SOME FOURIER INTEGRAL THEOREMS

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1. Introduction.

The well-known general convergence theorem in the theory of Fourier integral deals with the limit relation (Bochner [1])

$$\lim_{w\to\infty}\int_{-\infty}^{\infty} f\left(x+\frac{t}{w}\right)K(t)\,dt = f(x)\int_{-\infty}^{\infty} K(t)\,dt, \tag{1.1}$$

where $K(t)$ will naturally be supposed to be of $L_1(-\infty, \infty)$.

If $K(t)$ does not belong to the class $L_1(-\infty, \infty)$, the relation (1.1) will not be expected to be true. Instead it would be natural to suppose the relation

$$\int_{-\infty}^{\infty} f\left(x+\frac{t}{w}\right)K(t)\,dt = o(1) \quad \text{or} \quad O(1). \tag{1.2}$$

We shall consider an asymptotic behavior of the left hand side of (1.2) as w increases indefinitely when $K(t)$ and $f(t)$ are of $L_2(-\infty, \infty)$.

Similar problems concerning functions of two variables will be to find the orders of

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\left(x+\frac{s}{w},\ y+\frac{t}{w}\right)K(s)\overline{K(t)}\,ds\,dt, \tag{1.3}$$

when $w\to\infty$. We shall treat the problem in §2.

Now (1.3) is of the form

$$\frac{1}{\alpha^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K\left(\frac{x-u}{\alpha}\right)\overline{K\left(\frac{y-v}{\alpha}\right)}f(u, v)\,du\,dv \tag{1.4}$$

with changes of variables and $\alpha = 1/w$. We shall consider a rather special function, in place of $f(u, v)/\alpha^2$,

$$\left(\frac{1}{A\cdot\alpha\pi}\int_{-\infty}^{\infty} p(\xi)\frac{\sin A(u+\xi)\sin A(v+\xi)}{(u+\xi)(v+\xi)}d\xi\right)^2, \tag{1.5}$$

where $p(\xi)\in L_1(-\infty, \infty)$, $\alpha=\alpha(A)$, $A\to\infty$.

Here the following theorem should be noted:

$$\lim_{A\to\infty}\frac{1}{\pi}\int_{-\infty}^{\infty} p(\xi)\frac{\sin A(\xi+u)\sin A(\xi-u)}{A(\xi+u)(\xi-u)}d\xi = p(0), \quad \text{if} \quad u=0, \tag{1.6}$$
$$= 0, \quad \text{if} \quad u\neq 0,$$

provided that $p(\xi)$ is continuous and of $L_1(-\infty, \infty)$.

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Grenander ([1], Grenander and Rosenblatt [1]) proved this relation, applying to the estimation theory of spectral density of a stationary stochastic process.

(1.5) is the Fejér integral if $u=v$ and tends to $p(-u)$ when $A\to\infty$, $\alpha=1$. On the other hand the integral in (1.5) will become

$$\frac{1}{A\cdot\alpha\pi}\int_{-\infty}^{\infty}p(\eta-\eta_0)\frac{\sin A(\eta+w)\sin A(\eta-w)}{A(\eta+w)(\eta-w)}d\eta, \tag{1.7}$$

if we put

$$\eta_0=-\frac{u+v}{2},\qquad w=\frac{u-v}{2}.$$

(1.7) is just the left hand side of (1.6). Hence (1.4) is a type of average of the square of (1.6). And in fact the limit theorem connected with (1.4) will provide an important fact which will play an essential role in the theory of estimation of the spectral density of a stochastic process. We shall deal with the matter in the forthcoming paper.

I should like to mention that E. S. Parzen has shown the equivalent fact in his estimation theory (Parzen [1]). The theorem will be proved in §4 and thereafter.

2. Asymptotic bebaviors of certain integrals.

We shall give an estimation of

$$J(x,w)=\int_{-\infty}^{\infty}f\left(x+\frac{t}{w}\right)K(t)\,dt, \tag{2.1}$$

where $K(t)$ does not necessarily belong to $L_1(-\infty,\infty)$ and in fact we suppose $K(t)$ to be of $L_2(-\infty,\infty)$. Similarly we can treat the case where $K(t)\in L_p(-\infty,\infty)$, $p>1$, but we do not do it here.

Put

$$\int_{|t|\geqq A}|K(t)|^2dt=\eta(A) \tag{2.2}$$

and denote A such that

$$A/\eta(A)=w \tag{2.3}$$

as $A(w)$. Obviously $A(w)\to\infty$ if $w\to\infty$. We then get

THEOREM 1. *Let $f(t)$ and $K(t)$ belong to $L_2(-\infty,\infty)$. Then*

$$J(x,w)=O(\sqrt{A(w)})$$

for almost all x, where $A(w)$ is defined by (2.3) *and O may depend on x.*

Proof. We divide $J(x,w)=J(w)$ into two parts

$$
\begin{aligned}
J(w) &= \int_{-\infty}^{\infty} f\left(x+\frac{t}{w}\right)K(t)\,dt \\
&= \int_{|t|<A} + \int_{|t|\geqq A} \equiv J_1(w)+J_2(w),
\end{aligned} \tag{2.4}
$$

where $A=A(w)$. We have, for the first integral,

$$
\begin{aligned}
|J_1(w)|^2 &\leqq \int_{|t|<A} |K(t)|^2\,dt \int_{|t|<A} \left|f\left(x+\frac{t}{w}\right)\right|^2 dt \\
&= \int_{|t|<A} |K(t)|^2\,dt\cdot w\int_{|u|<A/w} |f(x+u)|^2\,du \\
&\leqq \int_{-\infty}^{\infty} |K(t)|^2\,dt\cdot O(A) = O(A(w)),
\end{aligned} \tag{2.5}
$$

for almost all x.

Next we have

$$
\begin{aligned}
|J_2(w)|^2 &\leqq \int_{|t|\geqq A} |K(t)|^2\,dt \int_{|t|\geqq A} \left|f\left(x+\frac{t}{w}\right)\right|^2 dt \\
&= \int_{|t|\geqq A} |K(t)|^2\,dt\cdot w\int_{-\infty}^{\infty} |f(u)|^2\,du \\
&= \eta(A)\cdot O(w) = O(w\eta(A)).
\end{aligned}
$$

Inserting this and (2.5) into (2.4), we get

$$
\begin{aligned}
J^2(w) &\leqq 2(|J_1(w)|^2+|J_2(w)|^2) \\
&\leqq O(A+w\eta(A)) = O(A)
\end{aligned}
$$

owing to (2.3), which proves the theorem.

The analogous argument leads us to

THEOREM 2. *If $K(t)\in L_2(-\infty,\infty)$ and $f(x,y)\in L_2(E_2)$, E_2 being two dimensional Euclidean space, then*

$$
\begin{aligned}
J(x,y,w) &= \iint K(s)\overline{K(t)}\, f\left(x+\frac{s}{w},\, y+\frac{t}{w}\right) ds\,dt \\
&= O(A(w))
\end{aligned} \tag{2.6}
$$

for almost all x, y where $A(w)$ is the one defined in (2.3) *and O may depend on x and y.*

3. Lemmas.

We shall prove some lemmas which will be useful in the sequel.

LEMMA 1. *Consider the integral*

$$
S = S_T(\alpha) = \int_{-\infty}^{\infty} \left|\frac{\sin T(\xi+\alpha)\sin T(\xi-\alpha)}{(\xi+\alpha)(\xi-\alpha)}\right| d\xi, \tag{3.1}
$$

α being non-negative. Then we have the following estimations.

(i) *If $\alpha T<1$, then*

$$S \leqq C_1 T, \tag{3.2}$$

where C_1 is an absolute constant.

(ii) *If $\alpha T>1$, then*

$$S \leqq C_2 \frac{\log T\alpha}{\alpha}, \tag{3.3}$$

C_2 being an absolute constant.

We use only (3.3) in §6, but we shall prove the case (i) also for the completeness.

Proof. (i) We divide S into two parts:

$$S=\int_{|\xi|\leqq 2/T}+\int_{|\xi|>2/T}; \tag{3.4}$$

the first integral of this expression does not exceed

$$\int_{-2/T}^{2/T} T^2 d\xi = 4T. \tag{3.5}$$

We may suppose $\alpha>0$. Then the second part of the right hand side does not exceed

$$\int_{|\xi|>2/T}\left|\frac{1}{(\xi+\alpha)(\xi-\alpha)}\right|d\xi = 2\int_{\xi>2/T}\frac{d\xi}{(\xi+\alpha)(\xi-\alpha)},$$

noticing $1>\alpha T$ and $\xi-\alpha>0$, which is

$$\frac{1}{\alpha}\log\frac{2/T-\alpha}{2/T+\alpha}=O(T)$$

which with (3.5) proves (i).

(ii) The part (ii) will be shown in the following way. Put

$$\begin{aligned} S &= \int_{-\infty}^{-\alpha-1/T}+\int_{-\alpha-1/T}^{-\alpha+1/T}+\int_{-\alpha+1/T}^{\alpha-1/T}+\int_{\alpha-1/T}^{\alpha+1/T}+\int_{\alpha+1/T}^{\infty} \\ &= S_1+S_2+S_3+S_4+S_5, \end{aligned}$$

say. Since $\alpha T>1$ and then $\alpha<\xi+\alpha<3\alpha$, if $|\xi-\alpha|<1/T$, then

$$\begin{aligned} S_4 &= \int_{|\xi-\alpha|<1/T}\left|\frac{\sin T(\xi+\alpha)\sin T(\xi-\alpha)}{(\xi+\alpha)(\xi+\alpha)}\right|d\xi \\ &\leqq \frac{1}{\alpha}\int_{|\xi-\alpha|<1/T}\left|\frac{\sin T(\xi-\alpha)}{\xi-\alpha}\right|d\xi < \frac{T}{\alpha}\int_{|\xi-\alpha|<1/T}d\xi=\frac{1}{\alpha}; \end{aligned} \tag{3.6}$$

S_2 is equal to S_4 and hence

$$S_2 \leqq \frac{1}{\alpha}. \tag{3.7}$$

Furthermore

$$S_1 = S_5 \leqq \int_{\alpha+1/T}^{\infty} \frac{d\xi}{(\xi+\alpha)(\xi-\alpha)} \tag{3.8}$$
$$= \frac{1}{2\alpha} \log(2\alpha T + 1) = O\left(\frac{\log T\alpha}{\alpha}\right).$$

Lastly,

$$S_3 = \int_{-\alpha+2/T}^{\alpha-1/T} \left| \frac{\sin T(\xi+\alpha)\sin T(\xi-\alpha)}{(\xi+\alpha)(\xi-\alpha)} \right| d\xi \leqq \int_{-\alpha+1/T}^{\alpha-1/T} \frac{d\xi}{|(\xi+\alpha)(\xi-\alpha)|} \tag{3.9}$$
$$= \frac{1}{\alpha}\log(2\alpha T - 1) = O\left(\frac{\log T\alpha}{\alpha}\right).$$

(3.6), (3.7), (3.8) and (3.9) yield (ii).

LEMMA 2.

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin T(\xi+\alpha)\sin T(\xi-\alpha)}{(\xi+\alpha)(\xi-\alpha)} d\xi = \frac{\sin 2\alpha T}{2\alpha}.$$

The proof will be easy.

4. A limit theorem.

We want to prove the following theorem.

THEOREM 3. *Let $p(x)$ be a continuous bounded function of $L_1(-\infty, \infty)$ and let $K(x)$ be of $L_2(-\infty, \infty)$. We then have*

$$\lim_{T\to\infty} \frac{1}{TB_T} \frac{1}{\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K\left(\frac{x-u}{B_T}\right)\overline{K\left(\frac{y-v}{B_T}\right)} dxdy \tag{4.1}$$
$$\cdot\left(\int_{-\infty}^{\infty} p(\xi)\frac{\sin T(x+\xi)\sin T(y+\xi)}{(x+\xi)(y+\xi)} d\xi\right)^2$$
$$= 2\pi p^2(u)\int_{-\infty}^{\infty} |K(x)|^2 dx\cdot\delta(u, v),$$

where B_T tends to zero in such a way $TB_T \to \infty$ as $T\to\infty$, and $\delta(u, v)=1$ if $u=v$ and $=0$ if $u\neq v$.

As was stated in §1, the equivalent fact was proved by Parzen.

We proceed to prove Theorem 3. The proof of the existence of the integral in the left hand side will be included in the treatments of J_1 and J_2 below.

We shall divide the integral into two parts as

$$J = \frac{1}{TB_T}\frac{1}{\pi}\left(\iint_{|x-y|<\delta} + \iint_{|x-y|\geqq\delta}\right) \tag{4.2}$$
$$= J_1 + J_2,$$

say, δ being a small positive number.

If we put $(x-y)/2=\alpha$, $(x+y)/2=\beta$, J_1 will be

$$J_1=\pi TB_T{}^{-1}\iint_{|x-y|<\delta}K\Big(\frac{x-u}{B_T}\Big)\overline{K\Big(\frac{y-v}{B_T}\Big)}dxdy\Big(\int_{-\infty}^{\infty}p(\xi-\beta)\frac{\sin T(\xi+\alpha)\sin T(\xi-\alpha)}{\pi T(\xi+\alpha)(\xi-\alpha)}d\xi\Big)^2$$

$$=\pi TB_T{}^{-1}\iint_{|x-y|<\delta}K\Big(\frac{x-u}{B_T}\Big)\overline{K\Big(\frac{y-v}{B_T}\Big)}dxdy\Big(\int_{-\infty}^{\infty}p(\xi-\beta)L(\xi,\alpha,\beta;T)\,d\xi\Big)^2.$$

Here we have put

$$L(\xi,\alpha;T)=\frac{\sin T(\xi+\alpha)\sin T(\xi-\alpha)}{\pi T(\xi+\alpha)(\xi-\alpha)}. \tag{4.3}$$

Now we have

$$\begin{aligned} J_1&=\pi TB_T{}^{-1}\iint_{|x-y|<\delta}K\Big(\frac{x-u}{B_T}\Big)\overline{K\Big(\frac{y-v}{B_T}\Big)}dxdy\Big[p(-y)\int_{-\infty}^{\infty}L(\xi,\alpha;T)d\xi \\ &\qquad\qquad+\int_{-\infty}^{\infty}(p(\xi-\beta)-p(-y))L(\xi,\alpha;T)\,d\xi\Big]^2 \\ &=\pi TB_T{}^{-1}\iint_{|x-y|<\delta}K\Big(\frac{x-u}{B_T}\Big)\overline{K\Big(\frac{y-v}{B_T}\Big)}\,dxdy\,p^2(-y)\Big(\int_{-\infty}^{\infty}L(\xi,\alpha;T)d\xi\Big)^2+J_{12} \\ &=J_{11}+J_{12}, \end{aligned} \tag{4.4}$$

say.

Now we shall prove the theorem showing

$$\lim_{T\to\infty}J_{11}=-\pi p^2(-v)\int_{-\infty}^{\infty}|K(w)|^2dw \qquad \text{if} \quad u=v, \tag{4.6}$$
$$=0 \qquad \text{if} \quad u\neq v,$$

$$\lim_{T\to\infty}J_{12}=0 \tag{4.7}$$

and

$$\lim_{T\to\infty}J_2=0. \tag{4.8}$$

The proofs of (4.6) and (4.7) will be given in §5 and §6 respectively. The proof of (4.8) will be done in §7.

5. The proof of (4.6).

We shall prove (4.6) in this section. We have

$$J_{11}=\pi TB_T{}^{-1}\iint_{|x-y|<\delta}K\Big(\frac{x-u}{B_T}\Big)\overline{K\Big(\frac{y-v}{B_T}\Big)}p^2(-y)\frac{\sin^2T(x-y)}{T^2(x-y)}dxdy.$$

Here we have used Lemma 2 in §3. Putting $(x-u)/B_T=\lambda$, $(y-v)/B_T=\mu$, the last expression will get the form

$$\frac{\pi B_T}{T}\iint_{|(\lambda-\mu)B_T+(u-v)|<\delta}p^2(-v-\mu B_T)K(\lambda)\overline{K(\mu)}\frac{\sin^2T(B_T(\lambda-\mu)-(u-v))}{((\lambda-\mu)B_T-(u-v))^2}d\lambda d\mu$$

$$=\frac{\pi B_T}{T}\int_{-\infty}^{\infty}p^2(-v-B_T\mu)\,d\mu\int_{-\delta/B_T-(u-v)/B_T}^{\delta/B_T-(u-v)B_T}K(\mu+z)\overline{K(\mu)}\frac{\sin^2T(B_Tz-(u-v))}{(B_Tz-(u-v))^2}dz.$$

If $u=v$, then the last expression turns to

$$\frac{\pi B_T}{T}\int_{-\infty}^{\infty} p^2(-v-B_T\mu)\,d\mu\int_{-\delta/B_T}^{\delta/B_T} K(\mu-z)\,\overline{K(\mu)}\frac{\sin^2 TB_T z}{B_T{}^2 z^2}\,dz$$

$$=\frac{\pi}{TB_T}\int_{-\infty}^{\infty} p^2(-v-B_T\mu)\,|K(\mu)|^2\,d\mu\int_{-\delta/B_T}^{\delta/B_T}\frac{\sin^2 TB_T z}{z^2}\,dz$$

$$+\frac{\pi}{TB_T}\int_{-\infty}^{\infty} p^2(-v-B_T\mu)\overline{K(\mu)}\,d\mu\int_{-\delta/B_T}^{\delta/B_T}\{K(\mu+z)-K(\mu)\}\frac{\sin^2 TB_T z}{z^2}\,dz$$

$$=L_1+L_2,$$

say. Since $K(\mu)\in L_2$ and

$$\int_{-\delta/B_T}^{\delta/B_T}\frac{\sin^2 TB_T z}{TB_T z^2}\,dz=\int_{-\delta T}^{\delta T}\frac{\sin^2 z}{z^2}\,dz\to\pi$$

as $T\to\infty$, we have

$$\lim_{T\to\infty} L_1=\pi p^2(-v)\int_{-\infty}^{\infty}|K(\mu)|^2\,d\mu. \tag{5.1}$$

On the other hand L_2 converges to zero as $T\to\infty$, because denoting $|p(v)|\leqq C$, we have

$$|L_2|\leqq \pi C^2\int_{-\infty}^{\infty}|K(\mu)|\,d\mu\int_{-\delta T}^{\delta T}\left|K\left(\mu+\frac{z}{TB_T}\right)-K(\mu)\right|\frac{\sin^2 z}{z^2}\,dz$$

$$\leqq \pi^{3/2}C^2\int_{-\infty}^{\infty}|K(\mu)|\,d\mu\left(\int_{-\delta T}^{\delta T}\left|K\left(\mu+\frac{z}{TB_T}\right)-K(\mu)\right|^2\frac{\sin^2 z}{z^2}\,dz\right)^{1/2}$$

$$\leqq \pi^{3/2}C^2\left(\int_{-\infty}^{\infty}|K(\mu)|^2\,d\mu\right)^{1/2}\cdot\left(\int_{-\infty}^{\infty}\frac{\sin^2 z}{z^2}\,dz\cdot\int_{-\infty}^{\infty}\left|K\left(\mu+\frac{z}{TB_T}\right)-K(\mu)\right|^2 d\mu\right)^{1/2}$$

which obviously converges to zero as $T\to\infty$, since $1/TB_T\to 0$.

Hence we get the relation

$$\lim_{T\to\infty} J_{11}=\pi p^2(-v)\int_{-\infty}^{\infty}|K(\mu)|^2\,d\mu, \tag{5.2}$$

when $u=v$.

If, $u\neq v$, then (4.6) will be, putting $u-v=\theta$,

$$\frac{\pi B_T}{T}\int_{-\infty}^{\infty} p^2(-v-B_T\mu)\,d\mu\int_{-\delta/B_T-\theta/B_T}^{\delta/B_T-\theta/B_T} K(\mu+z)\overline{K(\mu)}\frac{\sin^2 T(B_T z-\theta)}{(zB_T-\theta)^2}\,dz$$

$$=\int_{-\infty}^{\infty} p^2(-v-B_T\mu)\,d\mu\int_{(-\delta-2\theta)T}^{(\delta-2\theta)T} K\left(\mu+\frac{w}{TB_T}+\frac{\theta}{B_T}\right)\overline{K(\mu)}\frac{\sin^2 w}{w^2}\,dw$$

which does not exceed in absolute value

$$\leqq C^2\int_{-(\delta+2\theta)T}^{(\delta-2\theta)T}\frac{\sin^2 w}{w^2}\,dw\int_{-\infty}^{\infty}\left|K(\mu)K\left(\mu+\frac{w}{TB_T}+\frac{\theta}{B_T}\right)\right|d\mu$$

$$\leqq C^2\cdot\int_{-\infty}^{\infty}|K(\mu)|^2\,d\mu\cdot\int_{-(\delta+2\theta)T}^{(\delta-2\theta)T}\frac{\sin^2 w}{w^2}\,dw.$$

This tends to zero as $T\to\infty$, if δ has been taken as $\delta<2|\theta|$ in advance. Thus

$$\lim_{T\to\infty} J_{11}=0 \tag{5.3}$$

when $u\neq v$.

6. Proof of (4.7).

In this section we shall prove (4.7). We have

$$\begin{aligned}J_{12}=&\pi TB_T^{-1}\iint_{|x-y|<\delta}K\Big(\frac{x-u}{B_T}\Big)\overline{K\Big(\frac{y-v}{B_T}\Big)}dxdy\\ &\cdot 2p(-y)\int_{-\infty}^{\infty}L(\xi,\alpha;T)d\xi\cdot\int_{-\infty}^{\infty}\{p(\xi-\beta)-p(-y)\}L(\xi,\alpha;T)d\xi\\ &+\pi TB_T^{-1}\iint_{|x-y|<\delta}K\Big(\frac{x-u}{B_T}\Big)\overline{K\Big(\frac{y-v}{B_T}\Big)}dxdy\\ &\cdot\Big[\int_{-\infty}^{\infty}\{p(\xi-\alpha)-p(-y)\}L(\xi,\alpha;T)d\xi\Big]^2\\ =&J_{121}+J_{122},\end{aligned}\tag{6.1}$$

say.

We have by Schwarz's inequality

$$\begin{aligned}|J_{122}|\leqq& 2\pi TB_T^{-1}\Big\{\iint_{|x-y|>\delta}\Big|K\Big(\frac{x-u}{B_T}\Big)K\Big(\frac{y-v}{B_T}\Big)\Big|^2dxdy\Big\}^{1/2}\\ &\cdot\Big[2\iint_{|x-y|<\delta}dxdy\Big(\int_{|y|<2\delta}\{p(\eta-y)-p(-y)\}\frac{\sin T\eta\sin T(\eta+2\alpha)}{T\eta(\eta+2\alpha)}d\eta\Big)^2\\ &+2\iint_{|x-y|<\delta}dxdy\Big(\int_{|\eta|>2\delta}\{p(\eta-y)-p(-y)\}\frac{\sin T\eta\sin T(\eta+2\alpha)}{T\eta(\eta+2\alpha)}d\eta\Big)^2\Big]^{1/2}\end{aligned}\tag{6.2}$$

in which

$$\begin{aligned}&\iint_{|x-y|<\delta}\Big|K\Big(\frac{x-u}{B_T}\Big)K\Big(\frac{y-v}{B_T}\Big)\Big|^2dxdy\\ &\leqq\int_{-\infty}^{\infty}dy\int_{y-\delta}^{y+\delta}\Big|K\Big(\frac{x-u}{B_T}\Big)K\Big(\frac{y-v}{B_T}\Big)\Big|^2dx\\ &\leqq\int_{-\infty}^{\infty}\Big|K\Big(\frac{y-v}{B_T}\Big)\Big|^2dy\int_{-\infty}^{\infty}\Big|K\Big(\frac{y-v}{B_T}\Big)\Big|^2dx\\ &=B_T^2\Big(\int_{-\infty}^{\infty}|K(\lambda)|^2d\lambda\Big)^2.\end{aligned}\tag{6.3}$$

The last term in the right hand side of (6.2) does not exceed

$$8\int_{-\infty}^{\infty}dy\int_{y-\delta}^{y+\delta}dx\Big(\int_{|\eta|>2\delta}|p(\eta-y)-p(-y)|\frac{d\eta}{T\eta^2}\Big)^2.$$

because of the inequality $\eta-2\alpha>\eta/2$ noticing $|\alpha|=|(x-y)/2|<\delta/2<y/2$. Furthermore the last expression is

$$
\begin{aligned}
&\frac{16\delta}{T^2}\int_{-\infty}^{\infty}dy\Big(\int_{|\eta|>2\delta}|p(\eta-y)-p(-y)|\frac{d\eta}{\eta^2}\Big)^2\\
&\leqq\frac{16\delta}{T^2}\Big[\int_{|\eta|>2\delta}\frac{d\eta}{\eta^2}\Big(\int_{-\infty}^{\infty}|p(\eta-y)-p(-y)|^2dy\Big)^{1/2}\Big]^2\\
&\leqq\frac{64\delta}{T^2}\Big(\int_{|\eta|>2\delta}\frac{d\eta}{\eta^2}\Big)^2\cdot\int_{-\infty}^{\infty}|p(w)|^2dw\\
&=\frac{C}{\delta T^2},
\end{aligned}
\tag{6.4}
$$

C being a constant and noticing $p(w)\in L_2$ owing to the fact that $p(w)$ is bounded and of $L_1(-\infty,\infty)$.

Now the remaining integral in the right hand side of (6.2) will now be treated. We have

$$
\begin{aligned}
M&=\int_{-2\delta}^{2\delta}\{p(\eta-y)-p(-y)\}\frac{\sin T\eta}{\eta}\,\frac{\sin T(\eta+2\alpha)}{\eta+2\alpha}d\eta\\
&=\int_{-2\delta}^{2\delta}\psi_T(\eta,y)\frac{\sin T\eta\cos\gamma}{\eta-\gamma}d\eta+\int_{-2\delta}^{2\delta}\psi_T(\eta,y)\frac{\cos T\eta\sin\gamma}{\eta-\gamma}d\eta,
\end{aligned}
$$

where we have put

$$\psi_T(\eta,y)=\{p(\eta-y)-p(-y)\}\frac{\sin T\eta}{\eta}$$

and $2\alpha=-\gamma$. Moreover

$$M=\cos\gamma\int_{-2\delta}^{2\delta}\psi_T(\eta,y)\sin T\eta\cdot\frac{d\eta}{\eta-\gamma}+\sin\gamma\int_{-2\delta}^{2\delta}\psi(\eta,y)\cos T\eta\frac{d\eta}{\eta-\gamma}.$$

We here appeal to the well known M. Riesz's theorem on the conjugate function, which leads us to

$$
\begin{aligned}
&\iint_{|x-y|<\delta}M^2dxdy\leqq\int_{-\infty}^{\infty}dy\int_{-\delta}^{\delta}M^2d\gamma\leqq\int_{-\infty}^{\infty}dy\int_{-2\delta}^{2\delta}M^2d\gamma\\
&\leqq2\int_{-\infty}^{\infty}dy\int_{-2\delta}^{2\delta}\Big|\int_{-2\delta}^{2\delta}\psi_T(\eta,y)\sin T\eta\frac{d\eta}{\eta-\gamma}\Big|^2d\gamma\\
&\quad+\int_{-\infty}^{\infty}dy\int_{-2\delta}^{2\delta}\Big|\int_{-2\delta}^{2\delta}\psi_T(\eta,y)\cos T\eta\frac{d\eta}{\eta-\gamma}\Big|^2d\gamma\\
&\leqq C'\int_{-\infty}^{\infty}dy\int_{-2\delta}^{2\delta}|\psi_T(\eta,y)|^2\sin^2T\eta\cdot d\eta+C'\int_{-\infty}^{\infty}dy\int_{-2\delta}^{2\delta}|\psi_T(\eta,y)|^2\cos^2Tyd\eta.
\end{aligned}
$$

where C' is an absolute constant. The last expression is

$$
\begin{aligned}
&2C'\int_{-\infty}^{\infty}dy\int_{-2\delta}^{2\delta}|p(\eta-y)-p(-y)|^2\frac{\sin^2T\eta}{\eta^2}d\eta\\
&=2C'\int_{-2\delta}^{2\delta}\int_{-\infty}^{\infty}|p(\eta-y)-p(-y)|^2dy\cdot\frac{\sin^2T\eta}{\eta}d\eta.
\end{aligned}
$$

Since

$$\int_{-\infty}^{\infty} |p(\eta - y) - p(-y)|^2 dy \to 0 \qquad \text{as} \quad \eta \to 0,$$

we get, by Fejér's theorem,

$$\lim_{T\to\infty} \frac{4}{T}\int_{-2\delta}^{2\delta}\left(\int_{-\infty}^{\infty} |p(\eta - y) - p(-y)|\, dy\right)\frac{\sin^2 T\eta}{\eta^2}\, d\eta. \tag{6.5}$$

Inserting (6.3), (6.4) and (6.5), we finally get

$$\begin{aligned} |J_{122}| &= O(TB_T^{-1})O(B_T)\left[o\left(\frac{1}{T}\right) + O\left(\frac{1}{T^2}\right)\right] \\ &= o(1). \end{aligned} \tag{6.6}$$

Next it is almost obvious that

$$\lim_{T\to\infty} J_{121} = 0, \tag{6.7}$$

because by Schwarz' inequality

$$\begin{aligned} |J_{121}|^2 \leqq 2\Big[\pi TB_T^{-1} &\iint_{|x-y|<\delta} \left|K\left(\frac{x-u}{B_T}\right)K\left(\frac{y-v}{B_T}\right)\right| \\ &\cdot p^2(-y)\left(\int_{-\infty}^{\infty} L(\xi, \alpha, T)\, d\xi\right)^2 dxdy\Big] \\ &\cdot\Big[\pi TB_T^{-1}\iint_{|x-y|<\delta}\left|K\left(\frac{x-u}{B_T}\right)K\left(\frac{y-v}{B_T}\right)\right| dxdy \\ &\cdot\left\{\int_{-\infty}^{\infty}\{p(\xi-\beta) - p(-y)\}\, L(\xi, \alpha, T)\, d\xi\right\}^2\Big]. \end{aligned}$$

The first factor of the right hand side is convergent as $T\to\infty$ since this is the same as J_{11} except that $|K|$ stands for K. The second factor is J_{122} with $|K|$ instead K which converges to zero as $T\to\infty$. These prove (6.7).

(6.6) and (6.7) complete the proof of (4.7).

7. The proof of (4.8).

We shall prove (4.8). We have

$$\begin{aligned} |J_2| &\leqq \frac{1}{\pi TB_T}\iint_{|x-y|>\delta}\left|K\left(\frac{x-u}{B_T}\right)K\left(\frac{y-v}{B_T}\right)\right| dxdy \\ &\qquad\cdot\left(\int_{-\infty}^{\infty} p(\xi)\frac{\sin T(\xi+x)\sin T(y+\xi)}{(x+\xi)(y+\xi)}\, d\xi\right)^2 \\ &\leqq \frac{C^2}{\pi TB_T}\iint_{|x-y|>\delta}\left|K\left(\frac{x-u}{B_T}\right)K\left(\frac{y-v}{B_T}\right)\right| dxdy\left(\int_{-\infty}^{\infty}\left|\frac{\sin T(\xi+x)\sin T(\xi+y)}{(x+\xi)(y+\xi)}\right| d\xi\right)^2, \end{aligned}$$

C being an upper bound of $p(\xi)$. By Lemma 1 (3.3), we have,

$$|J_2| \leqq \frac{C_2C^2}{\pi TB_T}\iint_{|x-y|>\delta}\left|K\left(\frac{x-u}{B_T}\right)K\left(\frac{y-v}{B_T}\right)\right|\cdot\frac{\log^2 2T(x-y)}{(x-y)^2}\, dxdy$$

$$\leqq \frac{C_2C^2}{\pi TB_T}\left(\iint_{|x-y|<\delta}\left|K\left(\frac{x-u}{B_T}\right)\right|^2\frac{\log^2 2T(x-y)}{(x-y)^2}dxdy\right)^{1/2}$$

$$\cdot\left(\iint_{|x-y|>\delta}\left|K\left(\frac{y-v}{B_T}\right)\right|^2\frac{\log^2 2T(x-y)}{(x-y)^2}dxdy\right)^{1/2}$$

which becomes, by the change of variables,

$$\frac{C_2C^2}{\pi TB_T}\left(\int_{-\infty}^{\infty}\left|K\left(\frac{x-u}{B_T}\right)\right|^2dx\cdot\int_{|\alpha|<\delta}\frac{\log^2 2T\alpha}{\alpha^2}d\alpha\right)^{1/2}$$

$$\cdot\left(\int_{-\infty}^{\infty}\left|K\left(\frac{y-v}{B_T}\right)^2dy\cdot\int_{|\alpha|>\delta}\frac{\log^2 2T\alpha}{\alpha^2}d\alpha\right)^{1/2}$$

$$=\frac{C_2C^2}{\pi TB_T}\int_{-\infty}^{\infty}|K(w)|^2dw\int_{|z|>2T\delta}\frac{\log^2 z}{z^2}dz$$

which converges to zero as $T\to\infty$. This proves (4.8).

Hence the proof of Theorem 3 is complete.

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