CONDITIONAL EXPECTATION IN AN OPERATOR ALGEBRA, III

By Hisaharu Umegaki

1. Introduction.

The theory of rings of operators of von Neumann has been developed by many authors, especially since it has been regarded as a non-commutative extension of the integration over a measure space by Dixmier [3],[1] Dye [4] and Segal [10], some fundamental theorems on measure theory have been extended. In the papers [3], [8] and [14], the authors have introduced the concept of the conditional expectation into some ring of operators of von Neumann (=von Neumann algebra in the sense of Dixmier [2] and we shall use this terminology below) which can be also regarded as a non-commutative extension of conditional expectation in the probability theory. The extension has also been made of the C. Moy characterization theorem (cf. [8] and [14]) and the martingale convergence theorem (cf. [15]). In the present note, as a part of a non-commutative extension of measure theoretic probability theory we shall prove for a von Neumann algebra of finite class a Halmos-Savage theorem (cf. [5]) with respect to sufficient statistics in probability theory which was reformulated under the terminology of Borel subfield by Bahadur [1].

For our purpose we shall depend upon as a basic theorem the Radon-Nikodym theorem due to Dye [4], in a von Neumann algebra. Firstly we shall introduce a space of some restricted normal states relative to a von Neumann subalgebra which will be called tracelet space (cf. Definition 1) and give an example of such space by the direct product of finite factors in the sense of Nakamura [7], and further we shall extend the existence theorem of the conditional expectation in a von Neumann algebra for normal states in the tracelet space (cf. Theorem 1). Under Theorem 1, we shall introduce a concept of sufficiency of von Neumann subalgebra for some restricted set of normal states and prove a Halmos-Savage characterizations of sufficient subalgebra (cf. Theorems 5 and 6). These have applications to find a characteristic property of subalgebra having unique expectation onto it (cf. Theorem 7), and to prove a von Neumann proposition (cf. Theorem 2 of [9]) for a von Neumann's operations which are stated in the final section.

2. Preliminary.

Let A be a countably decomposable ($= \sigma$-finite) von Neumann algebra of

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finite class acting on a Hilbert space H, and B, C and M von Neumann sub-algebras of A, and $B^c = B' \frown A$ where B' is the commutant of B. Denote the identity operator by 1 and a complex number by λ.

Let μ be arbitray but fixed faithful normal trace of A. Then for a subalgebra B, there exists a linear mapping e $(a \to a^e)$ from A onto B such that for any $a \in A$ and for any $b \in B$

(1) $\mu(ab) = \mu(a^e b)$,

(2) $a^e = 0$ $(a \geqq 0)$ *implies* $a = 0$,

(3) $a^e \geqq 0$ $(a \geqq 0)$, $a^{*e} = a^{e*}$ *and* $(ab)^e = a^e b$,

(4) $a_\alpha{}^e \uparrow a^e$ *if* $a_\alpha \uparrow a$.

(cf. [3], [8] and [14]). Such a linear mapping e is uniquely determined within the condition (1), that is, uniquely determined by μ and B, and it was called conditional expectation relative to B (cf. [14]) or expectation conditioned by B (cf. [8]). (Recently, Tomiyama [13] has proved that every linear idempotent mapping from A onto B with norm one has always the property (3)). Here we shall call a linear idempotent normal (=condition (4)) mapping from A onto B by *B-expectation*.

For the normal trace μ, denote $L^1(A, \mu)$ $(= L_\mu{}^1$ say) and $L^2(A, \mu)$ $(= L_\mu{}^2$ say) the L^1 and L^2 spaces with the norms $\| \ \|_1$ and $\| \ \|_2$ respectively, and similarily denote for the subalgebra B $L^1(B, \mu)$ and $L^2(B, \mu)$ which are considered as closed subspaces of $L_\mu{}^1$ and $L_\mu{}^2$ respectively. Then the conditional expectation e relative to B is uniquely extended to a positive linear idempotent mapping from $L_\mu{}^1$ onto $L^1(B, \mu)$ satisfying $\|x^e\|_1 \leqq \|x\|_1$ for every $x \in L_\mu{}^1$, and furthermore e extended to a projection operator from $L_\mu{}^2$ onto $L^2(B, \mu)$. We denote also the extended mappings of e to $L_\mu{}^1$ and $L_\mu{}^2$ respectively by same e.

If a measurable operator x (in the sense of Segal [10]) satisfies $x \eta B$ (i.e. $ux = xu$ for every unitary u in B') then x is called *B-measurable*.

For a normal state σ, a positive measurable operator x is called σ-*integrable* if for the spectral resolution $x = \int \lambda \, dE_\lambda$ the numerical integration $\int \lambda \, d\sigma(E_\lambda)$ is finite, and we denote it by $\sigma(x)$. A measurable operator x is σ-*integrable* if the absolute value $|x|$ of x is σ-integrable. For any pair of measurable operators x and y, if $|x - y|$ is σ-integrable and $\sigma(|x - y|) = 0$, then we call "$x = y$ σ-n.e.". Denote $L^2(A, \sigma)$ the space of all measurable operators x such that $\sigma(x^*x)$ is finite (i.e. x^*x is σ-integrable). For x and $y \in L^2(A, \sigma)$, y^*x is σ-integrable, and the space $L^2(A, \sigma)$ is a Hilbert space under the inner product $\langle x, y \rangle_\sigma = \sigma(y^*x)$ for $x, y \in L^2(A, \sigma)$ where the origin 0 in $L^2(A, \sigma)$ being σ-n.e. zero operator.

For any pair of normal states σ_1, σ_2, we call σ_1 being absolutely continuous with respect to σ_2 (denote $\sigma_1 \prec \sigma_2$), if $\sigma_2(a) = 0$ $(a \geqq 0)$ implies always $\sigma_1(a) = 0$. Further for any pair of sets of normal states S_1 and S_2, if $\sigma_2(a) = 0$ $(a \geqq 0)$ for every $\sigma_2 \in S_2$ implies $\sigma_1(a) = 0$ for every $\sigma_1 \in S_1$, we denote '$S_1 \prec S_2$', and if $S_1 \prec S_2$ and $S_2 \prec S_1$, denote '$S_1 \sim S_2$'. If $\rho \prec \sigma$, by Radon-

Nikodym theorem due to Dye [4] there exists uniquely an operator $d \in L^2(A, \sigma)$ within ρ-n.e. such that $\rho(a) = \sigma(d^*ad)$ for every $a \in A$, and we denote $D(\rho/\sigma) = d$ and call the Radon-Nikodym derivative in the sense of Dye (of ρ with respect to σ).

If S is a set of measurable operators, then denote the set of all positive operators in S by S^+.

Denote "s_σ" the supporting projection of a normal state σ, and for a von Neumann subalgebra B "b_σ" the supporting projection in B of the σ. For any von Neumann algebra M, denote the Banach space of all σ-weakly continuous linear functionals of M by M_* in the notation of [3].

3. Tracelet space and correspondence between normal states and expectations.

In the present section, firstly we shall introduce a some restricted set of normal states of A which will be referred throughout of this paper.

DEFINITION 1. For a von Neumann subalgebra B, let "S_B" be the set of all normal states σ of A satisfying

(5) $\sigma(ab) = \sigma(ba)$ for every $a \in A$ and for every $b \in B$,

and we call S_B the B-tracelet space.

The B-tracelet space contains obviously the space of all normal traces, and is weakly closed, convex set in A_*. The following lemma follows immediately from the definition of B-tracelet space and the Radon-Nikodym theorem due to Dye:

LEMMA 1. For any $\sigma \in S_B$, there exists $d_\sigma \in L^1(B^c, \mu)^+$ such that $\sigma(a) = \mu(d_\sigma a)$ for every $a \in A$. Further the derivative $D(\sigma/\mu)$ is self-adjoint, positive and $d_\sigma = D(\sigma/\mu)^2$. Conversely for any $t \in L^1(B^c, \mu)^+$ with $\|t\|_1 = 1$ $\sigma(a)$ $(= \mu(ta))$ belongs to S_B.

We shall now give an example of B-tracelet space:

EXAMPLE 1.[2] If A is a II_1-factor and B, C are subfactors of A such that $A = B \otimes C$ in the sense of Nakamura [7], where B and C are identifying with $B \otimes \{\lambda 1\}$ and $\{\lambda 1\} \otimes C$ respectively. Then obviously $B^c = C$ and $B = C^c$. Let μ_B and μ_C be the normal traces of B and C respectively, and $S(B)$ and $S(C)$ the sets of all normal states of B and C respectively. Then $\mu = \mu_B \times \mu_C$

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2) This example is suggested by Prof. M. Nakamura. Takesaki [12] has extended the theorem of Nakamura [7] to the direct product of factors without the assumption of finiteness. If we introduce formally the tracelet space into a von Neumann algebra, without assuming the finiteness, in the same way of Definition 1, then, when B is II_1-factor, C is a factor and $A = B \otimes C$ in the sense of [12], the B-tracelet space S_B is given by (6), where μ_B is the trace of B, and where B and $B \otimes \{\lambda 1\}$ are identifying as in this Example.

by [7], and

(6) $$S_B = \{\mu_B \times \rho_C;\ \rho_C \in S(C)\}$$

(6') $$S_C = \{\sigma_B \times \mu_C;\ \sigma_B \in S(B)\}$$

where $\mu_B \times \rho_C$ is defined as a state of A by

$$(\mu_B \times \rho_C)(\sum b_i \otimes c_i) = \sum \mu_B(b_i)\rho_C(c_i)$$

for any finite $b_i \in B$, $c_i \in C$ $(i = 1, 2, \cdots, n)$, that is, B- or C-tracelet spaces are represented by (6) or (6') respectively. We shall prove (6): the state $\mu_B \times \rho_C$ obviously satisfies (5) and belongs to S_B. Conversely, let $\sigma \in S_B$ and let σ_C be the state of C of the contraction of σ onto C. By Lemma 1 there is $d_\sigma \in L^1(C, \mu)^+$ such that $\sigma_C(c) = \mu_C(d_\sigma c)$ for every $c \in C$. Let $\{c_n\}$ be a sequence of operators in C satisfying $\|d_\sigma - c_n\|_1 \to 0$. Then

$$\sigma(b \otimes c) = \lim_{n \to \infty} \mu((1 \otimes c_n)(b \otimes c)) = \lim_{n \to \infty} \mu(b \otimes c_n c)$$
$$= \mu_B(b) \lim_{n \to \infty} \mu_C(c_n c) = \mu_B(b)\mu_C(d_\sigma c) = \mu_B(b)\sigma_C(c),$$

and we get (6).

In the following, for any fixed von Neumann subalgebra B of A we shall give a correspondence between normal states in S_B and B-expectations which is a generalization of the existence theorem of conditional expectation in A (cf. Theorem 1 of [14]).

THEOREM 1. *For any $\sigma \in S_B$, there corresponds a B-expectation ε_σ such that*

(7) $$\sigma(a) = \sigma(a^{\varepsilon_\sigma}) \qquad for\ every\ a \in A,$$

and ε_σ is uniquely determined by σ within the equation (7) and σ-n.e. Conversely any B-expectation ε is corresponding to certain $\sigma \in S_B$, that is, $\varepsilon = \varepsilon_\sigma$[3].

Proof. Putting $f_a(b) = \sigma(ab)$ for any fixed $a \in A$ and for any $b \in B$, f_a is a weakly* ($=\sigma$-weakly) continuous linear functional of B. For each $b \in B$, let u being the partially isometric operator in B such that $b = u|b|$,

$$|f_a(b)| = |\sigma(ab)| = |\sigma(au|b|)| = |\sigma(aub_1 b_1)| \qquad (b_1 = |b|^{1/2})$$
$$= |\sigma(b_1 aub_1)| \le \|au\|_\infty \sigma(b_1 b_1) \le \|a\|_\infty \sigma(|b|).[4]$$

Hence there exists uniquely $a' \in B$ such that $f_a(b) = \sigma(a'b)$ and $b_\sigma a' = a'$ where b_σ is the supporting projection in B of σ which belongs to the center of B. This implies $\sigma(ab) = \sigma(a'b)$ for every $a \in A$ and every $b \in B$. The mapping $a \to a'$ obtained above maps linearly A into B and satisfies (3) and (4). Indeed, for any a, a_1, $a_2 \in A$ and for any $b \in B$

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3) In general, for the B-expectation, there corresponds a number of states in the B-tracelet space S_B. Concerning this fact we shall discuss the sufficiency of subalgebra for a set of normal states in S_B.

4) $\|a\|_\infty$ is the operator bound of a.

$$\sigma((a_1 + a_2)'b) = \sigma((a_1 + a_2)b) = \sigma(a_1 b) + \sigma(a_2 b)$$
$$= \sigma(a'b) + \sigma(a'b) = \sigma((a' + a')b).$$

Since a_1', a_2' and $(a_1 + a_2)'$ belong to $b_\sigma B$, $(a_1 + a_2)' = a_1' + a_2'$. Similarily $(\lambda a)' = \lambda a'$ and the condition (3) is obviously satisfied. The condition of normality (4) for the mapping $a \to a'$ follows from that: If $a_\alpha \uparrow a$ for self-adjoint a_α, $a \in A$, then for $b \in B^+$

$$\sigma(a_\alpha' b) = \sigma(a_\alpha b) \uparrow \sigma(ab) = \sigma(a'b)$$

and $a_\alpha' \leq a'$ imply $a_\alpha' \uparrow b_0$ for some self-adjoint $b_0 \in b_\sigma' B$. Hence $\sigma(b_0 b) = \sigma(a'b)$ for all $b \in B$ and $a' = b_0$. Now we put

$$a^{\varepsilon_\sigma} = a' + a^e(1 - b_\sigma) \qquad \text{for every } a \in A,$$

where a^e is the conditional expactaion of a relative to B, then for every $b \in B$

$$b^{\varepsilon_\sigma} = b' + b^e(1 - b_\sigma) = bb_\sigma + b(1 - b_\sigma) = b$$

and $A^{\varepsilon_\sigma} = B$. Therefore ε_σ is obviously a B-expectation and satisfies (7). The uniquess of ε_σ: Taking a B-expectation ε satisfying $\sigma(a) = \sigma(a^\varepsilon)$ for all $a \in A$,

$$(8) \qquad \sigma(a^\varepsilon b) = \sigma((ab)^\varepsilon) = \sigma(ab) = \sigma((ab)^{\varepsilon_\sigma}) = \sigma(a^{\varepsilon_\sigma} b)$$

for every $b \in B$. (8) and $s_\sigma \leq b_\sigma$ imply that $b_\sigma a^{\varepsilon_\sigma} = b_\sigma a^\varepsilon$ and $s_\sigma a^\varepsilon = s_\sigma b_\sigma a^\varepsilon = s_\sigma b_\sigma a^{\varepsilon_\sigma}$ $= s_\sigma a^{\varepsilon_\sigma}$, and we obtain $a^\varepsilon = a^{\varepsilon_\sigma}$ σ-n.e.

Conversely, let ε be arbitrary B-expectation. Putting $\sigma(a) = \mu(a^\varepsilon)$, $\sigma(a^\varepsilon)$ $= \mu(a^{\varepsilon_\sigma}) = \mu(a^\varepsilon) = \sigma(a)$ and

$$\sigma(ab) = \mu((ab)^\varepsilon) = \mu(a^\varepsilon b) = \mu(ba^\varepsilon) = \mu((ba)^\varepsilon) = \sigma(ba).$$

These imply that $\sigma \in S_B$ and $a^\varepsilon = a^{\varepsilon_\sigma}$ σ-n.e..

REMARK 1. The B-expectation ε_σ corresponding to $\sigma \in S_B$ is uniquely determined by σ within (7) and without the terminology σ-n.e. if and only if σ is faithful on B, because the both parts of "*if*" and "*only if*" follow from that the support b_σ in B of σ is identity operator 1. In Theorem 1, even if the assumption on A—the countable decomposability and the finiteness —does not put, and if the set of states S_B defined by (5) is not empty, then for any $\sigma \in S_B$ there corresponds uniquely a linear idempotent mapping from A onto $b_\sigma B$ satisfying (3) and (4). This fact will be obtained by the same way as in the proof of Theorem 1.

REMARK 2. As Example 1, let A, B and C be II_1-factors such that $A = B \otimes C$, and let μ, μ_B and μ_C be the normal traces of A, B and C respectively. Then by Example 1 any $\sigma \in S_B$ is expressed by $\sigma = \mu_B \times \sigma_C$ where σ_C is the contraction of σ onto C (i.e. $\sigma_C(c) = \sigma(1 \otimes c)$). This implies that for any finite $b_i \in B$ and $c_i \in C$

$$(9) \qquad (\sum_i b_i \otimes c_i)^{\varepsilon_\sigma} = \sum_i b_i \otimes \sigma_C(c_i) 1.$$

Indeed, for any $b \in B$

$$\sigma((\textstyle\sum b_i \otimes \sigma_C(c_i)1)(b \otimes 1)) = \sigma(\textstyle\sum b_i b \otimes \sigma_C(c_i)1)$$

$$= \textstyle\sum (\mu_B(b_i b)\sigma_C(c_i)) = \sigma((\textstyle\sum b_i b \otimes c_i)) = \sigma((\textstyle\sum b_i \otimes c_i)(b \otimes 1))$$

and hence (9) holds.

In the following we prove that any B-expectation is representable by the conditional B-expectation e as (10) below.

THEOREM 2. *For any B-expectation ε, there exists uniquely $t_\varepsilon \in L^1(B^c, \mu)^+$ such that*

(10) $a^\varepsilon = (t_\varepsilon a)^e$ *for every $a \in A$.*

Conversely, for any $t \in L^1(B^c, \mu)^+$ with $t^e = 1$, putting $a^\varepsilon = (ta)^e$, then ε is a B-expectation. The correspondence $\varepsilon \to t_\varepsilon$ is one-to-one between the set of all B-expectations and $\{t \in L^1(B^c, \mu)^+; t^e = 1\}$.

Proof. Putting $\sigma(a) = \mu(a^\varepsilon)$, by the proof of Theorem 1 $\sigma \in S_B$. Then by Lemma 1 there exists uniquely $d_\sigma \in L^1(B^c, \mu)^+$ such that $\sigma(a) = \mu(d_\sigma a)$ for for all $a \in A$. Therefore for every $b \in B$

$$\mu(a^\varepsilon b) = \mu((ab)^\varepsilon) = \sigma(ab) = \mu(d_\sigma ab) = \mu((d_\sigma a)^e b)$$

and $a^\varepsilon = (d_\sigma a)^e$. The operator d_σ is the required one, i.e. $t_\varepsilon = d_\sigma$. The uniqueness will be followed from the final part in this proof.

Converse case: For any $t_1 \in L_\mu^2$, $t_2 \in L^2(B^c, \mu)$ and $b \in B$,

$$\mu((t_1 t_2)^e b) = \mu(t_1 t_2 b) = \mu(t_1 b t_2) = \mu(t_2 t_1 b) = \mu((t_2 t_1)^e b) \text{ }^{5)}$$

and $(t_1 t_2)^e = (t_2 t_1)^e$. Let $t \in L^1(B^c, \mu)^+$ be $t^e = 1$ and put $a^\varepsilon = (ta)^e$. Then $b^\varepsilon = (tb)^e = t^e b = b$ for any $b \in B$ and $(ta)^e = (t^{1/2}at^{1/2})^e \geqq 0$ for any $a \in A^+$. Hence for $a \in A^+$ $0 \leqq a^\varepsilon = (ta)^e \leqq (t\|a\|_\infty 1)^e = \|a\|_\infty 1$, and $a^\varepsilon \in B^+$. By the linearity of ε, $a^\varepsilon \in B$ for all $a \in A$. Since the normality of ε is obvious, ε is a B-expectation.

The final part follows immediately from the above parts and the following: If t_1, $t_2 \in L'$ satisfy $(t_1 a)^e = (t_2 a)^e$ for every $a \in A$, then $\mu(t_1 a) = \mu((t_1 a)^e) = \mu((t_2 a)^e) = \mu(t_2 a)$ and $t_1 = t_2$.

4. Sufficiency of subalgebra.

In this section, we shall introduce the concept of sufficient statistics of probability into the von Neumann algebra A.

DEFINITION 2. Let B be a von Neumann subalgebra and S_0 be a set of normal states of A. Then we call that B is *sufficient for* S_0, if the B-tracelet space contains S_0 (i.e. $S_0 \subset S_B$) and for each $a \in A$ there exists $a' \in B$ such that $a^{\varepsilon_\sigma} = a'$ σ-n.e. for every $\sigma \in S_0$.

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5) For any measurable operators t and t', tt' is the closure of the product operator, i.e., $tt' = t \cdot t'$ in the notation of [10].

This definition is slightly non-commutative extension of the case of suffi-cient statistics in probability theory (cf. [1] and [5]), which may be described as the following:

Let $(\Omega, \mathfrak{A})$ be a measurable space in the terminology of measure theory, and A the multiplication algebra of all bounded, measurable and complex valued functions defined on Ω. Let $\mathfrak{B}$ be an arbitary Borel subfield of $\mathfrak{A}$ and B the subalgebra of A consisting of all $\mathfrak{B}$-measurable functions. Then A is (in general homomorphically) represented by a commutative von Neumann algebra A acting on a Hilbert space $H^{6)}$ and B is similarily represented by a von Neumann subalgebra B of A. The space of all probability measures of $(\Omega, \mathfrak{A})$ can be identified with the space of all normal states of A by the natural way, which coincides with the B-tracelet space (likewise defined by (5) in Definition 1), because A is commutative. We denote the element in A and its corresponding operator in A by the same symbol a, etc.. Then for each probability measure σ, there corresponds a conditional expectation $E_\sigma(\mathfrak{F} \mid \mathfrak{B})$ $(\mathfrak{F} \in \mathfrak{A})$ relative to $\mathfrak{B}$ which is determined in σ-n.e. in the usual sense. In our notation, it is expressed by $E_\sigma(a \mid \mathfrak{B}) = a^{\varepsilon_\sigma}$ for $a \in A$ (cf. Theorem 1 and Remark 1). For a set S_0 of probability measures, the sufficiency of a Borel subfield $\mathfrak{B}$ is defined by such a way: for any subset $\mathfrak{F} \in \mathfrak{A}$ there is a function $a' \in B$ such that

$$(11) \qquad E_\sigma(\mathfrak{F} \mid \mathfrak{B}) = a' \qquad \sigma\text{-}n.e. \qquad \text{for every} \quad \sigma \in S_0.$$

For these cases the countable decomposability is not assumed, but since the equation (11) is defined within the "σ-n.e." part, these cases are covered by our case.

For $\sigma \in S_B$ if we put "$a^{\delta_\sigma} = b_\sigma a^{\varepsilon_\sigma}$" where b_σ is the supporting projection in B of σ, then $a^{\delta_\sigma} = a^{\varepsilon_\sigma}$ σ-n.e. and $\sigma(a) = \sigma(a^{\delta_\sigma})$ for all $a \in A$. The mapping δ_σ is a linear, idempotent from A onto $b_\sigma B$ satisfying (3) and (4). That is, linear mapping ε_σ and δ_σ can be considered as same one within σ-n.e.. By Theorem 2, taking

$$(12) \qquad t_\sigma = b_\sigma t_{\varepsilon_\sigma},$$

it is obvious that

$$(13) \qquad a^{\delta_\sigma} = (t_\sigma a)^e \qquad \text{for every} \quad a \in A \text{ and } \sigma \in S_B.$$

Under this notation, we have the following for any pair (B, S_0) of a von

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6) Let P be the set of all probability measures of $(\Omega, \mathfrak{A})$. For each $\sigma \in P$ A is homomorphically represented by a von Neumann algebra A_σ acting on the Hilbert space $L_\sigma^2(\Omega, \mathfrak{A})$ $(= H_\sigma$ say). The algebra A_σ is isomorphic to the quotient algebra A/N_σ, where $N_\sigma = \left\{ f \in A; \int |f| d\sigma = 0 \right\}$. The Hilbert space H is defined by the direct sum $\sum \{H_\sigma; \sigma \in P\}$ in the similar method of Takeda [11], and the von Neumann algebra A is defined by the discrete direct sum $\sum \{A_\sigma; \sigma \in P\}$ $(= \{(\sum f_\sigma \mid \sigma \in P); f \in A\})$ where the *operation, the addition and the multiplication are introduced into A by coordinatewise ones of $\{A_\sigma; \sigma \in P\}$, that is, $(\sum f_\sigma \mid \sigma \in P)^* = (\sum f_\sigma^* \mid \sigma \in P)$, $(\sum f_\sigma \mid \sigma \in P) + (\sum g_\sigma \mid \sigma \in P) = (\sum (f+g)_\sigma \mid \sigma \in P)$ etc.

Neumann subalgebra B and a set of normal states S_0 satisfying $S_0 \subset S_B$:

THEOREM 3. B is sufficient for S_0 if and only if $b_\rho a^{\delta_\sigma} = b_\sigma a^{\delta_\rho}$ for every $a \in A$ and for every pair $\sigma, \rho \in S_0$.[7]

THEOREM 4. B is sufficient for S_0 if and only if B is sufficient for the closed convex hull K of S_0 with respect to the weak topology of A_*.

Proof of Theorem 3.   The "*only if*" part is obvious, we prove the "*if*" part.   By the condition, for any $a \in A$

$$a^{\delta_\sigma} a^{\delta_\rho} = a^{\delta_\sigma} b_\sigma b_\rho a^{\delta_\rho} = a^{\delta_\sigma} b_\rho b_\sigma a^{\delta_\rho} = a^{\delta_\rho} a^{\delta_\sigma}.$$

Therefore for each fixed $a \in A^+$, there exists $a' \in B^+$ such that $a' = \sup\{a^{\delta_\sigma}; \sigma \in S_0\}$ where "sup" being taken under the ordering of operators.   Hence

$$b_\sigma a' = \sup\{b_\sigma a^{\delta_\rho}; \rho \in S_0\} = \sup\{b_\rho b_\sigma a^{\delta_\rho}; \rho \in S_0\}$$
$$= \sup\{b_\rho a^{\delta_\sigma}; \rho \in S_0\} = a^{\delta_\sigma}$$

and $a^{\delta_\sigma} = a'$ σ-n.e. for each $\sigma \in S_0$ and $a \in A^+$.   Since any $a \in A$ is a finite linear combination of operators in A^+ and δ_σ is linear, we get $a^{\delta_\sigma} = a'$ σ-n.e. for every $a \in A$ and for some corresponding $a' \in B$.

Proof of Theorem 4.   The "*if*" part is obvious, we prove the "*only if*" part.   Put K_0 the convex hull of S_0 and K the weak closure of K_0 in A_*. Any $\sigma \in K_0$ is expressed by $\sigma = \sum_{i=1}^n \lambda_i \sigma_i$ for $\sigma_i \in S_0$ and scalars $\lambda_i \geqq 0 (\sum_{i=1}^n \lambda_i = 1)$. If B is sufficient for S_0, then for each $a \in A$ there exists $a' \in B$ such that that $a^{\delta_\rho} = b_\rho a'$ for every $\rho \in S_0$.   Hence $\sigma(a') = \sum_{i=1}^n \lambda_i \sigma_i(a') = \sum \lambda_i \sigma_i(a) = \sigma(a)$ and $b_\sigma a' = a^{\delta_\sigma}$.   By a well-known theorem of Mazur, K coincides with the closure of K_0 in the strong topology in A_*, and for any $\sigma \in K$ there exists $\{\sigma_n\} \subset K_0$ such that $\sigma_n(a) \to \sigma(a)$ for all $a \in A$.   Therefore $\sigma(a) = \lim \sigma_n(a) = \lim \sigma_n(a')$ $= \sigma(a')$ and $a^{\delta_\sigma} = a'$ σ-n.e.

As Lemma 1, denote $d_\sigma = D(\sigma/\mu)^2$ for $\sigma \in S_B$ which belongs to $L^1(B^c, \mu)$, then

LEMMA 2.   *For any* $\sigma \in S_B$, $d_\sigma = t_\sigma d_\sigma{}^e = d_\sigma{}^e t_\sigma$, *where* t_σ *is defined by* (12).

Proof.   Since $a^{\delta_\sigma} = (t_\sigma a)^e$ for any $a \in A$,

(14) $$\mu(d_\sigma a) = \sigma(a) = \sigma(a^{\delta_\sigma}) = \mu(d_\sigma a^{\delta_\sigma}) = \mu(d_\sigma (t_\sigma a)^e).[8]$$

Taking $b_n' \in (B^c)^+$ which are commuting each other, satisfying $b_n' \leqq b_{n+1}'$ ($n = 1$, $2, \cdots$) and converging to t_σ in the $L_\mu{}^1$-mean, then $b_n'^e \uparrow t_\sigma{}^e = b_\sigma$ and

$$\mu(d_\sigma{}^e b_n') = \mu(d_\sigma b_n'^e) \uparrow \mu(d_\sigma b_\sigma) = \sigma(b_\sigma) = 1.$$

Since $0 \leqq d_\sigma{}^e b_n' \leqq d_\sigma{}^e b_{n+1}'$ ($n = 1, 2, \cdots$), it converges in the $L_\mu{}^1$-mean to some

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7)   This theorem is related with the concept of the pairwise sufficiency (cf. [5]) and it will be applied to the proof of Theorem 6.

8)   If d_σ, t_σ belong to $L_\mu{}^2$, then (15) (below) is clear from (14) (cf. §2).

operator in $L^1(A, \mu)$ and $\mu(d_\sigma{}^e b_n{}') \leq 1$, and since $d_\sigma{}^e$, $b_n{}'$ and t_σ commute each other, by Lebesgue convergence theorem, $d_\sigma{}^e t_\sigma \in L_\mu{}^1$, $0 \leq \mu(d_\sigma{}^e t_\sigma) \leq 1$ and for every $a \in A$

$$(15) \qquad \mu(d_\sigma(t_\sigma a)^e) = \lim \mu(d_\sigma(b_n{}'a)^e) = \lim \mu(d_\sigma{}^e b_n{}'a) = \mu(d_\sigma{}^e t_\sigma a).$$

Combining (14) and (15), $\mu(d_\sigma a) = \mu(d_\sigma{}^e t_\sigma a)$ and we get $d_\sigma = d_\sigma{}^e t_\sigma = t_\sigma d_\sigma{}^e$.

LEMMA 3. *The B-expectation ε_σ is uniquely extensible to a projection operator p_σ from $L^2(A, \sigma)$ onto $L^2(B, \sigma)$ and satisfies $\langle a^{\varepsilon_\sigma} b_1, b_2 \rangle_\sigma = \langle ab_1, b_2 \rangle_\sigma$ for b_1, $b_2 \in L^2(B, \sigma)$, where $L^2(B, \sigma)$ is a linear subspace of $L^2(A, \sigma)$ consisting of all B-measurable operators which is obviously closed.*

Proof. Since $a^{*\varepsilon_\sigma} a^{\varepsilon_\sigma} \leq (a^*a)^{\varepsilon_\sigma}$ for all $a \in A$ (cf. [13]), for any a, a_1 and $a_2 \in A$

$$\langle a_1{}^{\varepsilon_\sigma}, a_2 \rangle_\sigma = \sigma(a_2{}^* a_1{}^{\varepsilon_\sigma}) = \sigma(a_2{}^{\varepsilon_\sigma *} a_1{}^{\varepsilon_\sigma}) = \sigma(a_2{}^{\varepsilon_\sigma *} a_1) = \langle a_1, a_2{}^{\varepsilon_\sigma} \rangle_\sigma$$

and

$$\langle a^{\varepsilon_\sigma}, a^{\varepsilon_\sigma} \rangle_\sigma = \sigma(a^{\varepsilon_\sigma *} a^{\varepsilon_\sigma}) \leq \sigma((a^*a) = \sigma(a^*a) = \langle a, a \rangle_\sigma.$$

Therefore ε_σ is uniquely extended to the projection p_σ from $L^2(A, \sigma)$ onto $L^2(B, \sigma)$ because the restrictions of A and B onto the supporting domain of σ are dense in $L^2(A, \sigma)$ and $L^2(B, \sigma)$ respectively. The second part follows from that, $a^{\varepsilon_\sigma} b = p_\sigma(ab)$ for any $a \in A$ and $b \in B$ implies that

$$\langle a^{\varepsilon_\sigma} b_1, b_2 \rangle_\sigma = \langle p_\sigma(ab_1), b_2 \rangle_\sigma = \langle ab_1, p_\sigma b_2 \rangle_\sigma = \langle ab_1, b_2 \rangle_\sigma.$$

Hereafter, as in Theorems 3 and 4 *let (B, S_0) be a pair of a von Neumann subalgebra B and a set of normal states satisfying $S_0 \subset S_B$.* In the following, we prove Halmos-Savage theorem for our non-commutative case.

THEOREM 5. *It is a necessary and sufficient condition for B to be sufficient for S_0 that there exists a state $\pi \in S_B$ such that (i) $S_0 \sim \pi$, (ii) $\pi(as_\sigma) = \pi(s_\sigma a)$ for each $\sigma \in S_0$ and for every $a \in A$ and (iii) $D(\sigma/\pi)$ coincides with a B-measurable operator σ-n.e. for every $\sigma \in S$.*

Proof. Necessity: By the assumption, for each $a \in A$ there corresponds $a' \in B$ such that $a^{\delta_\sigma} = b_\sigma a'$, and by Theorem 2 there exists $t_\sigma \in L^1(B^c, \mu)$ such that $a^{\delta_\sigma} = (t_\sigma a)^e$ for all $a \in A$ and $t_\sigma{}^e = b_\sigma$, where the notations t_σ and δ_σ refer to the equations (12) and (13). Whence for each pair σ, $\rho \in S_0$

$$\mu(b_\rho t_\sigma a) = \mu(b_\rho(t_\sigma a)^e) = \mu(b_\rho a^{\delta_\sigma}) = \mu(b_\rho b_\sigma a')$$

$$= \mu(b_\sigma b_\rho a') = \mu(b_\sigma a^{\delta_\rho}) = \mu(b_\sigma(t_\rho a)^e) = \mu(b_\sigma t_\rho a)$$

and $b_\rho t_\sigma = b_\sigma t_\rho$. Since $b_\sigma t_\sigma = t_\sigma$ and $b_\rho t_\rho = t_\rho$,

$$t_\sigma t_\rho = b_\sigma t_\sigma b_\rho t_\rho = b_\rho t_\sigma b_\sigma t_\rho = t_\rho t_\sigma.$$

While for any b', $b'' \in B^c$ and $b \in B$, $\mu(b'^e b''^e b) = \mu(b' b'' b) = \mu(b'' b' b)$ and $b'^e b''^e = b'' ^e b'^e$. Since d_σ, $d_\rho \in L^1(B^c, \mu)$, d_σ and d_ρ are approximated by sequences in B^c respectively, and we have $d_\sigma{}^e d_\rho{}^e = d_\rho{}^e d_\sigma{}^e$. Since by Lemma 2 $d_\sigma = t_\sigma d_\sigma{}^e$

and $d_\rho = t_\rho d_\rho{}^e$,

(16) $d_\sigma d_\rho = t_\sigma d_\sigma{}^e t_\rho d_\rho{}^e = t_\sigma t_\rho d_\sigma{}^e d_\rho{}^e = t_\rho t_\sigma d_\rho{}^e d_\sigma{}^e = t_\rho d_\rho{}^e t_\sigma d_\sigma{}^e = d_\rho d_\sigma.$

Let $d_\sigma = \int \lambda dE_\lambda(\sigma)$ be the spectral resolution of $d_\sigma(\sigma \in S_0)$ and C the von Neumann subalgebra generated by $\{E_\lambda(\sigma); \lambda \geq 0,\ \sigma \in S_0\}$, then C is commutative and d_σ $(\sigma \in S_0)$ are C-measurable, moreover there exists a projection $p_0 \in C$ such that $p_0 = \sup\{s_\sigma;\ \sigma \in S_0\}$. Let F be a family of all projections $p \in C$ such that $p \in F$ if and only if $p \leq s_\sigma$ for some $\sigma \in S_0$. By the countable decomposability of C and by Zorn lemma there exist a countable $\{\sigma_n\} \subset S_0$ and $\{p_n\} \subset F$ such that $p_n \leq s_{\sigma_n}$ and $\sup_{n=1}^\infty p_n = p_0$. Putting $\pi = \sum_{n=1}^\infty \sigma_n/2^n$, π belongs to the closed convex hull of S_0 (which is contained in S_B) and $p_0 = s_\pi$, and hence $\pi \sim S_0$. Consequently by Theorem 4 $a^{\delta \pi} = b_\pi a'$ (a' is the operator in B, cf. Def. 2) and by $b_\sigma \leq b_\pi$ we get $b_\sigma t_\pi = t_\sigma$ and[9]

$$d_\sigma d_\pi{}^{-1} = (t_\sigma d_\sigma{}^e)(t_\pi d_\pi{}^e)^{-1} = t_\sigma t_\pi{}^{-1} d_\sigma{}^e d_\pi{}^{e-1} = s_\sigma b_\sigma d_\sigma{}^e d_\pi{}^{e-1} = s_\sigma d_\sigma{}^e d_\pi{}^{e-1}$$

and $d_\sigma d_\pi{}^{-1} = d_\sigma{}^e d_\pi{}^{e-1}$ σ-$n.e.$. Since d_σ commutes with d_π and $d_\mu(=1)$, $D(\sigma/\pi) \geq 0$ and $d_\sigma d_\pi{}^{-1} = D(\sigma/\pi)^2$. Therefore $D(\sigma/\pi)$ coincides with the B-measurable operator $(d_\sigma{}^e d_\pi{}^{e-1})^{1/2}$ σ-$n.e.$. The condition (ii) is obvious from (16) and the construction of π.

Sufficiency: For any fixed $\sigma \in S_0$, put $A_\sigma = \{s_\sigma a s_\sigma;\ a \in A\}$ and $B_\sigma = \{s_\sigma b s_\sigma;\ b \in B\}$. Then A_σ and B_σ are von Neumann algebras (acting on $s_\sigma H$) with the parallel properties (in §2) of A and B respectively. Since for any $a \in A$, $b \in B$ and $\rho \in S_B$

$$\rho(s_\sigma b s_\sigma a s_\sigma) = \rho(b s_\sigma a s_\sigma) = \rho(s_\sigma a s_\sigma b) = \rho(s_\sigma a s_\sigma b s_\sigma),$$

putting $\pi'(a) = \pi(a)/\pi(s_\sigma)$ and $\sigma'(a) = \sigma(a)$ for $a \in A_\sigma$, π' and σ' belong to S_{B_σ} as states of A_σ. By the assumptions (ii) and (iii), $D(\sigma'/\pi')$ belongs to $L^2(B_\sigma, \pi')$, and by Lemma 3 (for A_σ, B_σ and π') we get immediately

$$\sigma((s_\sigma a s_\sigma)^{\varepsilon \pi} b) = \langle (s_\sigma a s_\sigma)^{\varepsilon \pi} b D(\sigma/\pi),\ D(\sigma/\pi) \rangle_\pi$$
$$= \langle s_\sigma a s_\sigma b D(\sigma/\pi),\ D(\sigma/\pi) \rangle_\pi = \sigma(s_\sigma a b s_\sigma) = \sigma(ab) = \sigma(a^{\varepsilon \sigma} b)$$

and $b_\sigma (s_\sigma a s_\sigma)^{\varepsilon \pi} = b_\sigma a^{\varepsilon \sigma} = a^{\delta \sigma}$ for every $a \in A$. While the condition (ii) implies that $\pi(s_\sigma ab) = \pi(abs_\sigma) = \pi(as_\sigma b)$ for every $b \in B$ and $(s_\sigma a)^{\delta \pi} = (as_\sigma)^{\delta \pi}$ for every $a \in A$. Therefore again by (ii), for $a \in A^+$, $b \in B^+$ and $b_1 = b^{1/2}$

$$\pi(s_\sigma ab) = \pi(s_\sigma b_1 ab_1) = \pi(s_\sigma b_1 ab_1 s_\sigma) \leq \pi(b_1 ab_1) = \pi(ab)$$

and

$$a^{\delta \sigma} = b_\sigma a^{\varepsilon \sigma} = b_\sigma (s_\sigma a s_\sigma)^{\varepsilon \pi} \leq b_\pi (s_\sigma a s_\sigma)^{\varepsilon \pi} = (s_\sigma a s_\sigma)^{\delta \pi} = (s_\sigma a)^{\delta \pi} \leq a^{\delta \pi}.$$

Put $f(a) = \sigma(a^{\delta \pi}) - \sigma(a^{\delta \sigma})$. Then f is a positive linear functional of A and

$$f(1) = \sigma(1^{\delta \pi}) - \sigma(1^{\delta \sigma}) = \sigma(b_\pi) - \sigma(b_\sigma) = \sigma(1) - \sigma(1) = 0.$$

Consequently $f = 0$ and $\sigma(a^{\varepsilon \pi}) = \sigma(a^{\delta \pi}) = \sigma(a^{\delta \sigma}) = \sigma(a^{\varepsilon \sigma})$. Thus we obtain $a^{\varepsilon \sigma} = a^{\varepsilon \pi}$

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9) For any positive and measurable operator t, t^{-1} is defined such as the closure of the inverse operator in its supporting manifold and zero in its orthocomplimented manifold, which is also positive and measurable.

σ-n.e. for every $a \in A$. Since σ is arbitrary in S_0, B is sufficient for S_0.

REMARK 3. In the original case of Theorem 5 (cf. Theorem 1 of [5]), the countable decomposability has not been assumed, but it has been discussed for a set of probability measures dominated by a probability measure $\bar{p}$, which implies that the multiplication algebra of all bounded random variables defined $\bar{p}$-n.e. becomes a countably decomposable von Neumann algebra, and it reduces to our case.

In the rest part of this section, we shall describe for our case a related concept of a theorem of Halmos-Savage with respect to the pairwise sufficiency in probability theory. The following will be obtained immediately from Theorem 5:

COORLLARY 5.1. *If the set S_0 consists of the faithful normal states, then it is a necessary and sufficient condition for B to be sufficient for S_0 that $D(\sigma/\rho)$ is B-measurable for every pair of states σ, $\rho \in S_0$.*

DEFINITION 3. A closed linear subspace C of A_* is called *abelien subspace* if C is generated by $\{l_c\tau; c \in C\}$ for a certain commutative, weakly closed self-adjoint subalgebra C of A and for a certain normal trace τ of A, where the operation l_c is defined by $(l_c f)(a) = f(ca)$ ($f \in A^*$).

For example, the closed linear subspaces of A_* generated by the set of all normal traces of A is an abelian subspace. In general, the real part of any abelian subspace C of A_* is an abstract (L)-space in the sense of Kakutani (cf. [6]) where the ordering in C is defined from the positive definiteness.[10] By Radon-Nikodym theorem due to Dye, we can easily see that:

LEMMA 4. *If a closed linear and self-adjoint subspace C of A_* is an abelian subspace, then $C \frown A_*^+$ generates C and d_σ, d_ρ commute each other for any pair of normal states σ, ρ in C.*

Denote for any pair of normal states σ, ρ $D(\sigma, \rho) = D(\sigma/(\sigma + \rho)/2)$. Under these terminologies we prove the following:

THEOREM 6. *It is a necessary and sufficient condition for B to be sufficient for S_0 that ($1°$) S_0 is contained in an abelian subspace of A_* and ($2°$) $D(\sigma, \rho)$ coincides with a B-measurable operator σ-n.e. for every pair σ, $\rho \in S_0$.*

Proof. Necessity: We refer to the proof of Theorem 5. Let C be the von Neumann subalgebra and π the normal state in S_B which were defined in the part of the necessity of that proof. Then C is commutative, $d_\sigma \in L^1(C, \mu)$ for every $\sigma \in S_0$ and further by the construction of π $d_\pi \in L^1(C, \mu)$. Putting

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10) We shall give a characterization of the abelian subspace under the terminology of abstract (L)-space in another occasion.

$C = \{f_x;\ x \in L^1(C, \mu)\}$ (where $f_x(a) = \mu(xa)$), C is obviously an abelian subspace of A_* and contains S_0. Consequently the closed convex hull K of S_0 is contained in C and for any pair $\sigma,\ \rho \in K$ with $\sigma \prec \rho\ D(\sigma/\rho) \geqq 0$, and by Theorem 4 and the proof of Theorem 5 $d_\sigma d_\pi^{-1} = D(\sigma/\pi)^2 = s_\sigma v_\sigma$ for certain B-measurable positive operator v_σ, which depends on σ, for each $\sigma \in K$. Hence

$$d_\sigma d_\rho^{-1} = s_\sigma(v_\sigma d_\pi)(v_\rho d_\pi)^{-1} = s_\sigma(v_\sigma v_\rho^{-1})s_\pi = s_\sigma v_\sigma v_\rho^{-1}.$$

Since v_σ and v_ρ can be taken to be positive and commuting each other, $D(\sigma/\rho) = (d_\sigma d_\rho^{-1})^{1/2} = s_\sigma(v_\sigma v_\rho^{-1})^{1/2}$ and $(2°)$ follows immediately from this.

   Sufficiency: By the condition $(1°)$ and by Lemma 4, $d_\sigma d_\rho = d_\rho d_\sigma$ for each pair $\sigma,\ \rho \in S_0$ and $d_\sigma d_\tau = d_\tau d_\sigma$ (where $\tau = (\sigma + \rho)/2$), and further $\tau(as_\sigma) = \tau(s_\sigma a)$ for every $a \in A$. Therefore by the same manner of the proof of Theorem 5, $a^{\varepsilon_\sigma} = a^{\varepsilon_\tau}$ σ-n.e. and similarly $a^{\varepsilon_\rho} = a^{\varepsilon_\tau}$ ρ-n.e.. Consequently $a^{\delta_\sigma} = b_\sigma a^{\varepsilon_\sigma} = b_\sigma a^{\varepsilon_\tau}$, $a^{\delta_\rho} = b_\rho a^{\varepsilon_\tau}$ and

$$b_\rho a^{\delta_\sigma} = b_\rho b_\sigma a^{\delta_\tau} = b_\sigma b_\rho a^{\varepsilon_\tau} = b_\sigma a^{\delta_\rho}.$$

Thus by Theorem 3 we get the sufficiency of B for S_0.

5. Uniqueness of expectation.

   In this final section, we shall give a simple application of the sufficiency of a von Neumann subalgebra to the uniqueness of expectation on it. This is essential for non-commutative case.

   THEOREM 7. *The following conditions are equivalent each other*:
   (i) B *is sufficient for* S_B,
   (ii) $B^c \subset B$, *where* $B^c = B' \frown A$,
   (iii) *The B-expectation is unique*.

   Proof. (i)→(ii): For $b' \in (B^c)^+$, put $b = (b' + 1)/\mu(b' + 1)$ and $\sigma(a) = \mu(ba)$ $(a \in A)$. Then $\sigma \in S_B$ and is faithful. Hence $\varepsilon_\sigma = \varepsilon_\mu(= e)$, and $a^{\delta_\sigma} = b_\sigma a^{\varepsilon_\sigma} = a^{\varepsilon_\sigma}$ by $b_\sigma = 1$. By Theorem 2, taking t_σ such as (12), then for all $a \in A$

$$\mu(t_\sigma a) = \mu((t_\sigma a)^e) = \mu(a^{\varepsilon_\sigma}) = \mu(a^e) = \mu(a)$$

which implies $t_\sigma = 1$. While $b = d_\sigma = t_\sigma d_\sigma{}^e$ (by Lemma 2) $= d_\sigma{}^e$ and $b \in B$. Therefore $b' \in B$, and $B^c \subset B$, because each operator in B^c is expressed by a finite linear combination of the operators in $(B^c)^+$.

   (ii)→(iii): Let ε be a B-expectation. Putting $\sigma(a) = \mu(a^\varepsilon)$, then $\sigma \in S_B$ and $\varepsilon_\sigma = \varepsilon$. Since $a^\varepsilon = (t_\sigma a)^e$ for $t_\sigma \in L^1(B^c, \mu)$ with $t_\sigma{}^e = 1$ and by (ii) t_σ belongs to $L^1(B, \mu)$, we obtain $t_\sigma = 1$ and $a_\varepsilon = (t_\sigma a)^e = a^e$.

   (iii)→(i) follows immediately from Theorem 1.

   COROLLARY 7.1. *If $B = N^c$ for certain commutative von Neumann subalgebra N, then the B-expectation is unique*.

   Proof. $N \subset N'$ and $N \subset A$ imply $N \subset N^c = B$ and $B' \subset N'$. Hence $B^c \subset N' \frown A = N^c = B$ and we get the proof by Theorem 7.

From this corollary, Theorem 2 in [9] (which will be stated in Cor. 7.1') follows such as: For a projection $p \in A$, denote $C_p = \{p\}' \frown A$ and $a^{|p} = pap + (1-p)a(1-p)$ for every $a \in A$, $|p$ being a notation of von Neumann (e.g. see [9]). Then $|p$ is obviously a C_p-expectation and by Corollary 7.1 every C_p-expectation is unique and hence is represented by $|p$. Moreover let $\{p, q, \cdots, r, \cdots\}$ be a countable family of projections in A which are commuting each other, and put $B = C_p \frown C_q \frown \cdots$. Then $B = \{p, q, \cdots\}^c$, and the B-expectation is unique. While for every $a \in A$ $a^{|p|q|\cdots|r}$ converges in the strong operator topology to an operator $b \in B$ (cf. [9], [14]) and denote $a^{|p|q|\cdots} = b$. Then the mapping $|p|q|\cdots$ is a B-expectation, which has been called the *von Neumann's operation* defined by the von Neumann subalgebra $\{p, q, \cdots\}''$ (cf. [9]). Consequently for the subalgebra B, every B-expectation is represented by the von Neumann's operation $|p|q|\cdots$. Especially, if the Hilbert space H, on which A acts, is *separable*, and if N is *commutative*, then N is generated by a countable family of projections in it, therefore we obtain

COROLLARY 7.1'. *For the von Neumann subalgebra $B(=N^c)$, the B-expectation coincides with the von Neumann's operation defined by N, that is, for any countable family of projections $p, q, \cdots$ which generates N, $|p|q|\cdots$ defines the unique B-expectation and hence is determined only by N.*

This corollary contains a von Neumann proposition, which has been proved in [9], for our special von Neumann algebra A. Concerning the von Neumann's operation, we shall give a further remark (cf. Cor. 7·1''). For any von Neumann subalgebra M, M^c satisfies $M^c = M^{ccc}(= ((M^c)^c)^c)$. Indeed, $a \in M^c$ is equivalent to that $aa' = a'a$ for all $a' \in M^{c'} = M \smile A'$ which is also equivalent to $a \in (M^c \smile A') \frown A$ $(= M^{ccc})$, where $M \smile A'$ means the smallest von Neumann subalgebra containing M and A'. Furthermere if M is commutative, then M^{cc} is also commutative. This is clear by that $M^{cc} = (M' \frown A)' \frown A = (M \smile A') \frown A$. We take N, and the Hilbert space H as in the Cor. 7.1', then $N^{ccc} = N^c = B$, N^{cc} is commutative and is generated by a countable family of projections $p, q, \cdots$. Therefore the von Neumann's operation defined by N^{cc} is also B-expectation, that is, we obtain

COROLLARY 7.1''. *The von Neuman's operations defined by N and N^{cc} are identical.*

As a special case of Cor. 7.1 and Cor. 7.1', we have the following:

COROLLARY 7.2. *If B is a maximally abelian subalgebra, then the B-expectation is unique, and furthermore if the Hilbert space H is separable then it is a von Neumann's operation defined by B.*

If the converse of the last part of this corollary holds, then B is maximally abelian, that is, for commutative B, the equality "*the B-expectation*

$=$ *the von Neumann's operation defined by B "* is a characteristic property of the maximality of B (cf. Theorem 3 of [9]).

Let T be the set of all normal traces of A, then from the proof of Theorem 7 we have:

COROLLARY 7.3. *For a von Neumann subalgebra B, B is sufficient for T if and only if $A' \frown A \subset B$.*

This contains a result of Dixmier (cf. Proposition 8 of [3]) for the present restricted algebra A.

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DEPARTMENT OF MATHEMATICS,
TOKYO INTITUTE OF TECHNOLOGY.