ON EXTREMAL QUASICONFORMAL MAPPING

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Let $\{T_n\}$ be a sequence of quasiconformal mappings of a domain D onto a domain Δ converging uniformly to a quasiconformal mapping T of D onto Δ. Let K_n and K be the maximal dilatations of T_n and T, respectively, then K_n tends to K when n tends to infinity. Our first purpose in the present note is to establish that the quotient of two complex derivatives q_{T_n}/p_{T_n} of T_n converges to q_T/p_T of T in a certain sense under the above situation. Our result may possibly be not original. However, so far as we concern, there are no papers stating it explicity. Our second purpose is to establish that there exists an extremal quasiconformal mapping in a family with a boundary correspondence and it satisfies a differential equation of Beltrami type with some remarkable restrictions. To this end, we shall apply the result stated in the first part. According to the result obtained in this case, one can recognize that there exists an essential difference between the cases treated previously by Teichmüller [10, 11] and recently by Ahlfors [1] and a case presented here. Then there arise many unsolved problems, all of which are perhaps very difficult to settle. Situations are quite similar in a case with a countably infinite number of distinguished boundary points.

1. Definitions and known results on quasiconformal mappings.

Among various definitions of quasiconformality the one due to Pfluger-Ahlfors-Mori [1, 5, 6] is most convenient for our later purposes.

A topological mapping $w=T(z)$ of a planar region D onto another such region Δ is called quasiconformal with the parameter K, if (i) $w=T(z)$ preserves the orientation of the plane, and (ii) for any quadrilateral Ω contained in D together with its boundary, it satisfies

$$\operatorname{mod} T(\Omega) \leqq K \operatorname{mod} \Omega,$$

where K is a constant $\geqq 1$ and $\operatorname{mod}\Box$ denotes the modulus of the indicated quadrilateral $\Box$. The infimum of K satisfying the above condition is called the maximal dilatation K_T of the mapping T.

Mori proved in his theorem 1 [5, 6] that (i) $w=T(z)$ is totally differentiable almost everywhere in D; (ii) at each totally differentiable point z

$$(|p_T|+|q_T|)^2 \leqq K_T(|p_T|^2-|q_T|^2),$$

where $p_T=\partial T/\partial z$ and $q_T=\partial T/\partial \bar{z}$; (iii) $w=T(z)$ is absolutely continuous in Tonelli's sense, that is, for almost all $y=y_0$, the function $T(x, y_0)$ is absolutely

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continuous in x on any closed subinterval contained in the intersection of $y=y_0$ and similarly for $T(x_0, y)$. Inverse statement has been proved by Yûjôbô [13]. Let z_0 be a differentiable point of T at which the Jacobian $|p_T(z_0)|^2-|q_T(z_0)|^2$ of T does not vanish, then we shall define the dilatation $K_T(z_0)$ of T at z_0 by the quotient

$$\frac{|p_T(z_0)|+|q_T(z_0)|}{|p_T(z_0)|-|q_T(z_0)|}.$$

Morrey [7] and others have given analytic definitions of quasi-conformality and Bers [2] has proved the equivalence of their definitions and the Pfluger-Ahlfors-Mori's geometric definition. As an immediate consequence of this equivalence, any quasiconformal mapping is a measurable and locally L^2 mapping. Evidently a set at which the Jacobian vanishes is of measure zero and hence we can consider the essential supremum of $K_T(z_0)$. Then it is equal to K_T, since the Yûjôbô's inverse theorem holds.

By Morrey's lemma 7 in his paper [7], we can apply the Green's formula

$$\iint_D (|p_T|^2-|q_T|^2)\,dx\,dy=\frac{1}{2i}\int_{\partial D} T(z)\overline{dT(z)}$$

for any quasiconformal mapping T. General Green-Stokes' formula also remains valid. These formulas can be easily deduced by the so-called approximation method by integral means and the necessary tools for this have been precisely explained in a text book due to Rado-Reichelderfer [9], so that we shall here not discuss it.

2. Convergence theorems on complex derivatives.

LEMMA 1. *Let $\{U_n(z)\}$ be a sequence of quasiconformal mappings of D onto Δ_n converging uniformly to the identity mapping z in D, then there exists a subsequence $\{U_{n_\nu}(z)\}$ such that $\lim_{\nu\to\infty} p_{U_{n_\nu}}(z)=1$ and $\lim_{\nu\to\infty} q_{U_{n_\nu}}(z)=0$ hold almost everywhere in D, if*

$$\iint_B |p_{U_n}|^2\,dx\,dy<M(B)$$

holds for any n with a fixed constant $M(B)$ depending only on $B\subset D$.

Proof. In the first place we shall prove that $p_{U_n}(z)$ and $q_{U_n}(z)$ tend to 1 and 0 weakly in L^2-sense, respectively. Let ω be any continuously differentiable function with a compact carrier in D, then, for any given $\varepsilon>0$ there exists an integer n_0 such that for any integer $n>n_0$

$$\left|\iint_D \omega\frac{\partial(U_n-z)}{\partial z}\,dx\,dy\right|=\left|\iint_D (U_n-z)\frac{\partial\omega}{\partial z}\,dx\,dy\right|$$
$$\leqq\iint_D |U_n-z|\left|\frac{\partial\omega}{\partial z}\right|dx\,dy\leqq\varepsilon\iint_D\left|\frac{\partial\omega}{\partial z}\right|dx\,dy,$$

whence follows

$$\lim_{n\to\infty}\iint_D \omega(p_{U_n}(z)-1)\,dx\,dy=0.$$

Similarly we have

$$\lim_{n\to\infty}\iint_D \omega q_{U_n}(z)\,dx\,dy=0.$$

Now we shall prove that there exists a subsequence U_{n_ν} such that

$$\lim_{n\to\infty} q_{U_{n_\nu}}(z)=0 \qquad \text{a.e. in } D.$$

Since $|q_{U_n}(z)|\leqq k_{U_n}|p_{U_n}(z)|$ holds almost everywhere in D with $k_{U_n}=(1-K_{U_n})\div(1+K_{U_n})$, we have

$$0\leqq\iint_B |q_{U_n}(z)|^2dx\,dy\leqq k_{U_n}^2\iint_B |p_{U_n}(z)|^2dx\,dy\leqq k_{U_n}^2M(B).$$

By a theorem due to Ahlfors or Mori stating that the uniform convergence of U_n to z implies $\lim_{n\to\infty}K_{U_n}=1$ or equivalently $\lim_{n\to\infty}k_{U_n}=0$, we have a limit relation

$$\lim_{n\to\infty}\iint_B |q_{U_n}(z)|^2dx\,dy=0,$$

which shows that $q_{U_n}(z)$ converges to 0 strongly in L^2-sense. Therefore, we can select a subsequnce $\{U_{n_\nu}\}$ such that $q_{U_{n_\nu}}(z)$ converges to zero almost everywhere in D. By retaining the original indices for simplicity, we shall further prove that there exists a subsequence $\{U_{n_\nu}\}$ such that $\lim_{\nu\to\infty}p_{U_{n_\nu}}(z)=1$ a. e. in D. Let $A_m(r)$ be an integral

$$\iint_{|z|<r}(|p_{U_m}-1|^2-|q_{U_m}|^2)\,dx\,dy,$$

which can be transformed into a contour integral

$$\frac{1}{2i}\int_{|z|=r}(U_m-z)\,d(\bar{U}_m-\bar{z})$$

by the Green's formula or by integration by parts. Then, for any sufficiently large m, we have

$$|A_m(r)|\leqq\frac{1}{2}\int_{|z|=r}|U_m-z|\,|d(\bar{U}_m-\bar{z})|\leqq\varepsilon\int_{|z|=r}|d(\bar{U}_m-\bar{z})|.$$

On the other hand, we have $|dU_m|\leqq(|p_{U_m}|+|q_{U_m}|)r\,d\theta$, and hence

$$|A_m(r)|\leqq\varepsilon r\int_0^{2\pi}(|p_{U_m}|+|q_{U_m}|+1)\,d\theta.$$

Integrating both members from 0 to R, we have

$$\begin{aligned}\int_0^R|A_m(r)|\,dr&\leqq\varepsilon\int_0^R r\,dr\int_0^{2\pi}(|p_{U_m}|+|q_{U_m}|+1)\,d\theta\\&=\varepsilon\iint_{|z|<R}(|p_{U_m}|+|q_{U_m}|+1)\,r\,dr\,d\theta\\&\leqq\varepsilon R\pi^{1/2}(2M(|z|<R)+\pi R^2)^{1/2}.\end{aligned}$$

Let m tend to infinity and successively ε to zero, then we have

$$\lim_{m\to\infty}\int_0^R |A_m(r)|\,dr = 0.$$

Thus, for almost all r in $[0, R]$, we see that $\lim_{m\to\infty} A_m(r) = 0$. Since the choice of the center of the circular disc is arbitrary, we get, for almost all r and for almost all z_0 in D,

$$\lim_{m\to\infty}\iint_{|z-z_0|<r}(|p_{U_m}-1|^2 - |q_{U_m}|^2)\,dx\,dy = 0.$$

This shows that

$$\lim_{m\to\infty}\iint_{|z-z_0|<r}|p_{U_m}-1|^2\,dx\,dy = \lim_{m\to\infty}\iint_{|z-z_0|<r}|q_{U_m}|^2\,dx\,dy = 0,$$

whence follows the strong convergence of $p_{U_m}(z)$ to 1, and hence we can select a subsequence U_{m_ν} such that $p_{U_{m_\nu}}(z)$ converges to 1 almost everywhere in D.

In the above lemma the condition of uniform boundedness of $\{p_{U_n}\}$ in the L^2-space is somewhat artificial. Therefore we should remove it for an important case appearing in certain applications.

LEMMA 2. *Let D be a bounded domain in the z-plane and $\{U_n(z)\}$ be a sequence of quasiconformal mappings such that $U_n(z)$ converges uniformly to z in D (or $\bar{D}$) and $U_n(D) = D$. Then*

$$\iint_D |p_{U_n}(z)|^2\,dx\,dy < M(D)$$

holds for any n.

Proof. It is well known that $U_n(z)$ is a measurable mapping of D onto itself whose density is equal to the Jocobian $|p_{U_n}(z)|^2 - |q_{U_n}(z)|^2$. Since $U_n(z)$ converges uniformly to z, the maximal dilatation K of U_n defined either globally or locally also converges to 1. Therefore, there exists a constant K_0 such that $K_{U_n} < K_0 < \infty$ for any n. Then $K_{U_n}(z) \leqq K_{U_n} < K_0$ holds. Thus for any closed domain D' ($\bar{D}' \subset D$), we have

$$\iint_{D'}(|p_{U_n}(z)|^2 + |q_{U_n}(z)|^2)\,dx\,dy \leqq \frac{K_0^2+1}{2K_0}\iint_{D'}(|p_{U_n}(z)|^2 - |q_{U_n}(z)|^2)\,dx\,dy$$
$$\leqq \frac{K_0^2+1}{2K_0}A(D),$$

where $A(D)$ is the area of D. Thus we conclude that

$$\iint_D(|p_{U_n}(z)|^2 + |q_{U_n}(z)|^2)\,dx\,dy \leqq \frac{K_0^2+1}{2K_0}A(D).$$

This implies the desired result.

THEOREM 1. *Let D be a bounded domain in the z-plane and $\{S_n(z)\}$ be* *a sequence of quasiconformal mappings of D onto a domain Δ such that $S_n(z)$ converges uniformly in $\bar{D}$ to a quasiconformal mapping $S(z)$ of D onto Δ, then there exist a subsequence $\{S_{n_\nu}\}$ and a point-sequence $\{z_{n_\nu}\}$ corresponding to $z \in D$ such that*

$$\lim_{\nu\to\infty} q_{S_{n_\nu}}(z_{n_\nu})/p_{S_{n_\nu}}(z_{n_\nu}) = q_S(z)/p_S(z) \quad \textit{and} \quad \lim_{\nu\to\infty} z_{n_\nu} = z$$

hold almost everywhere in D.

Proof. Let $U_n(z)$ be $S_n^{-1}S(z)$, then $U_n(z)$ tends uniformly to z in $\bar{D}$ and $U_n(z)$ is a quasiconformal mapping of D onto itself with maximal dilatation $K_{U_n} \leqq K_{S_n} K_S$. Since K_{S_n} tends to K_S when n tends to infinity, K_{S_n} is uniformly bounded. Thus, by lemmas 1 and 2, $p_{U_{n_\nu}}(z) \to 1$ and $q_{U_{n_\nu}}(z) \to 0$ hold almost everywhere in D for a suitable subsequence. On the other hand, we have

$$p_S(z) = p_{S_n}(z_n) p_{U_n}(z) + q_{S_n}(z_n)\overline{q_{U_n}(z)},$$
$$q_S(z) = p_{S_n}(z_n) q_{U_n}(z) + q_{S_n}(z_n)\overline{p_{U_n}(z)}$$

a. e. in D with $z_n = U_n(z)$. This can be easily proved as weak-derivatives of the respective mappings. Thus we have

$$h_S(z) = \frac{q_S(z)}{p_S(z)} = \frac{\overline{p_{U_n}(z)}(h_{S_{n_\nu}}(z_{n_\nu}) + q_{U_{n_\nu}}(z)/\overline{p_{U_{n_\nu}}(z)})}{p_{U_n}(z)(1 + h_{S_{n_\nu}}(z_n{}^\nu)\overline{q_{U_{n_\nu}}(z)}/p_{U_n}(z))},$$

with $h_{S_n}(z_n) = q_{S_n}(z_n)/p_{S_n}(z_n)$. By a simple calculation we have

$$|h_S(z) - h_{S_{n_\nu}}(z_{n_\nu})|$$
$$\leqq \left| h_{S_{n_\nu}}(z_{n_\nu}) - \frac{h_{S_{n_\nu}}(z_{n_\nu}) + A_{n_\nu}}{1 + h_{S_{n_\nu}}(z_{n_\nu})\bar{A}_{n_\nu}} \right| + \left| 1 - \frac{\bar{a}_{n_\nu}}{a_{n_\nu}} \right| \left| \frac{h_{S_{n_\nu}}(z_{n_\nu}) + A_{n_\nu}}{1 + h_{S_{n_\nu}}(z_{n_\nu})\bar{A}_{n_\nu}} \right|,$$

where $a_n = p_{U_n}(z)$, $A_{n_\nu} = q_{U_{n_\nu}}(z)/p_{U_{n_\nu}}(z)$. Let k_0 be a constant such that $|h_{S_n}(z_n)| \leqq k_0 < 1$ for any n and z_n almost everywhere in D. The existence of such k_0 is evident by the assumption and the first part in this paper. By lemmas 1 and 2, we have $|a_n - \bar{a}_n| \leqq 2\varepsilon$, $|A_n| \leqq \varepsilon$ and $1 - \varepsilon \leqq |a_n| \leqq 1 + \varepsilon$ for any sufficiently large n, whence follows that

$$|h_S(z) - h_{S_{n_\nu}}(z_{n_\nu})| \leqq \frac{2}{1-k_0}\left(|A_{n_\nu}| + \frac{|a_{n_\nu} - \bar{a}_{n_\nu}|}{|a_{n_\nu}|} \right) \leqq \frac{10\varepsilon}{1-k_0}$$

holds at almost all z and corresponding z_{n_ν} in D. Thus our result has been completely proved.

We shall give here a convergence theorem more convenient for our later purpose.

THEOREM 2. *Let D be a domain in the z-plane and $\{S_n(z)\}$ be a sequence of quasiconformal mappings of D onto itself such that $S_n(z)$ converges uniformly in D to a likewise mapping $S(z)$ of D onto itself and $S_n(z)$ is of* *constant excentricity or dilatation in any differentiable points. Then complex derivatives p_{S_n} and q_{S_n} converges strongly to p_S and q_S in L^2-sense, respectively, and hence their suitable subsequences converge almost everywhere in D.*

Proof. By the Green's formula, for any circular disc D_r: $|z-z_0|\leqq r<1$ contained in D, we have

$$\iint_{D_r}(|p_{S_n}-p_S|^2-|q_{S_n}-q_S|^2)\,dx\,dy=\frac{1}{2i}\int_{\partial D_r}(S_n-S)\,d(\bar{S}_n-\bar{S}).$$

Starting from this formula, we see by a similar method as in lemma 1 that

$$\lim_{n\to\infty}\iint_{D_r}(|p_{S_n}-p_S|^2-|q_{S_n}-q_S|^2)\,dx\,dy=0$$

for almost all r, and therefore

$$\lim_{n\to\infty}\left[\iint_{D_r}(|p_{S_n}|^2-|q_{S_n}|^2)\,dx\,dy-2\mathrm{Re}\iint_{D_r}(p_{S_n}\bar{p}_S-q_{S_n}\bar{q}_S)\,dx\,dy\right]$$
$$=-\iint_{D_r}(|p_S|^2-|q_S|^2)\,dx\,dy.$$

By the weak convergence of p_{S_n} and q_{S_n} to p_S and q_S, respectively, which are also easy to verify and actually similar to lemma 1, the second integral in the left hand converges to the right hand integral. Hence we have

$$\lim_{n\to\infty}\iint_{D_r}(|p_{S_n}|^2-|q_{S_n}|^2)\,dx\,dy=\iint_{D_r}(|p_S|^2-|q_S|^2)\,dx\,dy.$$

Let k_n and k be the excentricities of S_n and S, respectively, then k_n and k remain constant almost everywhere in D and k_n converges to k, by the uniform convergence of S_n to S in D. Thus we have

$$\lim_{n\to\infty}(1-k_n{}^2)\iint_{D_r}|p_{S_n}|^2\,dx\,dy=(1-k^2)\iint_{D_r}|p_S|^2\,dx\,dy$$

or

$$\lim_{n\to\infty}\iint_{D_r}|p_{S_n}|^2\,dx\,dy=\iint_{D_r}|p_S|^2\,dx\,dy.$$

Therefore, by the weak convergence of p_{S_n} to p_S, we have

$$\lim_{n\to\infty}\iint_{D_r}|p_{S_n}-p_S|^2\,dx\,dy=0$$

for almost all r. Therefore, the strong convergence of p_{S_n} to p_S has been proved. The strong convergence of q_{S_n} to q_S is verified similarly.

3. Preparation on symmetric homotopy.

Let $\mathfrak{W}$ be a hyperelliptic Riemann surface defined by an algebraic equation

$$w^2 = \prod_{l=0}^{2n-1} (z - e^{l\pi i/n}).$$

For our later discussions it will be sufficient to restrict here that n is a sufficiently large integer, although we can discuss the problem in case of an arbitrary genus $n-1$ under some needed modifications. Let $\mathfrak{G}$ be a group consisting of any directly conformal mappings of $\mathfrak{W}$ onto itself. Recently Tsuji [12] has shown in his theorem 1 that the order of $\mathfrak{G}$ is at most $8((n-1)+1) = 8n$. Let G, R and S be three directly conformal mappings of $\mathfrak{W}$ onto itself defined by birational transformations

$$\begin{pmatrix} z \to z \\ w \to -w \end{pmatrix}, \quad \begin{pmatrix} z \to z e^{\pi i/n} \\ w \to w \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} z \to 1/z \\ w \to w/z^n \end{pmatrix},$$

respectively. Evidently we have $G^2 = S^2 = R^{2n} = I$ (the identity mapping). By Tsuji's theorem $\mathfrak{G}$ is generated by G, R and S. Let J be an involutory indirectly conformal mapping of $\mathfrak{W}$ onto itself defined by an indirect birational transformation

$$\begin{pmatrix} z \to 1/\bar{z} \\ w \to \bar{w}/\bar{z}^n \end{pmatrix}.$$

Let T_* be a topological sense-preserving mapping of the schlicht unit disc $|z| \leqq 1$ onto itself with distinguished point condition $T_*(e^{l\pi i/n}) = e^{l\pi i/n}$, $l = 0, \cdots, 2n-1$. Then T_* can be extended to a topological sense-preserving mapping T of $\mathfrak{W}$ onto itself with two symmetry conditions (G): $TG(\mathfrak{p}) = GT(\mathfrak{p})$ and (J): $TJ(\mathfrak{p}) = JT(\mathfrak{p})$ for any point $\mathfrak{p}$ on $\mathfrak{W}$ and moreover with distinguished point condition (D): $T(\mathfrak{p}_l) = \mathfrak{p}_l$ for any integer l, where $\mathfrak{p}_l$ is the l-th ramification point $(e^{l\pi i/n}, 0)$ of $\mathfrak{W}$. Evidently T is homotopic to the identity mapping I on $\mathfrak{W}$ with the connecting mapping T_t which provides the continuous passage from T to I and satisfies two symmetry as well as distinguished point conditions.

Let $\{I\}$ be a homotopy class of topological mappings of $\mathfrak{W}$ onto itself which involves the identity mapping I, and $\{I\}_{GJ}$ a subclass of $\{I\}$ any member of which satisfies two symmetry conditions (G) and (J) and any two elements of which can be continuously connected by elements belonging to $\{I\}$. When two given elements T_1 and T_2 of $\{I\}_{GJ}$ can be continuously connected in $\{I\}_{GJ}$, then we say that T_1 and T_2 are mutually symmetrically homotopic. Ordinary homotopy relation is denoted by $T_1 \sim T_2$ and symmetric homotopy by $T_1 \overset{S}{\sim} T_2$. Mangler's theorem on homotopy and isotopy tells us that $\{I\}$ and $\{I\}_{GJ}$ can be considered as isotopy classes.

Suppose that $T \in \{I\}_{GJ}$, then any of $\mathfrak{W}$ lying on $|z| = 1$ corresponds to a likewise point on $|z| = 1$ by (J) and any other point does not correspond to a point on $|z| = 1$, since T is topological on $\mathfrak{W}$. By (G), any ramification point $\mathfrak{p}_l$ of $\mathfrak{W}$ corresponds to some ramification point $\mathfrak{p}_m$. Since m is uniquely determined by l, we denote this correspondence by $m(l)$. Then, since T is topological on $|z| = 1$, an intermediate circular arc $\widehat{\mathfrak{p}_l \mathfrak{p}_{l+1}}$ on $|z| = 1$ corresponds topologically to a likewise arc $\widehat{\mathfrak{p}_{m(l)}\mathfrak{p}_{m(l+1)}}$ in either sense-preserving or -reversing manner. Thus T induces a permutation $(m(0), \cdots, m(2n-1))$ of $(0, \cdots, 2n-1)$. Moreover T induces a topological sense-preserving mapping T_* of the z-sphere onto itself. The mapping degree or the orientation of T_* is homotopically or rather isotopically invariant, thus T_* carries the unit disc $|z| \leqq 1$ onto either itself or $|z| \geqq 1$, according as the orientation of $|z|=1$ has been preserved or not by T or equivalently by T_*, that is, $(m(0), \cdots, m(2n-1))$ is either a cyclic permutation of $(0, \cdots, 2n-1)$ or that of $(2n-1, \cdots, 0)$. Therefore there are only four possibilities: (1) T or (2) GT or (3) ST or (4) GST carry two-sheeted unit discs onto themselves without exchanging two sheets. If T induces a cyclic permutation $(m(0), \cdots, m(2n-1))$ of $(0, \cdots, 2n-1)$, then only two cases (1) and (2) can occur. When the case (1) occurs, then there exists an integer α such that $(R^\alpha T)_*$ induced by the identification process $R^\alpha T$ mod G has an associate permutation $(0, \cdots, 2n-1)$. $(R^\alpha T)_*$ restricted on $|z| \leqq 1$ is homotopic to I_* (I mod G) on $|z| \leqq 1$ and hence $R^\alpha T$ is symmetrically homotopic to I on $\mathfrak{W}$, that is, $R^\alpha T \overset{S}{\simeq} I$. On the other hand, $T \sim I$ by hypothesis, and hence $R^\alpha \sim I$. It is well known that, in any homotopy class of topological sense-preserving mappings of any closed Riemann surface onto itself or another one, there is at most one conformal mapping. Therefore α must reduce to zero mod $2n$. Thus $T \overset{S}{\simeq} I$ and T satisfies the distinguished point condition (D). When the case (2) occurs, then there is an integer α such that $(R^\alpha GT)_*$ induces a permutation $(0, \cdots, 2n-1)$. Thus $(R^\alpha GT)_* \sim I_*$ on $|z| \leqq 1$ so that $R^\alpha G \sim I$ on $\mathfrak{W}$, since $T \sim I$ on $\mathfrak{W}$ by hypothesis. Therefore $R^\alpha G = I$, that is, $R^\alpha = G$, which is absurd. Two other cases (3) and (4) also lead to contradictions in a quite similar manner as in the case (2). Therefore, we have

LEMMA 3. *If* $T \in \{I\}_{GJ}$, *then* T *induces a topological sense-preserving mapping* T_* *of* $|z| \leqq 1$ *onto itself with distinguished point condition* $T_*(e^{l\pi i/n}) = e^{l\pi i/n}$, $l=0, \cdots, 2n-1$. *Moreover,* $\{I\}_{GJ}$ *is symmetric homotopy or rather isotopy class, that is, any two elements* $T_1, T_2 \in \{I\}_{GJ}$ *are symmetrically homotopic*: $T_1 \overset{S}{\simeq} T_2$.

Let $\mathfrak{W}'$ be another hyperelliptic Riemann surface defined by an algebraic equation

$$w'^2 = \prod_{l=0}^{2n-1} (z' - e^{i\theta_l}), \qquad \theta_l < \theta_{l+1}.$$

We shall here denote the corresponding conformal and indirectly conformal mappings of $\mathfrak{W}'$ onto itself by G' and J' as G and J on $\mathfrak{W}$, respectively. Let T_1 and T_2 be two topological sense-preserving mappings of $\mathfrak{W}$ onto $\mathfrak{W}'$ which are homotopic mutually and satisfy two symmetry conditions $(G)'$: $G'T_j(\mathfrak{p}) = T_jG(\mathfrak{p})$ and $(J)'$: $J'T_j(\mathfrak{p}) = T_jJ(\mathfrak{p})$ for any $\mathfrak{p} \in \mathfrak{W}$. Then $T_2^{-1}T_1$ belongs to $\{I\}_{GJ}$ on $\mathfrak{W}$, since, by $G^2 = J^2 = I$, we have $JT_2^{-1}T_1 = T_2^{-1}T_1J$ and $GT_2^{-1}T_1 = T_2^{-1}T_1G$. Therefore, we have $T_1(\mathfrak{p}_l) = T_2(\mathfrak{p}_l)$ for any l, and moreover $T_2^{-1}T_1 \overset{S}{\simeq} I$ on $\mathfrak{W}$ and hence $T_1 \overset{S}{\simeq} T_2$, that is, there exists a family of mappings $T(\mathfrak{p}; t)$ providing continuous passage from T_1 to T_2 with two symmetry conditions $(D)'$ and $(J)'$. Therefore, a homotopy class with two symmetricities $(D)'$ and $(J)'$ gives rise to a fixed distinguished point condition and is a symmetric homotopy class. Induced distinguished point condition $\mathfrak{p}_l \to (e^{i\theta_m}, 0)$ can be denoted by a permutation $(m(0), \cdots, m(2n-1))$. This permutation is either a cyclic or reversely cyclic permutation of $(0, \cdots, 2n-1)$. Let $\mathfrak{H}$ be any given symmetric homotopy class of topological sense-preserving mappings of $\mathfrak{W}$ onto $\mathfrak{W}'$ and $(m(0), \cdots, m(2n-1))$ be induced permutation. If $(m(0), \cdots, m(2n-1))$ is a cyclic permutation of $(0, \cdots, 2n-1)$, then there is an integer α such that TR^α gives the permutation $(0, \cdots, 2n-1)$. Then, according as TR^s for any s brings the upper sheet of the unit disc $|z| \leqq 1$ onto either the upper sheet of the unit disc $|z'| \leqq 1$ or the lower one, we select either a homotopy class $\mathfrak{H}R^\alpha$ or $\mathfrak{H}R^\alpha G$. Evidently, TR^α in the former case and $TR^\alpha G$ in the latter case give a desired distinguished point condition $\mathfrak{p}_l \to (e^{i\theta_l}, 0)$ and carry the upper sheet of the unit disc $|z| \leqq 1$ of $\mathfrak{W}$ onto that of $|z'| \leqq 1$ of $\mathfrak{W}'$. If $(m(0), \cdots, m(2n-1))$ is an inversely cyclic permutation of $(0, \cdots, 2n-1)$, then, for a suitable integer m, $\mathfrak{H}R^\alpha S$ (or $\mathfrak{H}R^\alpha SG$) induces a fixed distinguished point condition $\mathfrak{p}_l \to (e^{i\theta_l}, 0)$ and any member of this class carries the upper sheet of the unit disc of $\mathfrak{W}$ onto that of the unit disc of $\mathfrak{W}'$.

THEOREM 3. *When we select a suitable homotopy class $\mathfrak{H}$ of topological sense-preserving symmetric mappings of $\mathfrak{W}$ onto $\mathfrak{W}'$, then $\mathfrak{H}$ is a symmetric homotopy class and any member of $\mathfrak{H}$ induces a topological sense-preserving mapping of $|z| \leqq 1$ onto $|z'| \leqq 1$ with a given distinguished point condition: $\mathfrak{p}_l \to e^{i\theta_l}$.*

In our considerations in theorem 3, we have made strong use of the analytic structures of the hyperelliptic surfaces. However, it can be perhaps deduced by a purely topological consideration. We cannot yet settle whether it is true or not in any arbitrary Riemann surface of higher topological and algebraic characters, though it seems intuitively evident.

Now we should mention here a remark on our motivation which obliges to establish our theorem. Let $\mathfrak{W}$ and $\mathfrak{W}'$ be two general Riemann surfaces of the same genus and $(\mathfrak{p}_j)_{j=1,\dots,2n}$ and $(\mathfrak{q}_j)_{j=1,\dots,2n}$ be two systems of distinguished points lying on $\mathfrak{W}$ and $\mathfrak{W}'$, respectively. Let H be a homotopy class of topological mappings of $\mathfrak{W}$ onto $\mathfrak{W}'$ with distinguished point condition $\mathfrak{p}_j \to \mathfrak{q}_j$. Teichmüller and Ahlfors reduced a problem to seek an extremal quasiconformal mapping in H to a corresponding problem without distinguished point condition in such a manner that the original surfaces are replaced by their two-sheeted coverings with ramification points $(\mathfrak{p}_j)$ and $(\mathfrak{q}_j)$ and H is replaced by a suitable homotopy class of topological mappings between two-sheeted coverings. Although this process brings many simplifications of the matter, an ambiguity happens. In fact, in the inverse process the prescribed distinguished point condition $\mathfrak{p}_j \to \mathfrak{q}_j$ would be destroyed and the condition has simply a meaning as a total system, that is, a $\mathfrak{p}_j$ corresponds to some $\mathfrak{q}_l$. A cause for this ambiguity would be a rough enumeration of symmetric homotopy types, though to perform perfectly this enumeration would be very difficult. Our theorem 3 shows that in our hyperelliptic case there is no ambiguity with respect to the distinguished point condition and corresponding symmetric homotopy class. In fact, we have made a sufficiently precise enumeration of symmetric homotopy classes in the proof of theorem 3.

4. Differential equation of Beltrami type gratified by an extremal quasiconformal mapping.

Let T be a quasiconformal mapping of $|z|<1$ onto $|z'|<1$, then by Ahlfors' or Mori's theorem T can be extended to a topological mapping of $|z|\leqq 1$ onto $|z'|\leqq 1$. Let $e^{i\theta_l}$ be the image $T(\exp(il\pi/2^{n-1}))$. Let $\mathfrak{F}\{T\}_n$ be a homotopy class of topological mappings of $|z|\leqq 1$ onto $|z'|\leqq 1$ with distinguished point condition $(D)_n$: $\exp(il\pi/2^{n-1})\to\exp(i\theta_l(n))$ for any integer l, $0\leqq l\leqq 2^{n-1}$. Then by theorem 3 there is a suitable symmetric homotopy class $\mathfrak{H}_n$ of topological mappings of $\mathfrak{W}_n$ onto $\mathfrak{W}_n'$, where $\mathfrak{W}_n$ and $\mathfrak{W}_n'$ are two hyperelliptic Riemann surfaces defined by algebraic equations

$$w^2=\prod_{l=0}^{2^n-1}(z-e^{i\pi l/2^{n-1}}) \quad \text{and} \quad w'^2=\prod_{l=0}^{2^n-1}(z'-e^{i\theta_l(n)}),$$

respectively, and any member of $\mathfrak{H}_n$ induces a topological mapping of $|z|\leqq 1$ onto $|z'|\leqq 1$ which gives rise to the given distinguished point condition $(D)_n$. Again the integer n is supposed to be sufficiently large. Then by Teichmüller-Ahlfors' theorem we see that in $\mathfrak{F}\{T\}_n$ there exists a unique extremal quasiconformal mapping E_n: $z'=\zeta_n(z)$ or conformal mapping. When the former case occurs, then E_n satisfies a differential equation of Beltrami type

$$\frac{q_n}{p_n}=k_n\frac{\bar{f}_n}{|f_n|},\qquad q_n=\frac{\partial\zeta_n}{\partial\bar{z}},\qquad p_n=\frac{\partial\zeta_n}{\partial z}$$

except at most 2^{n+1} points in $|z|<1$, where k_n with $0<k_n<1$ is a constant excentricity of E_n and f_n is a regular function in $|z|<1$. Certainly $f_n(z)dz^2$ is a regular quadratic differential on $\mathfrak{W}_n$.

Some parts in the following discussions have been once suggested to the present author by Mr. Z. Yanagihara. Evidently $\mathfrak{F}\{T\}_n$ contains $\mathfrak{F}\{T\}_{n+1}$, and therefore $k_n\leqq k_{n+1}$ remains valid. Thus $\lim k_n=k$ exists and k is not equal to 1, since a family $\mathfrak{F}\{T\}$ of quasiconformal mappings of $|z|\leqq 1$ onto $|z'|\leqq 1$ with a fixed boundary correspondence $T(e^{i\varphi})=e^{i\theta}$ is not empty and actually contains an element T. Let $\mathfrak{F}\{T\}_n^{K_T}$ be a subfamily of $\mathfrak{F}\{T\}_n$ consisting of members with the maximal dilatation less than K_T, then the family $\mathfrak{F}\{T\}_n^{K_T}$ forms a compact family by its distinguished boundary point condition. To that end, we first prove that $|S(0)|\leqq\rho<1$ holds for any $S\in\mathfrak{F}\{T\}_n^{K_T}$, where ρ depends only on K. Suppose contrarily that there is a sequence

$S_m(0)$ in $\mathfrak{F}\{T\}_n^{K_T}$ such that $\lim_{m\to\infty} S_m(0) = e^{i\theta_0}$. Let $LS_m(z)$ be a quasiconformal mapping defined by

$$\frac{S_m(z) - S_m(0)}{1 - \overline{S_m(0)} S_m(z)},$$

then $LS_m(0) = 0$ and $K_{LS_m} \equiv$ max dilatation of $LS_m(z) = K_{S_m} \leqq K_T$. Therefore, we have by Mori's theorem

$$\left(\frac{|z_1 - z_2|}{16}\right)^{K_T} \leqq |LS_m(z_1) - LS_m(z_2)|$$

for any $z_1, z_2 \in \{|z| \leqq 1\}$. Let m tend to ∞, then $\lim_{m\to\infty} S_m(0) = e^{i\theta_0}$ and hence $\lim_{m\to\infty} LS_m(z_j) = -e^{i\theta_0}$. Hence $|z_1 - z_2|$ must be zero, what is absurd by the distinguished point condition. Thus there exists a number ρ (<1) such that, for any $S \in \mathfrak{F}\{T\}_n^{K_T}$, $|S(0)| \leqq \rho$ holds. Again by Mori's theorem, we have

$$|LS(z_1) - LS(z_2)| \leqq 16 |z_1 - z_2|^{1/K_T},$$

and hence

$$|S(z_1) - S(z_2)| \leqq \frac{64 |z_1 - z_2|^{1/K_T}}{1 - \rho^2}$$

for any $S \in \mathfrak{F}\{T\}_n^{K_T}$ and any $z_1, z_2 \in \{|z| \leqq 1\}$. This implies that the family $\mathfrak{F}(T)_n^{K_T}$ is equicontinuous and hence compact. Evidently E_m belongs to the class $\mathfrak{F}\{T\}_n^{K_T}$ for any $m \geqq n$. Thus we can select a subsequence $\{E_{m_\nu}\}$ of $\{E_m\}$ such that E_{m_ν} converges uniformly on $|z| \leqq 1$. Then the limit mapping E belongs to any $\mathfrak{F}\{T\}_n^{K_T}$ and hence E is a quasiconformal mapping of $|z| \leqq 1$ onto $|z'| \leqq 1$ with constant dilatation $(1+k)/(1-k)$ and with distinguished point condition $E(\exp(il\pi/2^{n-1})) = \exp(i\theta_l(n))$ for any n and l, which implies that $E(e^{i\varphi}) = e^{i\theta} = T(e^{i\varphi})$. Now we shall make use of our theorem 2 (or directly theorem 1). Then we can select subsequences $\{q_{m_{\nu_\mu}}\}$ and $\{p_{m_{\nu_\mu}}\}$ of $\{q_{m_\nu}\}$ and $\{p_{m_\nu}\}$, respectively, which converge to q and p of E almost everywhere in the unit disc $|z| < 1$. Therefore

$$q_{m_{\nu_\mu}} / p_{m_{\nu_\mu}} \to q/p$$

holds almost everywhere in $|z| < 1$, since the measure of a set at which the Jacobian is equal to zero is zero. Now we shall prove that the limit quasiconformal mapping E is also extremal in $\mathfrak{F}\{T\}$. Let $U(z)$ be an arbitrary element in $\mathfrak{F}\{T\}$, then $U(z)$ also belongs to $\mathfrak{F}\{T\}_n$. Therefore, $k_n \leqq$ maximal excentricity k_U of U remains valid, whence $k \leqq k_U$. Thus we have the following theorem 4.

THEOREM 4. *In $\mathfrak{F}\{T\}$, there exists either a conformal mapping or an extremal quasiconformal mapping E whose dilatation is constant. Moreover in the latter case E satisfies a differential equation*

$$\frac{q}{p} = k \lim_{n\to\infty} \frac{\bar{f}_n}{|f_n|}$$

almost everywhere in $|z|<1$, where k is a constant excentricity of E such that $0<k<1$ and $\{f_n(z)\}$ is a sequence of suitable regular functions of z in $|z|<1$, which can be extended to a hyperelliptic Riemann surface $\mathfrak{W}_n$ as a regular quadratic differential $f_n(z)dz^2$.

In the above formulation of our theorem 4, the properties of $\lim f_n$ are not clear and hence we shall study the behaviors of $\{f_n\}$ in the sequel.

LEMMA 4. *Let $\{f(z)\}$ be a family of regular functions in $|z|<1$ satisfying a normalization condition*

$$\iint_{|z|<1} |f(z)|\,dx\,dy = 1,$$

then $\{f(z)\}$ forms a normal family.

Proof. Let $D_\rho(z_0)$ be a disc $|z-z_0|\leqq\rho$ contained in $|z|<1$. Then we have

$$f(z_0+re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0+\rho e^{i\varphi})\frac{\rho^2-r^2}{\rho^2-2\rho r\cos(\theta-\varphi)+r^2}d\varphi,$$

from which we can conclude that, with $r=|z-z_0|$,

$$|f(z)-f(z_0)| \leqq \frac{r}{\pi}\int_0^{2\pi}|f(z_0+\rho e^{i\varphi})|\frac{|r-\rho\cos(\theta-\varphi)|}{|\rho^2-2\rho r\cos(\theta-\varphi)+r^2|}d\varphi$$
$$\leqq \frac{r}{\pi}\frac{\rho+r}{(\rho-r)^2}\int_0^{2\pi}|f(z_0+\rho e^{i\varphi})|\,d\varphi.$$

Multiplying this by ρ and successively integrating with respect to ρ from ρ_1 to $\rho_2\,(>\rho_1)$, for which $|z-z_0|\leqq\rho_2$ is contained in $|z|<1$, we have

$$\frac{1}{2}|f(z)-f(z_0)|(\rho_2{}^2-\rho_1{}^2) \leqq \int_{\rho_1}^{\rho_2}\frac{\rho+r}{(\rho-r)^2}\int_0^{2\pi}|f(z_0+\rho e^{i\varphi})|\,d\varphi\rho d\rho$$
$$\leqq \frac{r}{\pi}\frac{\rho_1+r}{(\rho_1-r)^2}\int_0^1\int_0^{2\pi}|f(z_0+\rho e^{i\theta})|\rho d\rho d\theta,$$

since $(\rho_1+r)/(\rho_1-r)^2>(\rho_2+r)/(\rho_2-r)^2$ holds, if $r<\rho_1<\rho_2$. Therefore we obtain

$$|f(z)-f(z_0)| \leqq \frac{2}{\pi}|z-z_0|\frac{\rho_1+r}{(\rho_1-r)^2}\frac{1}{\rho_2{}^2-\rho_1{}^2}.$$

Let z and z_0 belong to a disc $|z|<R<1$, then we can put $r=(1-R)/3$, $\rho_1=2(1-R)/3$ and $\rho_2=1-R$. Then we have

$$|f(z)-f(z_0)| \leqq \frac{162}{5\pi}\frac{|z-z_0|}{(1-R)^3}.$$

This implies that $\{f\}$ forms an equicontinuous family on any compact subdomain in $|z|<1$. Thus $\{f\}$ is a normal family in $|z|<1$. For the family $\{f\}$ we can easily see by its normalization that any subsequence extracted from $\{f\}$ does not diverge uniformly to the constant infinity.

Now we shall return to our original intention. In theorem 4, we can evidently impose a normalization condition

$$\iint_{|z|<1} |f_n(z)|\,dx\,dy = 1.$$

Then, by lemma 4, we can extract a uniformly convergent subsequence in any compact subdomain $|z|<1$. For simplicity's sake, we shall retain the original indices. Then two possibilities can occur: either (1) limit function $f(z)=\lim_{n\to\infty} f_n(z)$ is not the constant zero or (2) $f(z)$ reduces to the constant zero.

When the case (1) occurs, then the corresponding limit mapping E satisfies a differential equation

$$\frac{q}{p} = k\frac{\bar{f}}{|f|}$$

almost everywhere in $|z|<1$. In this case, if moreover f is not constant, f must have at least one singularity on $|z|=1$. In fact, a quadratic differential $d\zeta_n(\mathfrak{p})^2 = f_n(z)\,dz^2$ on $\mathfrak{W}_n$ can be considered as the one on the whole Riemann sphere in view of its symmetry condition $d\zeta_n(\mathfrak{p})^2 = d\zeta_n(G\mathfrak{p})^2$ and moreover it satisfies a symmetry condition $d\zeta_n(\mathfrak{p})^2 = \overline{d\zeta_n(J\mathfrak{p})^2}$, where J should be considered as the reflection with respect to $|z|=1$. If f is regular on $|z|=1$, that is, f is analytically continuable beyond $|z|=1$, then f must reduce to a non-zero constant, since it is regular on the whole Riemann sphere and is not constant zero in $|z|<1$. But, this is absurd. Moreover, we have the following fact. When f or precisely its analytic continuation by reflection is regular on the whole unit circumference with exception of a finite number of simple poles lying there, the extremal mapping E is unique. To that end, we shall consider an extremal problem seeking the extremal quasiconformal mapping in a family $\mathfrak{F}\{f\}$ any member of which is a topological mapping of $|z|\leqq 1$ onto $|w|\leqq 1$ with distinguished point condition $S(\mathfrak{p}_j)=T(\mathfrak{p}_j)=E(\mathfrak{p}_j)$, where all the $\mathfrak{p}_j$ are simple poles of $f(z)$ lying on $|z|=1$. In this case an analogue of theorem 3 also remains valid, although some complicated factors should be modified. Then the formal solution in the sense of Ahlfors is also the unique extremal quasiconformal mapping in $\mathfrak{F}\{f\}$ by Teichmüller-Ahlfors' theorem in the form formulated by Ahlfors [1]. On the other hand, E has a form of a formal solution, since f is analytically continuable onto the whole Riemann sphere and onto its two-sheeted covering. Therefore, E must coincide with this extremal quasiconformal mapping and hence it must be unique. Now we shall exclude the cases explained above. Then $f(z)$ has at least one essential singularity. However, $f(z)$ may be bounded on $|z|<1$. An example for this is the extremal quasiconformal mapping of a universal covering surface $|z|<1$ of a given closed Riemann surface $\mathfrak{W}$ onto another universal covering surface $|w|<1$ of another closed surface $\mathfrak{W}'$ of the same genus $g>1$ in some homotopy class α, that is, the extremal mapping of $\mathfrak{W}$ onto $\mathfrak{W}'$ in the corresponding homotopy class H_α formulated in terms of uniformization. Its extremality with respect to a fixed boundary correspondence has been proved in our previous paper [8].

When the case (2) occurs, then the situation is still ambiguous as in theorem 4. By theorem 4, $e^{i\theta_n(z)}$ converges almost everywhere in $|z|<1$, where $\theta_n(z)=\arg f_n(z)$ is a suitable continuous branch, that is, $\theta_n(z)$ is a function harmonic in $|z|<1$. However, it is not sure whether $\lim_{n\to\infty}\theta_n(z)$ does exist or not. When we choose $\arg f_n(z)$ as in a manner such that $0\leqq\Theta_n(z)<2\pi$, $\Theta_n(z)=\arg f_n(z)$, then the limit of $\Theta_n(z)$ exists, or more precisely $\theta_n(z)$ cannot move ergodically.

5. Remarks and open problems.

Now we should point out here that, between Teichmüller-Ahlfors' theorem in the case of compact Riemann surface and our theorem 4 in the case of the unit disc with fixed boundary correspondence which gives at least one quasiconformal mapping, there is an essential difference. According to Bers' equivalence proof [2] of various sorts of definitions of quasiconformality, E satisfies evidently a sort of Beltrami equation. However, our theorem 4 states more explicit fact. In the case of closed Riemann surface, the corresponding differential equation is the most explicit one, and there is no ambiguity in the theoretical view-point. In this case, that a linear vector space of quadratic differentials on a given closed surface is of finite dimension plays an essential role for its non-degeneracy of the final form. However, in our case $\{f(z)\}$ any member of which is regular in $|z|<1$ and satisfies a normalization

$$\iint_{|z|>1}|f(z)|\,dx\,dy=1$$

is of infinite dimension, therefore degeneration or the case (2) may occur in the most general case. An analogous method as carried by Ahlfors [1] also meets the same difficulty. Now we shall list here two open problems.

(1) Does the degeneration occur inevitably? This problem leads to a classification problem of continuous boundary correspondences, if it is affirmative. A perfect criterion of Beurling-Ahlfors [3] in order to exist a quasiconformal mapping having a given continuous boundary correspondence is the most elegant one in the classification problem of continuous boundary correspondences and their example is the decisive one in the first step. Our problem should be considered as a subclassification problem.

(2) Does the uniqueness of extremal quasiconformal mapping hold? If this is affirmative, then there are some important applications in the global theory of quasiconformal mappings. In this problem the form of final differential equation is out of question, although it is desirable.

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