\Re C;\Im C)_1^n$.

THEOREM 1. *Let $\Omega(u;v)_1^n$ be a non-constant real-valued function harmonic in the $2n$ variables $(u;v)_1^n=(u_1,v_1,\cdots,u_n,v_n)$. Then, regarded as a functional defined for an argument function*

$$\Phi(z)=1+\sum_{\nu=1}^{\infty}C_\nu z^\nu\in\Re,$$

$\Omega(\Re C;\Im C)_1^n$ is maximized within the class $\Re$ if and only if $\Phi(z)$ is of the form

$$\Phi^*(z)=\sum_{\mu=1}^{m}\rho_\mu\frac{\varepsilon_\mu+z}{\varepsilon_\mu-z}$$

where m is an integer with $1\leqq m\leqq n$, the ε_μ $(1\leqq\mu\leqq m)$ are complex constants with $|\varepsilon_\mu|=1$ and the ρ_μ are positive real constants satisfying $\sum_{\mu=1}^{m}\rho_\mu=1$.

Proof. The coefficient region of $(\Re C;\Im C)_1^n$ for $\Re$ is compact and the class $\Re$ is normal and compact with respect to the uniform convergence in the wider sense. Hence there exist some functions maximizing $\Omega(\Re C;\Im C)_1^n$. Since $\Omega(u;v)_1^n$ is supposed non-constant and harmonic, its maximum on any compact region is attained only on its boundary. Consequently, the maximizing functions must be of the form $\Phi^*(z)$ by virtue of a classical theorem of Carathéodory [1, 2]; cf., for instance, also Rogosinski [10] or Nishimiya [9].

As shows the proof of theorem 1, the essential part in the assumption on $\Omega(u;v)_1^n$ is that its maximum on the coefficient-region for $\Re$ is attained on the boundary. Hence, for instance, we might only assume that $\Omega(u;v)_1^n$ is subharmonic in $(u;v)_1^n$ on the coefficient-region. However, in this case its range may possibly have a flat part and we can then conclude only that $\Omega(u;v)_1^n$ is maximized within $\Re$ by certain functions of the form $\Phi^*(z)$ and eventually also by different functions.

3. The class $\mathfrak{St}$.

Let $\mathfrak{St}$ be the class which consists of analytic functions $f(z)$ univalent and starlike in $|z|<1$ with respect to the origin and normalized by $f(0)=0$ and $f'(0)=1$. The close connection between this class and the class $\Re$ considered just above is well known. Accordingly, once a theorem on $\mathfrak{R}$ having been established, the corresponding theorem on $\mathfrak{St}$ follows readily and vice versa. But, in transferring theorem 1 into the class $\mathfrak{St}$, we shall formulate the result in restricting, for the sake of brevity, to a special functional instead of describing the immediate transform of this theorem. In fact, we take here a pluriharmonic function, i.e. the real part of an analytic function, as a functional to be maximized.

THEOREM 2. *Let $F(a)_2^n$ be a non-constant analytic function in the $n-1$ complex variables $(a)_2^n=(a_2, \cdots, a_n)$. Then, regarded as a functional defined for an argument function*

$$f(z)=z+\sum_{\nu=2}^{\infty}a_\nu z^\nu \in \mathfrak{St},$$

$\mathfrak{R}F(a)_2^n$ is maximized within the class $\mathfrak{St}$ if and only if $f(z)$ is of the form

$$f^*(z)=z\prod_{\mu=1}^{m}(1-\bar{\varepsilon}_\mu z)^{-2\rho_\mu}$$

where m is an integer with $1\leqq m\leqq n-1$, the ε_μ $(1\leqq\mu\leqq m)$ are complex constants with $|\varepsilon_\mu|=1$ and the ρ_μ are positive real constants satisfying $\sum_{\mu=1}^{m}\rho_\mu=1$.

Proof. Two classes $\mathfrak{St}=\{f(z)\}$ and $\mathfrak{R}=\{\Phi(z)\}$ are connected one-to-one by the relation

$$\frac{zf'(z)}{f(z)}=\Phi(z), \qquad \text{i.e.} \quad f(z)=z\exp\int_0^z\frac{\Phi(z)-1}{z}dz.$$

Hence, putting

$$\Phi(z)=1+\sum_{\nu=1}^{\infty}C_\nu z^\nu,$$

we get the identity

$$z+\sum_{\nu=2}^{\infty}a_\nu z^\nu=z\exp\sum_{\nu=1}^{\infty}\frac{C_\nu}{\nu}z^\nu.$$

It is readily seen that, for any integer $\nu\geqq 2$, a_ν is a polynomial in $C_1, \cdots, C_{\nu-1}$ while $C_{\nu-1}$ is a polynomial in $a_2, \cdots, a_\nu$. Thus we may put

$$\mathfrak{R}F(a)_2^n=\Omega(\mathfrak{R}C;\mathfrak{J}C)_1^{n-1}$$

of which the right-hand member is a non-constant function harmonic in the $2(n-1)$ variables $(\mathfrak{R}C;\mathfrak{J}C)_1^{n-1}$. Applying theorem 1 to $\Omega(\mathfrak{R}C;\mathfrak{J}C)_1^{n-1}$ with $n-1$ instead of n, we see that the maximum of $\mathfrak{R}F(a)_2^n$ is attained if and only if $f(z)$ is of the form $f^*(z)$ which satisfies

$$\frac{zf^{*\prime}(z)}{f^*(z)}=\sum_{\mu=1}^{m}\rho_\mu\frac{\varepsilon_\mu+z}{\varepsilon_\mu-z}=1+2z\sum_{\mu=1}^{m}\frac{\rho_\mu}{\varepsilon_\mu-z},$$

m, ε_μ and ρ_μ being of the nature stated in the theorem, whence follows after integration

$$f^*(z) = z \exp \int_0^z \left(\frac{f^{*\prime}(z)}{f^*(z)} - \frac{1}{z} \right) dz = z \prod_{\mu=1}^m (1 - \bar{\varepsilon}_\mu z)^{-2\rho_\mu}.$$

In general, the function $w = f^*(z)$ given in theorem 2 maps the unit circle onto the whole w-plane cut along m radial slits such that each ε_μ corresponds to a boundary element lying on $w = \infty$ where the consecutive slits make an angle with the aperture $2\pi\rho_\mu$; cf. a previous paper [4] and also Mori [7]. This mapping function can be expressed in an alternative form. In fact, let the points on $|z| = 1$ which correspond to the finite ends of the image-slits be denoted by ζ_μ $(\mu = 1, \cdots, m)$. Then, by Schwarz-Christoffel's formula, we may write

$$f^*(z) = \int_0^z \prod_{\mu=1}^m (1 - \bar{\zeta}_\mu z)(1 - \bar{\varepsilon}_\mu z)^{-1-2\rho_\mu} dz.$$

Comparing this expression with that given in theorem 2, we obtain after differentiation the identity

$$\prod_{\mu=1}^m (1 - \bar{\zeta}_\mu z) = \left(1 + 2z \sum_{\mu=1}^m \frac{\rho_\mu \bar{\varepsilon}_\mu}{1 - \bar{\varepsilon}_\mu z} \right) \prod_{\mu=1}^m (1 - \bar{\varepsilon}_\mu z).$$

Consequently, we see that the points ζ_μ $(\mu = 1, \cdots, m)$ are determined as the roots of the algebraic equation

$$1 + 2z \sum_{\kappa=1}^m \frac{\rho_\kappa}{\varepsilon_\kappa - z} = 0.$$

On the other hand, by putting $z = \varepsilon_\kappa$ in the above identity, we get

$$\prod_{\mu=1}^m (1 - \bar{\zeta}_\mu \varepsilon_\kappa) = 2\rho_\kappa \prod_{\substack{\mu=1 \\ \mu \neq \kappa}}^m (1 - \bar{\varepsilon}_\mu \varepsilon_\kappa) \qquad (\kappa = 1, \cdots, m),$$

which may be regarded as a system of simultaneous equations also satisfied by ζ_μ $(\mu = 1, \cdots, m)$.

By the way, we notice here that a theorem obtained by Nehari and Netanyahu [8][1] follows readily from theorem 2. It will now be re-stated as a corollary in a slightly general form.

COROLLARY. *Let* $\mathfrak{St}^c$ *be the class of analytic functions which map* $|z| < 1$ *univalently onto the complements of point-sets starlike with respect to the origin. Let* $\Psi(\alpha)_0^n$ *be any non-constants analytic function in the* $n+1$ *complex variables* $(\alpha)_0^n = (\alpha_0, \cdots, \alpha_n)$. *Then, regarded as a functional defined for an argument function*

$$\phi(z) = \frac{1}{z} + \sum_{\nu=0}^\infty \alpha_\nu z^\nu \in \mathfrak{St}^c,$$

$\Re\Psi(\alpha)_0^n$ *is maximized within the class* $\mathfrak{St}^c$ *if and only if* $\phi(z)$ *is of the form*

1) In this occasion the present author should express his regret for missing out this paper [8] in which a result in a previous paper [6] had been already obtained. But the proof is not the same so that it might be regarded as an alternative one.

$$\phi^*(z) = \frac{1}{z} \prod_{\mu=1}^{m} (1 - \bar{\varepsilon}_\mu z)^{2\rho_\mu}$$

where m is an integer with $0 \leqq m \leqq n+1$, the ε_μ $(1 \leqq \mu \leqq m)$ are complex constants with $|\varepsilon_\mu| = 1$ and the ρ_μ are positive real constants satisfying $\sum_{\mu=1}^{m} \rho = 1$.

Proof. As readily seen, two classes $\mathfrak{St}^c = \{\phi(z)\}$ and $\mathfrak{St} = \{f(z)\}$ are connected one-to-one by the relation

$$\frac{1}{\phi(z)} = f(z), \quad \text{i. e.} \quad \phi(z) = \frac{1}{f(z)}.$$

Hence, putting

$$f(z) = z + \sum_{\nu=2}^{\infty} a_\nu z^\nu$$

we get the identity

$$\frac{1}{z} + \sum_{\nu=0}^{\infty} \alpha_\nu z^\nu = \left(z + \sum_{\nu=2}^{\infty} a_\nu z^\nu\right)^{-1}.$$

This shows that, for any integer $\nu \geqq 0$, α_ν is a polynomial in $a_2, \cdots, a_{\nu+2}$ while, for any integer $\nu \geqq 2$, a_ν is a polynomial in $\alpha_0, \cdots, \alpha_{\nu-2}$. Thus we may put

$$\Psi(\alpha)_0^n = F(a)_2^{n+2}$$

of which the right-hand member is a non-constant analytic function in its $n+1$ complex arguments. The assertion of the corollary follows readily by applying theorem 2 with $n+2$ instead of n.

This corollary may be derived more directly in quite a similar manner as theorem 2 by applying theorem 1. In fact, $\phi(z) \in \mathfrak{St}^c$ is equivalent to $-z\phi'(z)/\phi(z) \in \mathfrak{R}$.

In the original Nehari-Netanyahu's theorem, by considering a subclass of $\mathfrak{St}^c$ normalized by

$$\alpha_0 = 0,$$

they have observed a particular functional $|\alpha_n|$. In the problem of maximizing

$$\lg |\alpha_n| = \Re \lg \alpha_n,$$

the function $\lg \alpha_n$ is not regular throughout the coefficient-region of α_n for the class. However, a sufficiently small vicinity of $\alpha_n = 0$ surely belongs to this coefficient-region and such a vicinity is indifferent to the maximizing problem so that the problem is equivalent to maximizing $\Re \lg \alpha_n$ on the part of the coefficient-region not belonging to this vicinity where $\lg \alpha_n$ is regular. More generally, we may consider a functional $\Re(\varepsilon \alpha_n)$ with an arbitrary complex constant ε. It is readily verified that our assertion in the corollary remains valid by supplementing a condition

$$\sum_{\mu=0}^{m} \rho_\mu \varepsilon_\mu = 0$$

which corresponds to the normalization $\alpha_0 = 0$.

4. The class $\mathfrak{K}$.

Let finally $\mathfrak{K}$ be the class which consists of analytic functions $g(z)$ univalent and convex in $|z| < 1$ and normalized by $g(0) = 0$ and $g'(0) = 1$. This class is connected with $\mathfrak{St}$ very closely. Accordingly, theorem 2 can be readily transferred to this class.

THEOREM 3. *Let $G(b)_2^n$ be a non-constant analytic function in the $n-1$ complex variables $(b)_2^n = (b_2, \cdots, b_n)$. Then, regarded as a functional defined for an argument function*

$$g(z) = z + \sum_{\nu=2}^{\infty} b_\nu z^\nu \in \mathfrak{K},$$

$\Re G(b)_2^n$ is maximized within the class $\mathfrak{K}$ if and only if $g(z)$ is of the form

$$g^*(z) = \int_0^z \prod_{\mu=1}^{m} (1 - \varepsilon_\mu z)^{-2\rho_\mu} dz$$

where m, ε_μ and ρ_μ are of the same nature as in theorem 2.

Proof. The class $\mathfrak{K} = \{g(z)\}$ is connected one-to-one with the class $\mathfrak{St} = \{f(z)\}$ by the relation

$$zg'(z) = f(z), \qquad \text{i. e.} \quad g(z) = \int_0^z \frac{f(z)}{z} dz.$$

Hence putting

$$f(z) = z + \sum_{\nu=2}^{\infty} a_\nu z^\nu,$$

we get the relations

$$b_\nu = \frac{a_\nu}{\nu} \qquad (\nu = 2, 3, \cdots).$$

Hence the assertion of the theorem follows immediately from theorem 2.

REFERENCES

[1] CARATHÉODORY, C., Über den Variabilitätsbereich der Koeffizienten von Potenzreihen, die gegebene Werte nicht annehmen. Math. Ann. 64 (1904), 95–115.
[2] CARATHÉODORY, C., Über den Variabilitätsbereich der Fourierschen Konstanten von positiven harmonischen Funktionen. Rend. di Palermo 32 (1911), 193–217.
[3] HUMMEL, J. A., A variational method for starlike functions. Proc. Amer. Math. Soc. 9 (1958), 82–87.
[4] KOMATU, Y., Einige Darstellungen analytischer Funktionen und ihre Anwendungen auf konforme Abbildung. Proc. Imp. Acad. Tokyo 20 (1944), 536–541.

[5] KOMATU, Y., On analytic functions with positive real part in a circle. Kōdai Math. Sem. Rep. 10 (1958), 64–83.
[6] KOMATU, Y., On convolution of power series. Kōdai Math. Sem. Rep. 10 (1958), 141–144.
[7] MORI, A., On a conformal mapping with certain boundary correspondences. Journ. Math. Soc. Japan 2 (1950), 129–132.
[8] NEHARI, Z., AND E. NETANYAHU, On the coefficients of meromorphic schlicht functions. Proc. Amer. Math. Soc. 8 (1957), 15–23.
[9] NISHIMIYA, H., On coefficient-regions of Laurent series with positive real part. Kōdai Math. Sem. Rep. 11 (1959), 25–39.
[10] ROGOSINSKI, W., Über positive harmonische Entwicklungen und typisch-reelle Potenzreihen. Math. Zeitschr. 35 (1932), 93–121.

DEPARTMENT OF MATHEMATICS,
TOKYO INSTITUTE OF TECHNOLOGY.