REMARKS ON HOLOMORPHIC AUTOMORPHISMS OF A SIMPLY-CONNECTED NORMAL DOMAIN IN SEVERAL COMPLEX VARIABLES

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1. In the theory of several complex variables, a domain is called rigid ("starr") if it admits no holomorphic automorphism other than the identity; see Behnke and Thullen [1]. Holomorphic automorphisms and rigidity of a simply-connected normal domain have been studied by several authors. Though some examples of rigid domains were expicitly constructed for the first time by Cartan and Thullen [4], they are not domains of holomorphy. Behnke and Peschl [2] have then succeeded to construct domains of holomorphy which admit only the identical holomorphic automorphism by means of Carathéodory's metric and Lindelöf's inequality. Rothstein [7] has also shown important properties of holomorphic automorphisms of a normal domain in the case of two variables, but his results do not completely characterize the properties of holomorphic automorphisms. Recently, Hedtfeld [5] has given sufficient conditions for rigidity of simply-connected normal domains by means of Rothstein's results together with analytic projection.

In this paper, we shall give a complete characterization of all holomorphic automorphisms of a simply-connected normal domain and then establish a necessary and sufficient condition in order that a simply-connected normal domain is to be rigid. By the same method, we can solve the problem how to determine all holomorphic homeomorphisms of a simply-connected normal domain onto another.

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2. We first define a normal domain in the space of n complex variables, which has been defined by Rothstein [7] and Hedtfeld [5] in case of two variables.

DEFINITION 1. *A domain $D \subset C^n$ is said normal when it satisfies the following conditions:*

1) *D is bounded.*

2) *The boundary of D consists of a finite number of smooth hypersurfaces $\Phi_i(x_1, y_1, \cdots, x_n, y_n) = 0$, $z_j = x_j + iy_j$, $i = 1, \cdots, p$; $j = 1, \cdots, n$; $p \geqq n$.*

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3) *Each* $\Phi_i = 0$ *is defined by analytic hypersurface* $f_i(z_1, \cdots, z_n) = \gamma_i(t)$ *with* $A \leqq t \leqq B$, $\gamma_i(t)$ *being continuous. All* $f_i(z_1, \cdots, z_n)$ *are holomorphic in* $\bar{D}$, *the closure of* D.

4) *For every* $f_{i_1}(z_1, \cdots, z_n)$ *with* $1 \leqq i_1 \leqq p$, *there exist at least* $n-1$ *holomorphic functions* $f_{i_2}, \cdots, f_{i_n}$ *with* $1 \leqq i_j \leqq p$, $j = 2, \cdots, n$, *such that*

$$\frac{\partial(f_{i_1}, \cdots, f_{i_n})}{\partial(z_1, \cdots, z_n)} \not\equiv 0 \quad in \ \bar{D}.$$

By a work of Sommer [8], we can characterize any normal domain as a domain whose each boundary component $\Phi_i(x_1, y_1, \cdots, x_n, y_n) = 0$ satisfies $L(\Phi_i) = 0$, $L(\Phi_i)$ being the Levi's symbol, and is locally represented by $f_i(z_1, \cdots, z_n) = \gamma_i(t)$ where f_i is holomorphic in $\bar{D}$.

Now, we consider a function $f(z_1, \cdots, z_n)$ holomorphic in $\bar{D}$ and an analytic projection defined by it. The notion of analytic projection has been given precisely by Koch in his Dissertation [6].

Definition 2. *Analytic projection, or holomorphic basis,* $(\Re, \varphi)$ *defined by a holomorphic map f of D into an analytic space X is a collection of an analytic space* $\Re$ *and a holomorphic map* φ *of D onto* $\Re$ *such that for any holomorphic map g which is holomorphically dependent ("holomorph abhängig") in the sense of Stein* [9, 10, 11] *on f, there exists a holomorphic map* $\chi(p)$, $p \in \Re$, *satisfying* $g(x) = \chi(\varphi(x))$, $x \in D$. *We call* φ *a projection map of D onto* $\Re$.

We say that four points P_1, P_2, Q_1, Q_2 in D satisfy the relation:

$$(1) \qquad A(P_1, P_2; Q_1, Q_2)/(\Re, \varphi)$$

if the following conditions hold:

1) There exist two Jordan curves $C_1(t)$ and $C_2(t)$, $0 \leqq t \leqq 1$, connecting P_1, Q_1 and P_2, Q_2, respectively, and $C_1(0) = P_1$, $C_1(1) = Q_1$, $C_2(0) = P_2$, $C_2(1) = Q_2$.

2) All points on $C_1(t)$ and $C_2(t)$ are ordinary points of φ except their end points Q_1, Q_2.

3) For all t, $0 \leqq t \leqq 1$, $\varphi[C_1(t)] = \varphi[C_2(t)]$.

In the construction of a holomorphic basis, the equivalent classes of points in D (see Koch [6], Stein [9, 10, 11]) have been considered and it is known that two points Q_1 and Q_2 are equivalent, if P_1 and P_2 are equivalent and $A(P_1, P_2; Q_1, Q_2)/(\Re, \varphi)$ holds.

Now, we consider analytic projections $(\Re_i, \varphi_i)$ defined by $f_i(z_1, \cdots, z_n)$ $(i=1, \cdots, p)$, where f_i are given in 3) of Definition 1, and a holomorphic automorphism F of a simply-connected normal domain D. We then have the following Rothstein's theorem.

Lemma 1. (Rothstein [7]) *Let F be a holomorphic automorphism of* *a simply-connected normal domain D. Then there exist holomorphic maps* $h_{i\tau(i)}$ *of* $\mathfrak{R}_i$ *onto* $\mathfrak{R}_{\tau(i)}$ *such that*

$$f_{\tau(i)}(z_1, \cdots, z_n) = h_{i\tau(i)}(f_i(z_1, \cdots, z_n)).$$

For a precise characterization of all holomorphic automorphisms of a simply-connected normal domain, we consider a holomorphic map $\Phi_{1\cdots n} = (\varphi_1, \cdots, \varphi_n)$ of D into a product space of Riemann surfaces $\mathfrak{R}_1, \cdots, \mathfrak{R}_n$ defined by projection maps $\varphi_1, \cdots, \varphi_n$, and we denote the image of $\Phi_{1\cdots n}$ by $M_{1\cdots n} = \Phi_{1\cdots n}(D)$.

We now define a notion of an admissible system of n holomorphic functions.

DEFINITION 3. *A system of n functions* $f_1(z_1, \cdots, z_n), \cdots, f_n(z_1, \cdots, z_n)$ *holomorphic in* $\bar{D}$ *is said to be admissible when*

$$J(f_1(z_1, \cdots, z_n), \cdots, f_n(z_1, \cdots, z_n)) = \frac{\partial(f_1, \cdots, f_n)}{\partial(z_1, \cdots, z_n)} \not\equiv 0 \ \ in \ \bar{D}.$$

If $(j_1, \cdots, j_n)$ *is a permutation of* $(i_1, \cdots, i_n)$, *then* $(f_{j_1}, \cdots, f_{j_n})$ *and* $(f_{i_1}, \cdots, f_{i_n})$ *are regarded as the same admissible system.*

We consider for all admissible systems taken from $(f_1, \cdots, f_p)$ the maps $\Phi_{i_1\cdots i_n}$ defined by $\varphi_{i_1}, \cdots, \varphi_{i_n}$. We have a set of all $M_{i_1\cdots i_n}$ and the number of $M_{i_1\cdots i_n}$ is at most $\binom{p}{n}$. If F is a holomorphic automorphism of D, then by Lemma 1 F induces a holomorphic automorphism $h_{1\tau(1)} \times \cdots \times h_{p\tau(p)}$ of $\mathfrak{R}_1 \times \cdots \times \mathfrak{R}_p$ onto itself such that

$$f_{j_k} = h_{i_l\tau(i_l)}(f_{i_l}) \qquad (j_k = \tau(i_l)).$$

Hence, if $(f_{i_1}(z_1, \cdots, z_n), \cdots, f_{i_n}(z_1, \cdots, z_n))$ is an admissible system, then $(f_{j_1}(z_1, \cdots, z_n), \cdots, f_{j_n}(z_1, \cdots, z_n))$ is also admissible, and

$$M_{j_1\cdots j_n} = h_{i\tau(i)}(M_{i_1\cdots i_n}), \qquad \begin{cases} M_{i_1\cdots i_n} = \Phi_{i_1\cdots i_n}(D), \\ M_{j_1\cdots j_n} = \Phi_{j_1\cdots j_n}(D). \end{cases}$$

Thus we have:

LEMMA 2. $\{M_{i_1\cdots i_n}\}$ *is a set of* $M_{i_1\cdots i_n}$ *defined by admissible systms. Let F be a holomorphic automorphism of D, then F induces a holomorphic automorphism* $\{h_{i\tau(i)}\}$ *of the product space* $\mathfrak{R}_1 \times \cdots \times \mathfrak{R}_p$ *such that each* $h_{i_1\tau(i_1)} \times \cdots \times h_{i_n\tau(i_n)}$ *is a holomorphic map of* $M_{i_1\cdots i_n}$ *onto* $M_{j_1\cdots j_n}$.

For a characterization of holomorphic automorphisms of a simply-connected normal domain D, the following Lemma 3 is essential.

LEMMA 3. *Let* $\Phi_{i_1\cdots i_n} = (\varphi_{i_1}, \cdots, \varphi_{i_n})$ *be a holomorphic map of D into* $\mathfrak{R}_{i_1} \times \cdots \times \mathfrak{R}_{i_n}$, *then* $(M_{i_1\cdots i_n}, \Phi_{i_1\cdots i_n})$ *is a holomorphic basis for the map* $\Phi_{i_1\cdots i_n}$.

Proof. We first remark that, if H is a holomorphic automorphism of

$\mathfrak{R}_1 \times \cdots \times \mathfrak{R}_p$ such that

$$H(M_{i_1\cdots i_n}) = M_{j_1\cdots j_n}$$

holds for admissible systems $(f_{i_1}, \cdots, f_{i_n})$ and $(f_{j_1}, \cdots, f_{j_n})$, then H induces a locally one-to-one holomorphic map of $D - \{z;\ J(f_{i_1}, \cdots, f_{i_n}) = 0,\ J(f_{j_1}, \cdots, f_{j_n}) = 0\}$ onto itself.

The global rank of $\Phi_{i_1\cdots i_n}$ is n, and therefore the existence of a holomorphic basis has been proved by Stein [10]. Let Φ be a holomorphic map of D onto an analytic space X and (X, Φ) be a holomorphic basis for the map $\Phi_{i_1\cdots i_n}$. Then $\Phi_{i_1\cdots i_n}$ and Φ are holomorphically dependent on each other, that is,

$$\text{Rank}\,(\Phi_{i_1\cdots i_n}) = \text{Rank}\,(\Phi, \Phi_{i_1\cdots i_n}) = \text{Rank}\,(\Phi).$$

On the other hand, from Definition 2 of a holomorphic basis, there exists a holomorphic map χ of X onto $M_{i_1\cdots i_n}$ such that

$$\Phi_{i_1\cdots i_n} = \chi \circ \Phi.$$

Therefore

$$\text{Rank}\,\chi = n$$

at every point (z) with $\text{Rank}\,(\Phi_{i_1\cdots i_n}(z)) = n$. Now $f_{i_1}\cdots f_{i_n}$ are holomorphic in $\bar{D}$, therefore the fibres of $\Phi_{i_1\cdots i_n}$ over a point p of $M_{i_1\cdots i_n}$ consist of only a finite number of analytic sets. This shows that the set $\chi^{-1}(p)$ consists of a finite number of points in X and χ is a proper map. By a theorem on holomorphic maps (see Cartan [4], Theorem 1) χ is locally homeomorphic except the set

$$\{p;\ p \in \Phi_{i_1\cdots i_n}[J(\Phi_{i_1\cdots i_n}(z)) = 0]\}.$$

Hence we have local holomorphic inverses $\chi^{-1}_{U(p)}$ of χ in a neighborhood $U(p)$ of $p \in M_{i_1\cdots i_n}$. The aggregate of all $\chi^{-1}_{U(p)}$ with $p \in M_{i_1\cdots i_n} - [\Phi_{i_1\cdots i_n}\{J(\Phi_{i_1\cdots i_n}) = 0\}]$ defines a many-valued holomorphic map of $M_{i_1\cdots i_n} - [\Phi_{i_1\cdots i_n}\{J(\Phi_{i_1\cdots i_n}) = 0\}]$ onto $X - [\Phi\{J(\Phi) = 0\}]$.

Now we prove that $M_{i_1\cdots i_n}$ is simply-connected. Consider any closed curve C on $M_{i_1\cdots i_n}$. We may suppose that C consists of the image of ordinary points of the holomorphic map $\Phi_{i_1\cdots i_n}$, since the dimension of singularities of $M_{i_1\cdots i_n}$ is at most $2n-3$. Then the inverse image of C by $\Phi_{i_1\cdots i_n}$ consists of a finite number of curves $C_1, \cdots, C_m$. Two end points (z^1) and (z^2) of C_1 are equivalent with respect to the functions $f_{i_1}(z_1, \cdots, z_n), \cdots, f_{i_n}(z_1, \cdots, z_n)$, i.e.,

$$f_{i_1}(z^1) = f_{i_1}(z^2),\ \cdots,\ f_{i_n}(z^1) = f_{i_n}(z^2)$$

and there exist n points $(z^{i_1}), \cdots, (z^{i_n})$ in D and $2n$ curves $C_{i_1}, \cdots, C_{i_n}, C_{i_1}', \cdots, C_{i_n}'$ such that C_{i_k} and C_{i_k}' $(k = 1, \cdots, n)$ satisfy the relation (1) of a holomorphic basis. Therefore we have a closed curve $C_{i_1}^{-1}C_1C_{i_1}'$, which is retractible to a point in D. Now we consider the image of $C_{i_1}^{-1}C_1C_{i_1}'$ by $\Phi_{i_1\cdots i_n}$. It is retractible to a point by a deformation $F(p, t)$ $(0 \leqq t \leqq 1)$. The restriction of $F(p, t)$ on C is a deformation of C. Thus $M_{i_1\cdots i_n}$ is simply-connected.

Therefore the aggregate $\{\chi_{U(p)}^{-1}\}$ defines a one-valued holomorphic function χ^{-1} on $M_{i_1\cdots i_n}-[\Phi_{i_1\cdots i_n}\{J(\Phi_{i_1\cdots i_n})=0\}]$ with its value in X.

Now we prove that χ^{-1} is a continuous map of $M_{i_1\cdots i_n}$ onto X. For this it is sufficient to prove that χ^{-1} is continuous on $\{\Phi_{i_1\cdots i_n}(J(\Phi_{i_1\cdots i_n})=0)\}$. Suppose contrarily that χ^{-1} is discontinuous at a point $p_0\in\{\Phi_{i_1\cdots i_n}(J(\Phi_{i_1\cdots i_n})=0)\}$. Then there exist two distinct sequences of points $\{p_r\}$ and $\{p_s\}$ both converging to p_0 such that

$$x=\lim(\chi^{-1}(p_r))\neq\lim(\chi^{-1}(p_s))=\tilde{x},\qquad x,\ \tilde{x}\in X.$$

χ is holomorphic at x and $\tilde{x}$, we get

$$\chi(x)=\chi(\lim\chi^{-1}(p_r))=\chi(\lim\chi^{-1}(p_s))=\chi(\tilde{x})=p_0.$$

Therefore for disjoint neighborhoods $U(x)$ and $U(\tilde{x})$, there exist two neighborhoods $U_1(p_0)$ and $U_2(p_0)$ such that

$$\chi_{U_1(p_0)}^{-1}(U_1(p_0)-[\Phi_{i_1\cdots i_n}\{J(\Phi_{i_1\cdots i_n})=0\}])\subset U(x),$$

$$\chi_{U_2(p_0)}^{-1}(U_2(p_0)-[\Phi_{i_1\cdots i_n}\{J(\Phi_{i_1\cdots i_n})=0\}])\subset U(\tilde{x}).$$

In the intersection $U_1(p_0)-[\Phi_{i_1\cdots i_n}\{J(\Phi_{i_1\cdots i_n})=0\}]$ and $U_2(p_0)-[\Phi_{i_1\cdots i_n}\{J(\Phi_{i_1\cdots i_n})=0\}]$ there exists at least a point p for which χ^{-1} is one-valued in a neighborhood of p. On the other hand, we have

$$\chi_{U_1(p_0)}^{-1}(p)\in U(x),\qquad \chi_{U_2(p_0)}^{-1}(p)\in U(\tilde{x})$$

This contradicts the one-valuedness of χ^{-1} proved above. Thus we have a continuous map of $M_{i_1\cdots i_n}$ onto X. By a theorem of removable singularities, we have a holomorphic map χ^{-1} of $M_{i_1\cdots i_n}$ onto X such that

$$\Phi=\Phi_{i_1\cdots i_n}\circ\chi^{-1}.$$

Thus (X,Φ) and $(M_{i_1\cdots i_n},\Phi_{i_1\cdots i_n})$ are equivalent and our Lemma 3 has been proved.

3. Now we can state the following

THEOREM 1. *Let D be a simply-connected normal domain. Then for each holomorphic automorphism of D there exists a holomorphic automorphism of $\{M_{i_1\cdots i_n}\}$. Conversely, any holomorphic automorphism H of $\mathfrak{R}_1\times\cdots\times\mathfrak{R}_p$:*

$$w_j=h_{i\tau(i)}(w_i),\quad w_i\in\mathfrak{R}_i,\quad w_j\in\mathfrak{R}_j,$$

which is also a holomorphic homeomorphism of $\{M_{i_1\cdots i_n}\}$, induces a holomorphic automorphism of D.

Proof. The first assertion is the same as Lemma 2. Therefore we prove the second. We first prove that a holomorphic map $\Phi_{i_1\cdots i_n}$: $D\to M_{i_1\cdots i_n}$ is a holomorphic homeomorphism. Suppose contrarily that there exist two distinct points (z^0), $(z^1)\in D$ such that $\Phi_{i_1\cdots i_n}(z^0)=\Phi_{i_1\cdots i_n}(z^1)$, then (z^0) and (z^1) are equivalent. By Lemma 3, $(M_{i_1\cdots i_n}, \Phi_{i_1\cdots i_n})$ is a holomorphic basis, therefore there exist a point (z) and two curves C_0 and C_1 such that C_0 connects the point (z^0) to (z) and C_1 does the point (z^1) to (z) and $A((z), (z); (z^0), (z^1))/(M_{i_1\cdots i_n}, \Phi_{i_1\cdots i_n})$ holds. Now in a neighborhood $U(z)$ of (z) there exist two points (z^2) and (z^3) such that (z^2) lies on C_0, (z^3) on C_1 and $\Phi_{i_1\cdots i_n}(z^2) = \Phi_{i_1\cdots i_n}(z^3)$ and therefore $J(\Phi_{i_1\cdots i_n}(z)) = 0$. On the other hand, since (z) is an ordinary point of $\Phi_{i_1\cdots i_n}$, we have $J(\Phi_{i_1\cdots i_n}(z)) \neq 0$. This is a contradiction.

Now we consider a holomorphic map of D defined by

$$F_{ij} = \Phi^{-1}_{j_1\cdots j_n} \circ H \circ \Phi_{i_1\cdots i_n}$$

which is a holomorphic automorphism induced by H. It is necessary to prove that F_{ij} and $F_{kl} = \Phi^{-1}_{l_1\cdots l_n} \circ H \circ \Phi_{k_1\cdots k_n}$ are the same holomorphic automorphism of D. But we can see this easily by considering two systems $\{h_{i\tau(i)}\}$ and $\{h_{k\tau(k)}\}$ defined by F_{ij} and F_{kl}, respectively. Thus our theorem has been proved.

Theorem 1 gives us a complete characterization of all holomorphic automorphisms of a simply-connected normal domain, and can be used for construction and determination of rigid simply-connected normal domains.

By using the same terminologies as in theorem 1, we have

COROLLARY 1. *A necessary and sufficient condition that a simply-connected normal domain is rigid is that all holomorphic automorphisms H in theorem 1 are identical maps.*

4. We consider a holomorphic homeomorphism of a normal domain onto another. Let D and D' be simply-connected normal domains defined by $f_i(z_1, \cdots, z_n) = \gamma_i(t)$ $(i = 1, \cdots, p)$ and $f_j'(z_1, \cdots, z_n) = \gamma_j'(t)$ $(j = 1, \cdots, q)$, respectively, then we have two sets $\{M_{i_1\cdots i_n}\}$ and $\{M_{j_1}'{}_{\cdots j_n}\}$ and two sets of maps $\Phi_{i_1\cdots i_n}\colon D \to M_{i_1\cdots i_n}$ and $\Phi'_{j_1\cdots j_n}\colon D' \to M'_{j_1\cdots j_n}$. If F is a holomorphic homeomorphism of D onto D', then we have a holomorphic homeomorphism of $M_{i_1\cdots i_n}$ onto $M'_{j_1\cdots j_n}$ defined by

$$\Phi'_{j_1\cdots j_n} \circ F \circ \Phi^{-1}_{i_1\cdots i_n}$$

and $p = q$ (see Rothstein [7]).

Conversely, we can state the following

THEOREM 2. *Any holomorphic homeomorphism H of $\mathfrak{R}_1 \times \cdots \times \mathfrak{R}_p$ onto $\mathfrak{R}_1' \times \cdots \times \mathfrak{R}_p'$:*

$$w_j' = h_{i\tau(i)}(w_i), \quad j = \tau(i), \quad w_i \in \mathfrak{R}_i, \quad w_j' \in \mathfrak{R}_j',$$

which is also a holomorphic homeomorphism of $\{M_{i_1\cdots i_n}\}$ onto $\{M'_{j_1\cdots j_n}\}$, induces a holomorphic homeomorphism $\Phi'^{-1}_{j_1\cdots j_n} \circ H \circ \Phi_{i_1\cdots i_n}$ of D onto D'.

The proof of this theorem is obtained by the same method as that of theorem 1.

By use of this theorem, we have a necessary and sufficient condition for the possibility of a holomorphic homeomorphism between two simply-connected normal domains.

COROLLARY 2. *A necessary and sufficient condition that a simply-connected normal domain D can be mapped one-to-one holomorphically onto another D' is that there exists at least a holomorphic homeomorphism H in theorem* 2.

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