ON THE CONVOLUTION TRANSFORM

By Yûkichi Tanno

1. Introduction.

In this paper we shall study the inversion theory for the class of convolution transforms

(1) $$f(x)=\int_{-\infty}^{\infty} G(x-t)\varphi(t)\,dt$$

for which the kernel $G(t)$ is of the form

(2) $$G(t)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{F(s)}e^{st}ds.$$

Here

(3) $$F(s)=\prod_{k=1}^{\infty}\frac{(1-s^2/a_k^2)}{(1-s^2/c_k^2)},$$

where $\{a_k\}_1^\infty$ and $\{c_k\}_1^\infty$ are positive constants such that

(4) $$0<a_1\leqq a_2\leqq\cdots;\quad c_1\leqq c_2\leqq\cdots;\quad a_k\leqq c_k,$$

$$\lim_{n\to\infty}\frac{n}{a_n}=\Omega>\Omega'=\lim_{n\to\infty}\frac{n}{c_n}.$$

We agree that from certain point on, all c_k may $=\infty$. In fact, the case was extensively studied by Hirschman and Widder [1] Chapter IX. We shall follow after their arguments to consider the generalization.

If we set $a_k=(2k-1)/2$, $c_k=\infty$ $(k=1, 2, 3, \cdots)$, Theorem 7 and Theorem 8 below will give known results for the Stieltjes transform [1].

2. Properties of the kernel.

We suppose that

(1) $$E(s)=\prod_{k=1}^{\infty}\left(1-\frac{s^2}{a_k^2}\right)$$

where

(2) $$0<a_1\leqq a_2\leqq\cdots,\qquad \lim_{n\to\infty}\frac{n}{a_n}=\Omega.$$

Lemma 1. *If $E(s)$ is defined by equations* (1) *and* (2), *then*

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$$\lim_{r\to\infty} r^{-1} \log |E(re^{i\theta})| = \pi\Omega |\sin\theta|$$

uniformly for θ in any closed interval not containing an integral multiple of π.

This is known; see [1] p. 213.

LEMMA 2. *If $F(s)$ is defined by equations* (3) *and* (4) *of* §1, *then*

$$\lim_{r\to\infty} r^{-1} \log |F(re^{i\theta})| = \pi(\Omega - \Omega') |\sin\theta|$$

uniformly for θ in any closed interval not containing an integral multiple of π.

This is an immediate consequence of Lemma 1.

We define

$$h_k(t) = \left(1 - \frac{a_k^2}{c_k^2}\right)\frac{1}{2} a_k \int_{-\infty}^{t} e^{-a_k|u|} du + \frac{a_k^2}{c_k^2} j(t)$$

where $j(t)$ is the standard jump function, that is, $j(t)=0$ for $t<0$, $1/2$ for $t=0$ and 1 for $t>0$. It is easily verified that $h_k(t)$ is a distribution function with mean 0 and variance $2(a_k^{-2} - c_k^{-2})$ and that

$$\int_{-\infty}^{\infty} e^{-st} dh_k(t) = \frac{1 - s^2/c_k^2}{1 - s^2/a_k^2}, \tag{3}$$

the bilateral Laplace transform converging absolutely for $-a_k < \Re s < a_k$.

THEOREM 1. *If*

1. *$F(s)$ is defined by* (3) *and* (4) *of* §1,
2. *μ denotes the multiplicity of a_1 as a zero of $F(s)$,*

and

3. $G(t) = \dfrac{1}{2\pi i}\displaystyle\int_{-i\infty}^{i\infty} \frac{1}{F(s)} e^{st} ds \quad (-\infty < t < \infty)$,

then

A. *$G(t)$ is a frequency function with mean* 0 *and variance* $2(\sum_1^\infty a_k^{-2} - \sum_1^\infty c_k^{-2})$,

B. $\displaystyle\int_{-\infty}^{\infty} G(t)e^{-st}dt = 1/F(s)$, *the bilateral Laplace transform converging absolutely in the strip* $-a_1 < \Re s < a_1$,

C. $G(t) \in C^\infty$,

D. $G(t) = p(t)e^{-a_1 t} + R_+(t)$, $G(t) = p(-t)e^{a_1 t} + R_-(t)$

where $p(t)$ is a real polynomial of degree $\mu - 1$ and

$$\left(\frac{d}{dt}\right)^n R_+(t) = O(e^{-(a_1+\varepsilon)t}) \quad as \quad t\to\infty \qquad (n = 0, 1, 2, \cdots),$$

$$\left(\frac{d}{dt}\right)^n R_-(t) = O(e^{(a_1+\varepsilon)t}) \quad as \quad t\to-\infty \qquad (n = 0, 1, 2, \cdots)$$

for some $\varepsilon > 0$.

Proof. If we set

$$H_n(t) = h_1(t) \# h_2(t) \# \cdots \# h_n(t)$$

where operation $\#$ denotes the Stieltjes convolution for distribution functions, that is, $h \# k$ means

$$\int_{-\infty}^{\infty} h(t-u)\, dk(u),$$

then by the convolution theorem [2] $H_n(t)$ is a distribution function with the bilateral Laplace transform

$$\int_{-\infty}^{\infty} e^{-st} dH_n(t) = \prod_{k=1}^{n} \frac{1 - s^2/c_k^2}{1 - s^2/a_k^2}.$$

We have

$$\lim_{n\to\infty} \prod_{k=1}^{n} \frac{1 - s^2/c_k^2}{1 - s^2/a_k^2} = \frac{1}{F(s)}$$

uniformly for s in any compact set of the s-plane punctured at $\pm a_1, \pm a_2, \cdots$. Thus $1/F(i\tau)$ is the characteristic function of a distribution function $H(t) = \lim_{n\to\infty} H_n(t)$,

$$\int_{-\infty}^{\infty} e^{-i\tau t} dH(t) = \frac{1}{F(i\tau)}.$$

Further, by Lévy's theorem

$$H(t_1) - H(t_2) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{e^{st_1} - e^{st_2}}{sF(s)} ds.$$

Since by Lemma 2

(4) $$\log |F(i\tau)| \sim \pi(\Omega - \Omega') |\tau| \quad \text{as} \quad \tau \to \pm\infty,$$

it follows that $H(t)$ is infinitely differentiable. If $G(t) = dH(t)/dt$, then $G(t)$ is a frequency function, and

(5) $$G(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{e^{st}}{F(s)} ds.$$

From this the conclusion C follows.

To demonstrate the conclusion D, let us choose $\varepsilon > 0$ so small that no a_k $(k = 2, 3, \cdots)$ lies in the interval $-a_1 - \varepsilon \leqq \sigma < a_1$ $(s = \sigma + i\tau)$. Integrating about the rectangular contour with vertices at $\pm iT$, $-a_1 - \varepsilon \pm iT$ and letting T increase without limit, we obtain

$$G(t) = p(t) e^{-a_1 t} + R_+(t) \qquad R_+(t) = \frac{1}{2\pi i} \int_{-a_1-\varepsilon-i\infty}^{-a_1-\varepsilon+i\infty} \frac{e^{st}}{F(s)} ds.$$

Again by Lemma 2 if $\eta > 0$ then

$$\left| \frac{1}{F(s)} \right| = O(e^{-\pi(\Omega - \Omega' - \eta)|\tau|}) \quad \text{as} \quad \tau \to \pm\infty$$

uniformly for σ in any finite interval. From this it is easily seen that

$$\left(\frac{d}{dt}\right)^n R_+(t) = O(e^{-(a_1+\varepsilon)t}) \quad \text{as} \quad t \to \infty.$$

The second part of conclusion D will be established similarly.

From D we see that

$$\int_{-\infty}^{\infty} e^{st} G(t)\, dt$$

converges absolutely for $|\Re s| < a_1$ and defines in this strip an analytic function. Since

$$\int_{-\infty}^{\infty} e^{-i\tau t} G(t)\, dt = \frac{1}{F(i\tau)},$$

we have demonstrated the conclusion B; that is, for $|\Re s| < a_1$

$$\int_{-\infty}^{\infty} e^{-st} G(t)\, dt = \frac{1}{F(s)}. \tag{6}$$

From this equation the conclusion A follows by the the straightforward computations.

THEOREM 2. *If $G(t)$ is defined as in Theorem 1, then*

$$\operatorname{sgn} \frac{dG(t)}{dt} = -\operatorname{sgn} t.$$

This follows from the fact that the functions $h_k(t)$ are convex distribution functions.

3. Properties of the transform.

THEOREM 3. *If*

1. *$G(t)$ is defined as in Theorem 1,*
2. *$\alpha(t)$ is of bounded variation in every finite interval,*

and

3. $\int_{-\infty}^{\infty} G(x_0 - t)\, d\alpha(t)$ *converges,*

then

$$\int_{-\infty}^{\infty} G(x-t)\, d\alpha(t)$$

converges uniformly for x in any finite interval.

Proof. It is enough to show that

$$\lim_{A, B \to +\infty} \int_A^B G(x-t)\, d\alpha(t) = 0, \tag{1}$$

$$\lim_{A, B \to +\infty} \int_A^B G(x-t)\, d\alpha(t) = 0, \tag{1$'$}$$

uniformly for x in any finite interval. By Theorem 1 we have

$$(2) \qquad \frac{G(x-t)}{G(x_0-t)}=O(1) \quad \text{and} \quad \frac{d}{dt}\left[\frac{G(x-t)}{G(x_0-t)}\right]=O\left(\frac{1}{t^2}\right) \quad \text{as} \quad t\to\infty,$$

uniformly for x in any finite interval. If we set

$$L(t)=\int_t^\infty G(x_0-t)\,d\alpha(t),$$

then

$$(3) \qquad L(t)=o(1) \quad \text{as} \quad t\to+\infty.$$

We have

$$\int_A^B G(x-t)\,d\alpha(t)=\int_A^B \frac{G(x-t)}{G(x_0-t)}G(x_0-t)\,d\alpha(t)$$

$$=\left[-\frac{G(x-t)}{G(x_0-t)}L(t)\right]_A^B+\int_A^B\left(\frac{d}{dt}\frac{G(x-t)}{G(x_0-t)}\right)L(t)\,dt.$$

Using equations (2) and (3) we see that equation (1) holds uniformly for x. We can establish (1)′ similarly.

4. Operational calculus.

Denote by D the operation of differentiation. We define the operation $(1-D/a_k)^{-1}$ after Hirschman and Widder [1] by the following equation:

$$(1-D/a_k)^{-1}\varphi(x)=\int_{-\infty}^{\infty} e^{-yD/a_k}\varphi(x)h(y)dy,$$

where

$$h(y)=\begin{cases} e^y & (-\infty,\,0),\\ 0 & (0,\,\infty),\end{cases}$$

that is, by the equations

$$(1-D/a_k)^{-1}\varphi(x)=\begin{cases} a_k e^{a_k x}\displaystyle\int_x^\infty \varphi(y)e^{-a_k y}dy & \text{if} \quad a_k>0,\\[2ex] -a_k e^{a_k x}\displaystyle\int_\infty^x \varphi(y)e^{-a_k y}dy & \text{if} \quad a_k<0.\end{cases}$$

For example, if $a_k>0$ then

$$(1-D/a_k)^{-1}e^{st}=\frac{e^{st}}{1-s/a_k} \qquad \text{for} \quad \Re s<a_k.$$

Therefore

$$(1-D/a_k)^{-1}G(t)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{e^{st}}{F(s)(1-s/a_k)}ds,$$

the integral converging absolutely by Lemma 2.

Let us define

(1) $$F_n(s)=\prod_{k=n+1}^{\infty}\frac{1-s^2/a_k^2}{1-s^2/c_k^2},$$

(2) $$F_n^*(s)=\prod_{k=1}^{n}\frac{1-s^2/a_k^2}{1-s^2/c_k^2},$$

(3) $$G_n(t)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{e^{st}}{F_n(s)}ds.$$

THEOREM 4. *If*

1. $F_n(s)$ *is defined by equation* (1),
2. $G_n(t)$ *is defined by equation* (3),

then

A. $G_n(t)$ *is a frequency function of mean* 0 and *variance* $2(\sum_{n+1}^{\infty}a_k^{-2}-\sum_{n+1}^{\infty}c_k^{-2})$.

B. $\int_{-\infty}^{\infty}G_n(t)e^{-st}dt=1/F_n(s)$, *the bilateral Laplace transform converging absolutely in the strip* $-a_{n+1}<\Re s<a_{n+1}$,

C. $G_n(t)\in C^{\infty}$, $\quad -\infty<t<\infty$,

and

D. $G_n(t)=p_n(t)e^{-a_{n+1}t}+R_{n,+}(t)$, $\quad G_n(t)=p_n(-t)e^{a_{n+1}t}+R_{n,-}(t)$,

where $p_n(t)$ *is a polinomial of degree* μ_n-1, μ_n *denoting the multiplicity of* $s=a_{n+1}$ *as a zero of* $F_n(s)$, *and*

$$\left(\frac{d}{dt}\right)^m R_{n,+}(t)=O(e^{-(a_{n+1}+\varepsilon)t}) \quad as \quad t\to+\infty \ (m=0,1,2,\cdots),$$

$$\left(\frac{d}{dt}\right)^m R_{n,-}(t)=O(e^{(a_{n+1}+\varepsilon)t}) \quad as \quad t\to-\infty \ (m=0,1,2,\cdots)$$

for some $\varepsilon>0$.

This is an immediate consequence of Theorem 1.

From this theorem and Theorem 1 we have

(4) $$F_n^*(D)G(t)=G_n(t).$$

5. Inversion theorem.

THEOREM 5. *If*

1. $G(t)$ *is defined as in Theorem* 1,
2. $F_n^*(D)$ *and* $G_n(t)$ *are defined by* (2) *and* (3) *of* §4,
3. $\alpha(t)$ *is of bounded variation in every finite interval,*

and

4. $f(x)=\int_{-\infty}^{\infty}G(x-t)\,d\alpha(t)$ *converges,*

then

$$F_n^*(D)f(x)=\int_{-\infty}^{\infty}G_n(x-t)\,d\alpha(t),$$

the integral converging uniformly for x in any finite interval.

Proof. From the relation (4) of §4 it is enough to show that the integral

$$(1)\qquad \int_{-\infty}^{\infty}G_n(x-t)\,d\alpha(t)$$

converges uniformly for x in any finite interval. By Theorem 1 and Theorem 4 the integral

$$(2)\qquad \int_{-\infty}^{\infty}\frac{d}{dt}\frac{G_n(x-t)}{G(x-t)}\,dt$$

converges uniformly for x in any finite interval and we have

$$\lim_{t\to\pm\infty}\frac{G_n(x-t)}{G(x-t)}<\infty,$$

uniformly for x. For any x $(-\infty<x<\infty)$ we set

$$L(t)=\int_0^t G(x-t)\,d\alpha(t),$$

then by Theorem 4, $L(t)$ is bounded and $L(+\infty)$, $L(-\infty)$ exist. For arbitrary T_1, T_2 we have

$$\int_{T_1}^{T_2}G_n(x-t)\,d\alpha(t)=\int_{T_1}^{T_2}\frac{G_n(x-t)}{G(x-t)}dL(t)$$
$$=\left[\frac{G_n(x-t)}{G(x-t)}L(t)\right]_{T_1}^{T_2}-\int_{T_1}^{T_2}\left[\frac{d}{dt}\frac{G_n(x-t)}{G(x-t)}\right]L(t)\,dt.$$

The last two terms converge as $T_1\to-\infty$, $T_2\to+\infty$, uniformly for x in any finite interval.

COROLLARY 5. 1. *$G(t)$ is defined as in Theorem* 1,

2. *$F_n^*(D)$, $G_n(t)$ are defined by* (2) *and* (3) *of* §4,

3. *$\varphi(t)$ is integrable on every finite interval,*

4. $f(x)=\int_{-\infty}^{\infty}G(x-t)\varphi(t)\,dt$ *converges,*

then

$$F_n^*(D)f(x)=\int_{-\infty}^{\infty}G_n(x-t)\varphi(t)\,dt,$$

the integral converging absolutely for x in any finite interval.

In the previous theorem, set

$$\int_0^t\varphi(t)\,dt=\alpha(t).$$

Then the result follows immediately.

LEMMA 3. [1] *Let $\varphi(t)$ be continuous and $\alpha(t)$ of bounded variation in every finite subinterval of $a \leqq t < \infty$. If*

1. *$\varphi(t)$ is positive and monotonic,*

and

2. *$\int_a^\infty \varphi(t)\, d\alpha(t)$ converges,*

then $\lim_{t\to\infty} \varphi(t) = 0$ implies that

$$\alpha(t) = o\left(\frac{1}{\varphi(t)}\right) \text{ as } t \to +\infty.$$

THEOREM 6. *If*

1. *$G(t)$ is defined as in Theorem 1,*
2. *$F_n^*(D)$ and $G_n(t)$ are defined by (2) and (3) of §4,*
3. *$\alpha(t)$ is of bounded variatian in any finite interval,*

and

4. *$f(x) = \int_{-\infty}^\infty G(x-t)\, d\alpha(t)$ converges,*

then for n sufficiently large

$$\int_{x_1}^{x_2} F_n^*(D) f(x)\, dx = \int_{-\infty}^\infty G_n(x_2 - t)\alpha(t)\, dt - \int_{-\infty}^\infty G_n(x_1 - t)\alpha(t)\, dt.$$

Proof. By Theorem 2 and Lemma 3 we have

$$\alpha(t) = o[G(x-t)]^{-1} \quad \text{as} \quad t \to \pm\infty. \tag{1}$$

By Theorem 5

$$F_n^*(D) f(x) = \int_{-\infty}^\infty G_n(x-t)\, d\alpha(t),$$

the integral converging uniformly for x in any finite interval. Integrating by parts, we obtain

$$F_n^*(D) f(x) = \Big[G_n(x-t)\alpha(t)\Big]_{-\infty}^{\infty} - \int_{-\infty}^\infty \left[\frac{\partial}{\partial t} G_n(x-t)\right] \alpha(t)\, dt.$$

Theorem 1, Theorem 4 and the estimation (1) show that the integrated parts vanishes uniformly for $x_1 \leqq x \leqq x_2$. Thus

$$F_n^*(D) f(x) = -\int_{-\infty}^\infty \left[\frac{\partial}{\partial t} G_n(x-t)\right] \alpha(t)\, dt = \int_{-\infty}^\infty \left[\frac{\partial}{\partial x} G_n(x-t)\right] \alpha(t)\, dt,$$

the integral converging uniformly for $x_1 \leqq x \leqq x_2$. We have

$$\int_{x_1}^{x_2} F_n^*(D) f(x)\, dx = \int_{x_1}^{x_2} dx \int_{-\infty}^\infty \left[\frac{\partial}{\partial x} G_n(x-t)\right] \alpha(t)\, dt.$$

Because of the uniform convergence of the inner integral we may invert the order of integration and we have

$$(2)\qquad \int_{x_1}^{x_2} F_n^*(D)f(x)\,dx = \int_{-\infty}^{\infty}\Big[G_n(x_2-t)-G_n(x_1-t)\Big]\alpha(t)\,dt.$$

Using Theorem 1, Theorem 4 and the estimate (1), we see that if n is sufficiently large, the integral (2) will converge absolutely.

LEMMA 4. *If $G_n(t)$ is defined as in* (3) *of* §4, *then*

$$\lim_{n\to\infty} G_n(t) = 0 \qquad (0<|t|<\infty).$$

Proof. Let t_0 be an arbitrary number different from zero. Then we have

$$\left|\int_{t_0}^{t_0/2} G_n(t)\,dt\right| \leqq \int_{|t|>|t_0|/2} G_n(t)\,dt \leqq \frac{|t_0{}^2|}{4}\int_{-\infty}^{\infty} t^2 G_n(t)\,dt$$

$$= \frac{|t_0|^2}{2}\left(\sum_{n+1}^{\infty} a_k{}^{-2} - \sum_{n+1}^{\infty} c_k{}^2\right).$$

Hence

$$\lim_{n\to\infty}\left|\int_{t_0}^{t_0/2} G_n(t)\,dt\right| = 0.$$

But by Theorem 2 $G_n(t)$ is monotonic over the range of this integral and takes its smallest value at t_0; i.e.

$$G_n(t_0)\frac{|t_0|}{2} \leqq \left|\int_{t_0}^{t_0/2} G_n(t)dt\right|.$$

From this inequality the result follows immediately.

THEOREM 7. *If*

1. *$G(t)$ is defined as in Theorem* 1,
2. *$\varphi(t)$ is integrable on every finite interval,*
3. *$f(x) = \int_{-\infty}^{\infty} G(x-t)\varphi(t)\,dt$ converges,*
4. *$F_n^*(D)$ is defined by* (2) of §4,

and

5. *$\varphi(t)$ is continuous at x,*

then

$$\lim_{n\to\infty} F_n^*(D)f(x) = \varphi(x).$$

Proof. By Corollary 5 we have

$$F_n^*(D)f(x) = \int_{-\infty}^{\infty} G_n(x-t)\varphi(t)\,dt.$$

Since $G_n(t)$ is a frequency function

$$F_n^*(D)f(x)-\varphi(x)=\int_{-\infty}^{\infty}G_n(x-t)[\varphi(t)-\varphi(x)]\,dt$$

By the condition 5, for an arbitrary $\varepsilon>0$ we may choose $\delta>0$ so small that

$$|\varphi(t)-\varphi(x)|\leqq\varepsilon \qquad |t-x|\leqq\delta.$$

Put

$$\int_{-\infty}^{\infty}G_n(x-t)[\varphi(t)-\varphi(x)]dt=\int_{-\infty}^{x-\delta}+\int_{x-\delta}^{x+\delta}+\int_{x+\delta}^{\infty}=I_1+I_2+I_3, \text{ say.}$$

We have

$$|I_2|\leqq\varepsilon\int_{x-\delta}^{x+\delta}G_n(x-t)\,dt\leqq\varepsilon\int_{-\infty}^{\infty}G_n(t)\,dt=\varepsilon.$$

$$I_3=\int_{x+\delta}^{\infty}G_n(x-t)[\varphi(t)-\varphi(x)]\,dt=\int_{-\infty}^{\infty}\left[\frac{G_n(x-t)}{G(x-t)}\right][G(x-t)\{\varphi(t)-\varphi(x)\}]\,dt.$$

By Theorem 1 and Theorem 4, for $\varepsilon>0$, there exists T_0 such that for sufficiently large n

$$\left|\frac{G_n(x-t)}{G(x-t)}\right|<\varepsilon \qquad (t>T_0).$$

Thus

$$\left|\int_{T_0}^{\infty}\left[\frac{G_n(x-t)}{G(x-t)}\right][G(x-t)\{\varphi(t)-\varphi(x)\}]dt\right|<\varepsilon O(1).$$

Furthermore by Lemma 4 we have

$$\lim_{n\to\infty}\left|\int_{x+\delta}^{T_0}G_n(x-t)\{\varphi(t)-\varphi(x)\}dt\right|=0.$$

Hence

$$\overline{\lim_{n\to\infty}}\,|I_3|\leqq\varepsilon(1+O(1)),$$

and similarly

$$\overline{\lim_{n\to\infty}}\,|I_1|\leqq\varepsilon(1+O(1)).$$

Thus we get

$$\overline{\lim_{n\to\infty}}\left|\int_{-\infty}^{\infty}G_n(x-t)[\varphi(t)-\varphi(x)]dt\right|\leqq\varepsilon O(1),$$

Since ε is arbitrary our theorem is proved.

THEOREM 8. *If*

1. $G(t)$ *is defined as in Theorem* 1,
2. $\alpha(t)$ *is of bounded variation in every finite interval,*
3. $f(x)=\int_{-\infty}^{\infty}G(x-t)\,d\alpha(t)$ *converges,*
4. $F_n^*(D)$ *is defined by* (2) of §4,

and

5. *$\alpha(t)$ is continuous at x_1 and x_2,*

then

$$\lim_{n\to\infty}\int_{x_1}^{x_2} F_n{}^*(D)f(x)\,dx = \alpha(x_2) - \alpha(x_1).$$

This is an immediate consequence of Theorem 6 and Theorem 7.

REFERENCES

[1] HIRSCHMAN, I. I., AND D. V. WIDDER, The convolution Transform. Princeton, 1955.
[2] WIDDER, D. V., The Laplace Transform. Princeton, 1941.

HIROSAKI UNIVERSITY.