ON JULIA-LINES OF DIRICHLET SERIES

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1. Introduction.

Let us put

(1.1) $$F(s)=\sum a_n \exp(-\lambda_n s) \quad (s=\sigma+it,\ 0\leqq\lambda_1<\lambda_2<\cdots<\lambda_n\to\infty).$$

We begin with some definitions.

DEFINITION 1. *Let* (1.1) *be simply convergent in the whole plane. We call the horizontal line* $t=t_0$ *the Julia-line, provided that* (1.1) *takes every value, except perhaps two* (∞ *included*), *infinitely many times in any strip* $|t-t_0|<\varepsilon$, ε *being any positive constant.*

DEFINITION 2. *Under the same assumptions as above, the horizontal line* $t=t_0$ *is called the argument-line, provided that* $|F(s)|$ *tends uniformly to infinity, and* $\arg F(s)$ *assumes every argument* $\theta \pmod{2\pi}$ *infinitely many times in any strip* $|t-t_0|<\varepsilon$, ε *being any positive constant.*

Mandelbrojt has established the following theorems.

THEOREM A. ([4] p. 16, [3] theorem 1.) *Let* (1.1) *with* $\varliminf_{n\to+\infty}(\lambda_n/n)\geqq G>0$ *be simply* (*necessarily absolutely*) *convergent in the whole plane. Then following alternatives are possible*:

[I] *there exists at least one Julia-line in the strip* $|t-t_0|\leqq\pi/G$, *where* t_0 *is an arbitrary but fixed constant, or*

[II] (1.1) *tends uniformly to infinity in the strip* $|t-t_0|\leqq\pi/G$.

THEOREM B. ([5] p. 268.) *Let* (1.1) *with* $\varliminf_{n\to+\infty}(\lambda_{n+1}-\lambda_n)>0$ *be simply* (*necessarily absolutely*) *convergent in the whole plane, and be of positive order* ρ.[1] *Then* (1.1) *has at least one Julia-line in any strip* $|t-t_0|\leqq \mathrm{Max}(\pi\bar{D}^{\cdot}, \pi/2\rho)$, *where* t_0 *is an arbitrary but fixed constant, and* $\bar{D}^{\cdot}$ *is the superior mean density of* $\{\lambda_n\}$ ([5] p. 51).

The object of this note is to prove the next theorem, which is a genera-

Received October 29, 1958.

1) The order ρ of (1.1) is defined by

$$\rho=\varlimsup_{\sigma\to-\infty}\frac{1}{-\sigma}\log^+\log^+ M(\sigma), \qquad \text{where} \quad M(\sigma)=\sup_{-\infty<t<+\infty}|F(\sigma+it)|.$$

lization of Mandelbrojt's theorem. The proof of our theorem is based upon Mandelbrojt's ideas ([2] pp. 185–188).

THEOREM. *Let* (1.1) *with* $\underline{\lim}_{n\to+\infty}(\lambda_{n+1}-\lambda_n)=h>0$ *and* $\overline{\lim}_{n\to+\infty} n/\lambda_n =\delta$ ($\leqq 1/h$) *be simply* (*necessarily absolutely*) *convergent in the whole plane, and be of order* ρ. *If* $\rho>0$, *then* (1.1) *has at least one Julia-line in any strip* $|t-t_0|\leqq\pi\delta$, t_0 *being arbitrary but fixed. If* $\rho=0$, *then there exists at least one Julia-line or an argument-line in any strip* $|t-t_0|\leqq\pi\delta$.

REMARK. (1) In our theorem, *the width of the horizontal strip is independent of* ρ.

(2) On the relation between δ and $\bar{D}^{\cdot}$, we know that

$$\bar{D}^{\cdot}\leqq\delta\leqq e\bar{D}^{\cdot} \qquad \text{([5] pp. 51–53).}$$

As a corollary, we get an analogue of Fabry's gap-theorem.

COROLLARY. *Let* (1.1) *with* $\underline{\lim}_{n\to+\infty}(\lambda_{n+1}-\lambda_n)>0$, $\lim_{n\to+\infty} n/\lambda_n=0$ *be simply* (*necessarily absolutely*) *convergent in the whole plane, and be of order* ρ. *If* $\rho>0$, *then* (1.1) *has every line* $t=t_0$ *as the Julia-line,* t_0 *being arbitrary but fixed. If* $\rho=0$, *then* (1.1) *has every line* $t=t_0$ *as the Julia-line or the argument-line.*

2. Lemmas.

To establish our theorem, we need some lemmas. The next lemma is a generalization of Mandelbrojt's lemma ([3] pp. 13–14).

LEMMA 1. (Tanaka [8] p. 424). *Under the same assumptions as in the theorem, we have*

$$\sup_{\Re(s)=\Re(s_0)}|F(s)|\leqq A\max_{|u-s_1|=\pi(\delta+\varepsilon)}|F(u)|,$$

where (i) ε *is any given positive constant,* (ii) s_0 *and* s_1 *are two arbitrary points satisfying* $\Re(s_1)=\Re(s_0)-\Delta(\varepsilon)$, *where* $\Delta(\varepsilon)=3\delta\log(e^6/h\delta)+2\varepsilon$, (iii) A *is a constant depending upon* ε, δ *and* $\{\lambda_n\}$.

LEMMA 2. *Let* (1.1) *be uniformly convergent in the whole plane*[2] *and be of positive order* ρ. *Then, for any positive* Δ_1, *there exists a constant* α *dependent upon* Δ_1 *and* ρ *such that*

$$\overline{\lim_{\sigma\to-\infty}}\frac{\log^+M(\sigma-\Delta_1)}{\log^+M(\sigma)}\geqq\alpha>1,$$

where

2) It means the uniform convergence-abscissa of (1.1) is equal to $-\infty$.

$$M(\sigma)=\sup_{-\infty<t<+\infty}|F(\sigma+it)|.$$

Proof. Suppose that

$$\overline{\lim_{\sigma\to-\infty}}\frac{\log^+ M(\sigma-\varDelta_1)}{\log^+ M(\sigma)}<\beta. \tag{2.1}$$

Then, there exists a constant σ_0 (<0) such that

$$\log^+ M(\sigma-\varDelta_1)<\beta\log^+ M(\sigma)\qquad \text{for}\quad \sigma\leqq\sigma_0.$$

Hence

$$\log^+ M(\sigma_0-\varDelta_1)<\beta\log^+ M(\sigma_0),$$
$$\log^+ M(\sigma_0-2\varDelta_1)<\beta\log^+ M(\sigma_0-\varDelta_1),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots,$$
$$\log^+ M(\sigma_0-n\varDelta_1)<\beta\log^+ M(\sigma_0-(n-1)\varDelta_1),$$

so that

$$\log^+ M(\sigma_0-n\varDelta_1)<\beta^n\log^+ M(\sigma_0). \tag{2.2}$$

By the definition of ρ, there exists a sequence $\{\sigma_k\}$ $(\sigma_1>\sigma_2>\cdots>\sigma_k\to-\infty)$ such that

$$\rho=\overline{\lim_{\sigma\to+\infty}}\frac{1}{-\sigma}\log^+\log^+ M(\sigma)=\lim_{k\to+\infty}\frac{1}{-\sigma_k}\log^+\log^+ M(\sigma_k). \tag{2.3}$$

We can easily choose a sequence of positive integers $\{n_k\}$ such that

$$\sigma_0-n_k\varDelta_1\leqq\sigma_k<\sigma_0-(n_k-1)\varDelta_1\qquad (k=1,2,\cdots). \tag{2.4}$$

Hence, by (2.4) and (2.2)

$$\log^+ M(\sigma_k)\leqq\log^+ M(\sigma_0-n_k\varDelta_1)<\beta^{n_k}\log^+ M(\sigma_0),$$

so that

$$\frac{1}{-\sigma_k}\log^+\log^+ M(\sigma_k)<\left\{1-\frac{\sigma_0+\varDelta_1}{\sigma_k}\right\}\cdot\log^+\frac{\beta}{\varDelta_1}+\frac{1}{-\sigma_k}\log^+\log^+ M(\sigma_0).$$

Letting $k\to+\infty$, we get

$$\exp(\rho\varDelta_1)\leqq\beta. \tag{2.5}$$

Hence (2.1) implies (2.5). Therefore, from $\exp(\rho\varDelta_1)>\alpha$, we can conclude that

$$\overline{\lim_{\sigma\to-\infty}}\frac{\log^+ M(\sigma-\varDelta_1)}{\log^+ M(\sigma)}\geqq\alpha.$$

Since $\rho>0$, we can evidently choose a constant α such that

$$\exp(\rho\varDelta_1)>\alpha>1,$$

which proves our lemma 2.

LEMMA 3. *Under the same assumptions as in the theorem, if $\rho>0$, then we can find two sequences $\{s_n\}$, $\{s_n'\}$ (Fig. 1) and a constant α such that*

$$\varlimsup_{n\to+\infty} \frac{\log|F(s_n')|}{\log|F(s_n)|} \geqq \alpha > 1,$$

where (i) $s_n=\sigma_n+it_n$, $s_n'=\sigma_n'+it_n$, $|t_n-t_0|\leqq\pi(\delta+\varepsilon)$, $|\sigma_n-\sigma_n'|\leqq\varDelta_1+\pi(\delta+\varepsilon)$; (ii) t_0 *is an arbitrary real constant,* $\varDelta_1$ *and* ε *are any positive constants,* $\varDelta(\varepsilon)$ *is the constant in lemma* 1, *and* α *is the constant in lemma* 2.

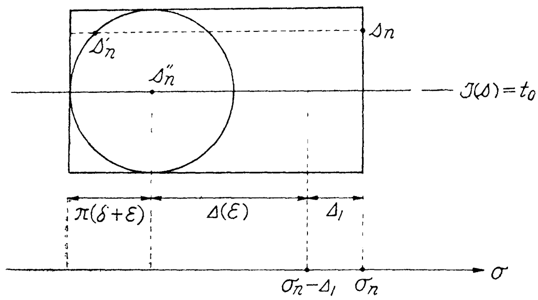


Fig. 1.

Proof. By lemma 2, there exists a sequence $\{\sigma_n\}$ $(\sigma_1>\sigma_2>\cdots>\sigma_n\to-\infty)$ such that

$$\varlimsup_{n\to+\infty}\frac{\log M(\sigma_n-\varDelta_1)}{\log M(\sigma_n)}\geqq\alpha>1. \tag{2.6}$$

On account of lemma 1, in which we put

$$\Re(s_0)=\sigma_n-\varDelta_1,\qquad s_1=s_n''=\{\sigma_n-\varDelta_1-\varDelta(\varepsilon)\}+it_0,$$

we can choose s_n' such that

$$M(\sigma_n-\varDelta_1)\leqq A\max_{|u-s_n''|=\pi(\delta+\varepsilon)}|F(u)|=A|F(s_n')|.$$

Hence

$$\log|F(s_n')|\geqq\log M(\sigma_n-\varDelta_1)-\log A. \tag{2.7}$$

Putting $s_n=\sigma_n+it_n$, $t_n=\Im(s_n')$, we have evidently

$$\log|F(s_n)|\leqq\log M(\sigma_n),$$

so that, by (2.7) and (2.6)

$$\varlimsup_{n\to+\infty}\frac{\log|F(s_n')|}{\log|F(s_n)|}\geqq\varlimsup_{n\to+\infty}\frac{\log M(\sigma_n-\varDelta_1)}{\log M(\sigma_n)}\geqq\alpha>1,$$

which proves lemma 3.

LEMMA 4. (Gronwall [1] pp. 316–318.) *Let $f(z)$ $(f(0)=0,\ f'(0)=1)$ be regular for $|z|<1$ and map $|z|<1$ conformally onto a convex domain. Then*

$$\frac{1}{(1+|z|)^2}\leqq|f'(z)|\leqq\frac{1}{(1-|z|)^2}.$$

LEMMA 5. (Mandelbrojt [2] p. 176, p. 197; [5] p. 265.)

(I) *Let* $\mathfrak{F} = \{f(z)\}$ $(|f(z)| > 1)$ *be the family of the analytic functions in* $\mathfrak{D}$. *Then, to any domain* $\mathfrak{D}_1$ *completely contained in* $\mathfrak{D}$, *there corresponds a constant* $\beta(\mathfrak{D}_1)$ (>1) *dependent upon* $\mathfrak{D}_1$ *such that*

$$\frac{1}{\beta} < \frac{\log |f(z_1)|}{\log |f(z_0)|} < \beta,$$

where z_0 *and* z_1 *are two arbitrary points contained in* $\mathfrak{D}_1$ *and* $f(z)$ *is any function belonging to* $\mathfrak{F}$.

(II) *Let* $\mathfrak{D}$ *be the unit circle* $|z| < 1$, *and* $\mathfrak{D}_1$ *be the circle* $|z| \leqq R$ (<1). *Then*

$$\lim_{R\to 0} \beta(\mathfrak{D}_1) = 1.$$

LEMMA 6. *Under the same conditions as in the theorem, if* $F(s) \neq 0$, *and* $|\arg F(s)| < 2m\pi$ *in the half-strip* $S(\varepsilon)$:

$$|t - t_0| \leqq \pi(\delta + \varepsilon), \qquad \sigma \leqq \sigma_0,$$

then

$$\begin{aligned} &\left|\sum_{\lambda_n < \omega} a_n \exp(-\lambda_n s)\{1 - \exp(\lambda_n - \omega)\}\right| \\ &\geqq |F(s)|\left\{1 - A\gamma^{2m}\exp\left(-\frac{\omega}{2}\right)\right\} \qquad \textit{for} \quad s \in S\left(\frac{\varepsilon}{2}\right), \end{aligned}$$

where (i) A *is the constant in lemma* 1 *and* (ii) γ (>1) *is a constant independent of* ω.

Proof. By the well-known Perron's formula ([7] p. 5), we get

$$F(s) = \sum_{\lambda_n < \omega} a_n \exp(-\lambda_n s)\{1 - \exp(\lambda_n - \omega)\} - R_\omega(s), \tag{2.8}$$

where

(i) $R_\omega(s) = \dfrac{1}{2\pi i}\displaystyle\int_{\beta - i\infty}^{\beta + i\infty} F(z) \frac{\exp(\omega(z - s))}{(z - s)(z + 1 - s)} dz,$

(ii) $\Re(s) = \sigma > \beta > \sigma - 1.$

Setting $\beta = \sigma - 1/2$, $M(\sigma) = \mathrm{Sup}_{-\infty < t < +\infty} |F(\sigma + it)|$,

$$\begin{aligned} |R_\omega(s)| &\leqq \frac{1}{\pi} M\left(\sigma - \frac{1}{2}\right)\exp\left(-\frac{\omega}{2}\right)\int_0^{+\infty} \frac{1}{\tau^2 + (1/2)^2} d\tau \\ &= M\left(\sigma - \frac{1}{2}\right)\exp\left(-\frac{\omega}{2}\right). \end{aligned}$$

Hence, by lemma 1 in which, replacing ε by $\varepsilon/2$, we put

$$s_0 = \left(\sigma - \frac{1}{2}\right) + it_0, \qquad s_1 = s_0 - \Delta \qquad \text{(Fig. 2)},$$

$$|R_\omega(s)| \leqq A\,|F(s_2)|\exp\left(-\frac{\omega}{2}\right), \tag{2.9}$$

where $|s_1 - s_2| = \pi(\delta + \varepsilon/2)$. Therefore, by (2.8) and (2.9),

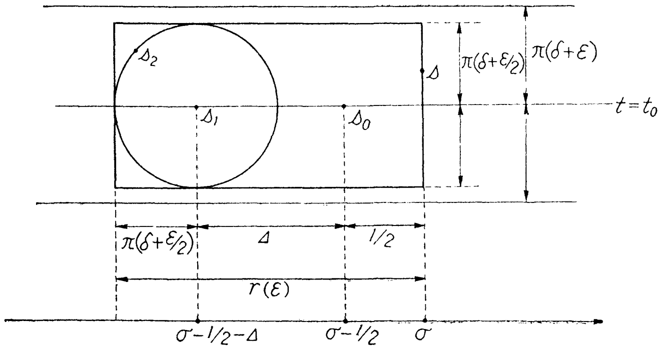


Fig. 2.

$$\text{(2.10)} \qquad \sum_{\lambda_n<\infty} a_n \exp(-\lambda_n s)\{1-\exp(\lambda_n-\omega)\} \geqq |F(s)|\left\{1-A\frac{F(s_2)}{F(s)}\exp\left(-\frac{\omega}{2}\right)\right\}.$$

Let us consider the function-family $\{F_n(s)\}=\{F(s-n)\}$ in the rectangle

$$D: \qquad |t-t_0|\leqq\pi\left(\delta+\frac{3\varepsilon}{4}\right), \qquad |\sigma|\leqq\frac{1}{2}+r(2\varepsilon),$$

where

$$r(\varepsilon)=\frac{1}{2}+\Delta+\pi\left(\delta+\frac{\varepsilon}{2}\right) \qquad \text{(Fig. 2).}$$

Since $|\Re(s_2)-\Re(s)|\leqq r(\varepsilon)$, there exist an integer n_k and two points s_{n_k}, s'_{n_k} such that

$$\text{(2.11)} \qquad F(s)=F_{n_k}(s_{n_k}), \qquad F(s_2)=F_{n_k}(s'_{n_k}),$$

where

(i) $s=s_{n_k}-n_k$, $s_2=s'_{n_k}-n_k$,

(ii) s_{n_k}, $s'_{n_k}\in D'$; D' denoting the rectangle

$$|t-t_0|\leqq\pi(\delta+\varepsilon/2), \qquad |\sigma|\leqq 1/2+r(\varepsilon).$$

Setting

$$f_n(s)=\left(\frac{F_n(s)}{F_n(it_0)}\right)^{1/4m}, \text{ [3]}$$

we get easily

$$f_n(it_0)=1, \qquad |\arg f_n(s)|<\pi \qquad \text{for} \quad s\in D.$$

Hence, the function-family $\{f_n(s)\}$ is normal in D, and bounded at it_0, so that $\{f_n(s)\}$ is uniformly bounded in D' ($\subset D$). By the entirely similar arguments, $\{1/f_n(s)\}$ is also uniformly bounded in D'. Thus, there exists a constant $\gamma(D')>1$ such that

$$\frac{1}{\gamma}<|f_n(s)|<\gamma \qquad \text{for} \quad s\in D' \qquad (n=1, 2, \cdots).$$

3) The following argument is due to Mandelbrojt ([2] p. 177).

Hence, by (2.11)

$$\left|\frac{F(s_2)}{F(s)}\right| < \gamma^{8m},$$

so that, by (2.0)

$$\sum_{\lambda_n<\omega} a_n \exp(-\lambda_n s)\{1-\exp(\lambda_n-\omega)\}$$
$$\geqq |F(s)|\left\{1-A\gamma^{8m}\exp\left(-\frac{\omega}{2}\right)\right\} \qquad \text{for} \quad s\in S\left(\frac{\varepsilon}{2}\right),$$

which proves our lemma 6.

LEMMA 7. *Under the same conditions as in the theorem, let us denote by* $N_\omega(t_1, t_2, \sigma_0)$ *the number of zeros of* $S_\omega(s)=\sum_{\lambda_k<\omega} a_k \exp(-\lambda_k s)\cdot\{1-\exp(\lambda_k-\omega)\}$ *contained in the half-strip* $t_1<t<t_2$, $\sigma<\sigma_0$. *Then*

$$N_\omega(t_1, t_2, \sigma_0) \geqq \frac{t_2-t_1}{2\pi}\lambda_{n(\omega)} - n(\omega) - K,$$

where

(i) $\lambda_{n(\omega)}$ *is the greatest exponent contained in* $0\leqq x<\omega$, *and* (ii) K *is a constant independent of* ω.

Proof. We have evidently

(2.12) $$N(t_1, t_2, \sigma_0) = \lim_{\sigma\to-\infty}(I_1+I_2+I_3+I_4),$$

where

$$I_1 = \frac{1}{2\pi}\int_{\substack{\sigma_0+it_1\\ \Re(s)=\sigma_0}}^{\sigma_0+it_2} d\arg S_\omega(s),$$
$$I_2 = \frac{1}{2\pi}\int_{\substack{\sigma_0+it_2\\ \Im(s)=t_2}}^{\sigma+it_2} d\arg S_\omega(s),$$
$$I_3 = \frac{1}{2\pi}\int_{\substack{\sigma+it_2\\ \Re(s)=\sigma}}^{\sigma+it_1} d\arg S_\omega(s),$$
$$I_4 = \frac{1}{2\pi}\int_{\substack{\sigma+it_1\\ \Im(s)=t_1}}^{\sigma_0+it_1} d\arg S_\omega(s),$$

Since $S_\omega(s)$ tends uniformly to $F(s)$ as $\omega\to+\infty$ on the segment $\Re(s)=\sigma_0$, $t_1\leqq\Im(s)\leqq t_2$, there exists a constant K independent of ω such that

(2.13) $$|I_1|<K.$$

Since

$$S_\omega(s) = a_{n(\omega)}\exp(-\lambda_{n(\omega)}s)\{1-\exp(\lambda_{n(\omega)}-\omega)\}(1+o(1))$$

on the segment $\Re(s)=\sigma$, $t_1\leqq\Im(s)\leqq t_2$, where $o(1)$ tends uniformly to 0 as $\sigma\to-\infty$, (2.14) $$\lim_{\sigma\to-\infty} I_3 = -\frac{1}{2\pi}\int_{t_2}^{t_1}\lambda_{n(\omega)}dt = \frac{t_2-t_1}{2\pi}\lambda_{n(\omega)}.$$

On the horizontal line $t=t_j$ $(j=1, 2)$, we can put

$$S_\omega(s) = X(t, \sigma) + iY(t, \sigma) \qquad (t=t_j,\ j=1, 2),$$

where

$$X(t, \sigma) = \sum_{i=1}^{n(\omega)}\gamma_i(t)\exp(-\lambda_i\sigma), \quad Y(t, \sigma) = \sum_{i=1}^{n(\omega)}\gamma_i^*(t)\exp(-\lambda_i\sigma)$$

We have easily

$$|I_2| \leqq \frac{m(t_2)}{2}, \qquad |I_4| \leqq \frac{m(t_1)}{2},$$

where $m(t)$ is the number of real roots of $X(t, \sigma)$ in $-\infty<\sigma<+\infty$. On the other hand, we know that the number of real roots of $\sum_{i=1}^{n}\gamma_i\exp(-\lambda_i\sigma)$ (γ_i and λ_i being real) does not exceed $n-1$ ([6] p. 49, problem 77). Hence

(2.15) $$|I_2| + |I_4| \leqq n(\omega) - 1 < n(\omega).$$

By (2.12), (2.13), (2.14) and (2.15),

$$N_\omega(t_1, t_2, \sigma_0) \geqq \frac{t_2-t_1}{2\pi}\lambda_{n(\omega)} - n(\omega) - K,$$

which proves our lemma 7.

3. Proof of the theorem.

We distinguish two cases.

Case I. $\rho>0$: By lemma 3, selecting suitable sub-sequences, if necessary, we can find two sequences $\{s_n\}$, $\{s_n'\}$ such tnat

(3.1) $$\lim_{n\to+\infty}\frac{\log|F(s_n')|}{\log|F(s_n)|} \geqq \alpha > 1,$$

where

(3.2)
$$\begin{aligned}
&(\text{i}) \quad s_n=\sigma_n+it, \qquad s_n'=(\sigma_n-\Delta_n)+it_n,\\
&(\text{ii}) \quad \lim t_n = T_0, \qquad |t_0-T_0| \leqq \pi(\delta+\varepsilon),\\
&(\text{iii}) \quad |\Delta_n| \leqq \Delta_1 + \Delta(\varepsilon) + \pi(\delta+\varepsilon).
\end{aligned}$$

Suppose that $w=f(z)$ maps $|z|<1$ conformally onto the rectangle R_1: $|\Re(w)|\leqq k$, $|\Im(w)|\leqq\varepsilon_1$ in such a manner that $f(0)=0$, $f'(0)=1$. Putting

$$r = \max_{0\leqq w\leqq p<k}|\varphi(w)|,$$

where $z=\varphi(w)$ denotes the inverse function of $w=f(z)$ (Fig. 3), by lemma 4, we get easily

$$r \leqq \int_{0\leqq w\leqq p}|\varphi'(w)||dw| = \int_{0\leqq w\leqq p}\frac{1}{|f'(z)|}|dw| \leqq (1+r)^2p,$$

so that

(3.3) $$r < 4p.$$

On account of lemma 5 (II), we can find sufficiently small $R\,(0 < R < 1)$ such that

$$1 < \beta(\mathfrak{D}_1) < \alpha,$$

where $\mathfrak{D}_1$ designates $|z| \leqq R$. Taking sufficiently small p, by (3.3) we can assume that

(3.4) $$r < 4p < R.$$

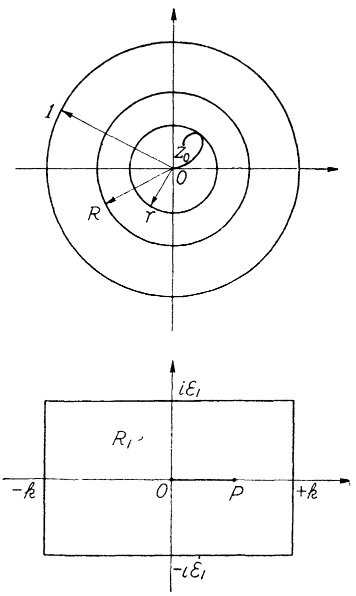
Now we map the rectangle R_1 in w-plane onto the rectangle $R_2(n)$ in s-plane by the linear transformation $s = g_n(w)$ (Fig. 3) such that

$$\text{(i)}\quad s = g_n(w) = \frac{\Delta_n}{p}(w - p) + s_n,$$

$$\text{(ii)}\quad s_n = g_n(p), \qquad s_n' = g_n(0).$$

Then the strip $|\Im(w)| \leqq \varepsilon_1$ corresponds to the strip $|\Im(s) - t_n| \leqq (\varepsilon_1 |\Delta_n|)/p$. Since Δ_n is bounded, and $\lim_{n\to\infty} t_n = T_0$, for any given $\varepsilon_2 > 0$, taking sufficiently small ε_1, we can assume that the rectangle $R_2(n)$ are contained in the strip $|\Im(s) - T_0| < \varepsilon_2$ for sufficiently large n.

Let us consider the function-family $\{F_n(z)\} = \{F(g_n(f(z)))\}$ in the domain $\mathfrak{D}$: $|z| < 1$. Then $\{F_n(z)\}$ is not normal in $\mathfrak{D}$. On the contrary,



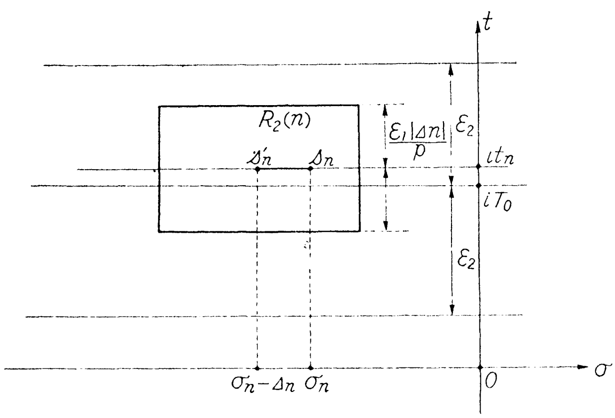


Fig. 3.

since $\lim_{n\to\infty} F_n(0) = \infty$ by (2.7), $F_n(z)$ tends uniformly to infinity in any domain completely contained in $\mathfrak{D}$. Then, by lemma 5 (I),

$$\frac{1}{\beta(\mathfrak{D}_1)} < \frac{\log|F_n(0)|}{\log|F_n(z_0)|} < \beta(\mathfrak{D}_1),$$

where $F_n(z_0) = F(s_n)$, so that

$$(3.5)\qquad \frac{1}{\beta(\mathfrak{D}_1)} < \frac{\log|F_n(s_n')|}{\log|F(s_n)|} < \beta(\mathfrak{D}_1) < \alpha,$$

which contradicts (3.1). Hence, $\{F_n(z)\}$ is not normal in $\mathfrak{D}$. In other words, $F(s)$ takes every value infinitely many times, except perphaps two (∞ included), in $\{R_2(n)\}$, a fortiori in $|\mathfrak{J}(s) - T_0| < \varepsilon_2$. Since ε_2 is arbitrary, $\mathfrak{J}(s) = T_0$ is a Julia-line. Letting $\varepsilon \to 0$ in (3.2) (ii), the first part of our theorem is established.

Case II. $\rho = 0$: By theorem A, it is sufficient to prove the existence of the argument-line in the case II of theorem A. In this case, we can determine σ_0 such that, in the half-strip $|t - t_0| \leqq \pi(\delta + \varepsilon)$, $\sigma \leqq \sigma_0$,

$$(3.6)\qquad |F(s)| > k > 0,$$

where k is a suitable constant. Applying lemma 7 to the half-strip $|t - t_0| \leqq \pi(\delta + \varepsilon/2)$, $\sigma \leqq \sigma_0$, we have

$$N_\omega(t_1, t_2, \sigma_0) \geqq \lambda_{n(\omega)}\left\{\left(\delta + \frac{\varepsilon}{2}\right) - \frac{n(\omega)}{\lambda_{n(\omega)}} - \frac{K}{\lambda_{n(\omega)}}\right\},$$

so that by $\overline{\lim}_{n\to\infty} n/\lambda_n = \delta$,

$$(3.7)\qquad N_\omega(t_1, t_2, \sigma_0) > \lambda_{n(\omega)} \cdot \frac{\varepsilon}{4}$$

for sufficiently large ω. If $|\arg F(s)| < 2m\pi$ in the half-strip $|t - t_0| \leqq \pi(\delta + \varepsilon)$, $\sigma \leqq \sigma_0$, then by (3.6) and lemma 6, for sufficiently large ω, we have

$$\left|\sum_{\lambda_n < \omega} a_n \exp(-\lambda_n s)\{1 - \exp(\lambda_n - \omega)\}\right| \geqq \frac{k}{2} > 0,$$

in the half-strip $|t - t_0| \leqq \pi(\delta + \varepsilon/2)$, $\sigma \leqq \sigma_0$, which contradicts (3.7). Hence, in the half-strip $|t - t_0| \leqq \pi(\delta + \varepsilon)$, $\sigma \leqq \sigma_0$, $|F(s)|$ tends uniformly to infinity, and moreover $\mathrm{Sup}\,|\arg F(s)| = +\infty$. Then there exists a sequence of points $\{s_n\}$ such that

$$\text{(i)}\quad \lim_{n\to\infty} |\arg F(s_n)| = +\infty,$$
$$\text{(ii)}\quad \lim_{n\to\infty} \mathfrak{J}(s_n) = t^*, \qquad |t^* - t_0| \leqq \pi(\delta + \varepsilon),$$

from which it follows that $\mathfrak{J}(s) = t^*$ is the argument-line. Letting $\varepsilon \to 0$, we can conclude the existence of the argument-line in the strip $|t - t_0| \leqq \pi\delta$. Thus the second part of our theorem is established.

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