ON ANALYTIC FUNCTIONS WITH POSITIVE REAL PART IN AN ANNULUS

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1. Introduction.

In a previous paper [5], considering a class of analytic functions regular and of positive real part in the unit circle, we have given an extension of Rogosinski's theorem [cf. 8] and further established several related mean distortion theorems. The purpose of the present paper is to show that corresponding results can be derived for an analogous class of analytic functions defined in an annulus. The previous results will then be regarded as limiting cases where the interior boundary component of the annulus degenerates to a single point.

Let $\mathfrak{R}=\{\Phi(z)\}$ be the class of analytic functions which are single-valued, regular and of positive real part in the annulus $(0<)\,q<|z|<1$ and normalized by the conditions

$$\Re\Phi(z)=1 \qquad \text{along} \quad |z|=q$$

and

$$\int_{-\pi}^{\pi}\Im\Phi(qe^{i\theta})\,d\theta=0.$$

The last normalization is supposed only for the unique determination of an inessential purely imaginary additive constant. These normalization conditions imply, in particular, that the constant term in the Laurent expansion o$\Phi(z)\in\mathfrak{R}$ is equal to unity. In fact, this quantity is given by

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi(qe^{i\theta})\,d\theta=\frac{1}{2\pi}\int_{-\pi}^{\pi}(\Re\Phi(qe^{i\theta})+i\,\Im\Phi(qe^{i\theta}))\,d\theta=1.$$

But, since the first condition depends not only on the mean value, it asserts, together with the second condition, more strong restriction.

2. Representation formula.

In case of the unit circle as a basic domain our method has depended on the Herglotz representation for the class considered. Correspondingly, we now derive an integral representation of Villat-Stieltjes type valid for any function of the class $\mathfrak{R}$. Though it is really a consequence of a more general

Received April 1, 1958.

representation formula given in a previous paper [3], it will play main role for the present purpose so that we formulate it here fully for the sake of completeness. Similar formulas are found also in various papers; for instance, cf. [1], [7], [6] etc.

According to circumstances we shall use notations in Weierstrassian theory of elliptic functions. Throughout the present paper, we suppose that they always concern those with primitive periods

$$2\omega_1 = 2\pi \qquad \text{and} \qquad 2\omega_3 = -2i \lg q,$$

unless the contrary is explicitly stated.

THEOREM 1. *Any $\Phi(z) \in \mathfrak{R}$ is representable by means of an integral of Villat-Stieltjes type in the form*

$$\Phi(z) = \frac{2}{i} \int_{-\pi}^{\pi} \left(\zeta(i \lg z + \varphi) - \frac{\eta_1}{\pi}(i \lg z + \varphi) \right) d\rho(\varphi)$$

where $\rho(\varphi)$ is a real-valued increasing function defined for $-\pi < \varphi \leqq \pi$ and with the total variation equal to unity, i.e.

$$d\rho(\varphi) \geqq 0 \ (-\pi < \varphi \leqq \pi) \quad \text{and} \quad \int_{-\pi}^{\pi} d\rho(\varphi) = 1.$$

Proof. We introduce a real-valued function defined by

$$\rho(r, \varphi) = \frac{1}{2\pi} \int_{-\pi}^{\varphi} \Re \Phi(re^{i\theta})\, d\theta \quad (-\pi < \varphi \leqq \pi, \ q < r < 1).$$

In view of assumption $\Re \Phi(re^{i\theta}) > 0$, it is an increasing function of φ for any fixed r with $q < r < 1$. Its total variation is equal to

$$\rho(r, \pi) = \Re \frac{1}{2\pi i} \int_{|z|=r} \frac{\Phi(z)}{z} dz = \Re c_0,$$

where c_0 denotes the constant term of the Laurent expansion of $\Phi(z)$; $\Phi(z)$ being single-valued in the annulus, the middle member of the last relation is independent of r. Remembering the boundary behavior of $\Phi(z)$ along $|z| = q$, it can be analytically prolonged across $|z| = q$. In view of its normalizations, we thus get

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi(qe^{i\theta})\, d\theta = 1.$$

Hence, $\rho(r, \varphi)$ is, qua function of φ, uniformly bounded for $q < r < 1$ and its total variation with respect to φ is always equal to unity; cf. the final remark in §1. Now, applying Villat's formula (cf. [9] or [2]) to $\Phi(z)$ with respect to a sub-annulus $q < |z| < r$, we get

$$\Phi(z) = \frac{1}{\pi i}\int_{-\pi}^{\pi} \Re\Phi(re^{i\theta})\,\zeta\Big(i \lg z + \theta,\ \frac{q}{r}\Big)\,d\theta$$

$$-\frac{1}{\pi i}\int_{-\pi}^{\pi} \Re\Phi(qe^{i\theta})\,\zeta_3\Big(i \lg z + \theta,\ \frac{q}{r}\Big)\,d\theta + ic$$

$$= \frac{2}{i}\int_{-\pi}^{\pi} \zeta\Big(i \lg z + \varphi,\ \frac{q}{r}\Big)\,d\rho(r, \varphi) - \frac{2}{\pi}\,\eta_1\Big(\frac{q}{r}\Big) \lg z + ic,$$

c being a real constant. Here the parameter q/r associated to zeta-functions means that their quasi-periods are 2π and $-2i \lg (q/r)$. Based on the properties of $\rho(r, \varphi)$ mentioned above, Helly's selection theorem shows that there exists a monotone increasing sequence $\{r_k\}$ with $r_k \to 1$ $(k \to \infty)$ for which the limit function

$$\rho(\varphi) = \lim_{k \to \infty} \rho(r_k, \varphi)$$

is defined throughout $-\pi < \varphi \leqq \pi$. Evidently, $\rho(\varphi)$ is increasing and its total variation is equal to unity. Now, on account of Lebesgue's convergence theorem, since the derivative $(\partial/\partial\varphi)\,\zeta(i \lg z + \varphi,\ q/r) = -\wp(i \lg z + \varphi,\ q/r)$ is continuous in φ as well as in r, we obtain the limit relation

$$\lim_{k \to \infty}\int_{-\pi}^{\pi} \zeta\Big(i \lg z + \varphi,\ \frac{q}{r_k}\Big)\,d\rho(r_k, \varphi)$$

$$= \lim_{k \to \infty}\left\{\Big[\zeta\Big(i \lg z + \varphi,\ \frac{q}{r_k}\Big)\,\rho(r_k, \varphi)\Big]_{\varphi=-\pi}^{\varphi=\pi} - \int_{-\pi}^{\pi} \rho(r_k, \varphi)\,\frac{\partial}{\partial\varphi}\zeta\Big(i \lg z + \varphi,\ \frac{q}{r_k}\Big)\,d\varphi\right\}$$

$$= \Big[\zeta(i \lg z + \varphi)\,\rho(\varphi)\Big]_{\varphi=-\pi}^{\varphi=\pi} - \int_{-\pi}^{\pi} \rho(\varphi)\,\frac{\partial}{\partial\varphi}\,\zeta(i \lg z + \varphi)\,d\varphi$$

$$= \int_{-\pi}^{\pi} \zeta(i \lg z + \varphi)\,d\rho(\varphi)$$

valid for any z in $q < |z| < 1$. On the other hand, $\eta_1(q/r)$ is continuous in r. Consequently, we obtain

$$\Phi(z) = \frac{2}{i}\int_{-\pi}^{\pi} \Big(\zeta(i \lg z + \varphi) - \frac{\eta_1}{\pi} i \lg z\Big)\,d\rho(\varphi) + ic.$$

It remains only to determine the value of the real constant c. For this purpose, we observe the integral of $\Phi(z)/z$ along $|z| = q$ which, after divided by $2\pi i$, becomes

$$1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi(qe^{i\theta})\,d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\left\{\frac{2}{i}\int_{-\pi}^{\pi} \Big(\zeta(i \lg q - \theta + \varphi) - \frac{\eta_1}{\pi}(i \lg q - \theta)\Big)\,d\rho(\varphi) + ic\right\}$$

$$= \int_{-\pi}^{\pi} d\rho(\varphi)\,\frac{1}{\pi i}\int_{-\pi}^{\pi} \Big(\zeta(i \lg q - \theta + \varphi) - \frac{\eta_1}{\pi}(i \lg q - \theta)\Big)\,d\theta + ic$$

$$= \int_{-\pi}^{\pi} \left\{ \left[-\frac{1}{\pi i} \lg \sigma(i \lg q - \theta + \varphi) \right]_{-\pi}^{\pi} - \frac{2\eta_1}{\pi} \lg q \right\} d\rho(\varphi) + ic$$

$$= \int_{-\pi}^{\pi} \left(1 - \frac{2\eta_1}{\pi} i\varphi \right) d\rho(\varphi) + ic.$$

Comparing the imaginary parts of both members, we get

$$0 = -\frac{2\eta_1}{\pi} \int_{-\pi}^{\pi} \varphi \, d\rho(\varphi) + c \qquad \text{i. e.} \quad c = \frac{2\eta_1}{\pi} \int_{-\pi}^{\pi} \varphi \, d\rho(\varphi).$$

Substituting this value of c, we finally get the desired representation for $\Phi(z)$.

The theorem just proved shows that any function of $\mathfrak{R}$ admits a representation of the type given there. It can be shown, moreover, that the representation is characteristic for the class $\mathfrak{R}$. In fact, the converse of theorem 1 holds also good.

THEOREM 2. *Let $\rho(\varphi)$ be a real-valued increasing function defined for $-\pi < \varphi \leqq \pi$ and with the total variation equal to unity. Then the function defined by*

$$\Phi(z) = \frac{2}{i} \int_{-\pi}^{\pi} \left(\zeta(i \lg z + \varphi) - \frac{\eta_1}{\pi}(i \lg z + \varphi) \right) d\rho(\varphi)$$

belongs to the class $\mathfrak{R}$.

Proof. The integrand $\zeta(i \lg z + \varphi) - (\eta_1/\pi)(i \lg z + \varphi)$ remains invariant under the substitution $\lg z \mid \lg z + 2\pi i$ and hence $\Phi(z)$ is single-valued in $q < |z| < 1$. It is evident that $\Phi(z)$ is regular in the annulus. We now observe the function defined by

$$H(z; \varphi) = \Re \frac{2}{i} \left(\zeta(i \lg z + \varphi) - \frac{\eta_1}{\pi}(i \lg z + \varphi) \right) = 2 \Im \zeta(i \lg z + \varphi) - \frac{2\eta_1}{\pi} \lg |z|.$$

It is regular and harmonic in z for $q < |z| < 1$. It can be expressed in two alternative ways:[1]

$$H(re^{i\theta}; \varphi) = \frac{1 - r^2}{1 - 2r \cos(\theta - \varphi) + r^2} + 2 \sum_{\nu=1}^{\infty} \frac{q^\nu (r^\nu - 1/r^\nu)}{1 - q^{2\nu}} \cos \nu(\theta - \varphi)$$

$$= 1 + 2 \sum_{\nu=1}^{\infty} \frac{r^\nu - q^{2\nu}/r^\nu}{1 - q^{2\nu}} \cos \nu(\theta - \varphi).$$

The first expression shows that $H(z; \varphi)$ has the boundary value vanishing everywhere along $|z| = 1$ except at a single point $z = e^{i\varphi}$ where it behaves like as the Poisson kernel $(1 - |z|^2)/|1 - z|^2$ which is positive for $|z| < 1$.

1) A similar argument has been made in [4]; cf. Lemma 4. 1. In this occasion, we wish to correct a misprint involved in [4]. In p. 114, l. 7 from bottom, $P(re^{i\theta}; \varphi)$ should be replaced by $2P(re^{i\theta}; \varphi)$.

The second expression shows that $H(z; \varphi)$ has the constant boundary value equal to unity along $|z| = q$. Consequently, by the assumption imposed on $\rho(\varphi)$, we conclude that

$$\Re \Phi(z) = \int_{-\pi}^{\pi} H(z; \varphi)\, d\rho(\varphi) > 0 \qquad (q < |z| < 1)$$

and $\Re\Phi(z) = 1$ along $|z| = q$. On the other hand, we further get, as shown in the proof of theorem 1,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi(qe^{i\theta})\, d\theta = \int_{-\pi}^{\pi} d\rho(\varphi) = 1,$$

whence follows, in particular,

$$\int_{-\pi}^{\pi} \Im \Phi(qe^{i\theta})\, d\theta = 0.$$

Thus, the function $\Phi(z)$ satisfies surely all the conditions imposed on the class $\Re$.

Throughout the present paper we shall observe exclusively the class $\Re$. However, the results which will be obtained for this class can be modified suitably so as to be valid for other analogous classes. The most essential point in our subsequent discussions is that any member of the class admits an integral representation of the form

$$\Phi(z) = \int_{-\pi}^{\pi} K(e^{-i\varphi}z)\, d\rho(\varphi)$$

with certain kernel $K(e^{-i\varphi}z)$ which is in itself a member of the class for every value of φ and corresponds to $(2/i)\,(\zeta(i \lg (e^{-i\varphi}z)) - (\eta_1/\pi) i \lg (e^{-i\varphi}z))$ in case of $\Re$. Cf. a similar remark stated at the end of §4 in [5].

3. Extremal mapping.

In case of the unit circle as a simply-connected basic domain, the linear function $(1+z)/(1-z)$ which maps $|z| < 1$ onto the right half-plane has played mostly the role of an extremal function for distortion theorems discussed in [5]. In our present case of the annulus $q < |z| < 1$ as a doubly-connected basic domain, the function

$$\Phi^*(z) = \frac{2}{i}\left(\zeta(i \lg z) - \frac{\eta_1}{\pi} i \lg z\right)$$

will play the corresponding role. Accordingly, in order to prevent the interruption of our main discourse, its mapping behavior will be investigated here preparatorily.

The function $\Phi^*(z)$ possesses as its associated function $\rho(\varphi)$ a particular one which remains unchanged except at a single jump with the height 1 at

$\varphi=0$. Hence, by theorem 2, it belongs surely to the class $\mathfrak{R}$. Further it is regular throughout the closed annulus $q \leqq |z| \leqq 1$ except at a simple pole at $z=1$.

Now, as shown in the previous paper [4; cf. Lemma 4. 2], for any fixed r with $q<r<1$ the quantity $\mathfrak{J}\zeta(i \lg r-\theta)$, qua function of θ, is an even periodic function with the period 2π which is strictly decreasing for $0 \leqq \varphi \leqq \pi$. Consequently, the quantity defined by

$$\mathfrak{R}\Phi^*(re^{i\theta}) = 2\mathfrak{J}\zeta(i \lg r - \theta) - \frac{2\eta_1}{\pi} \lg r$$

possesses the same properties. On the other hand, for any fixed r with $q<r<1$, the quantity defined by

$$\mathfrak{J}\Phi^*(re^{i\theta}) = -2\mathfrak{R}\zeta(i \lg r - \theta) - \frac{2\eta_1}{\pi}\theta$$

is evidently an odd function in θ. Hence, the function $\Phi^*(z)$ is univalent in $q<|z|<1$. Along the unit circumference it satisfies

$$\mathfrak{R}\Phi^*(e^{i\theta}) = 2\mathfrak{J}\zeta(-\theta) = 0 \qquad (0<|\theta| \leqq \pi).$$

Consequently, the image of $q<|z|<1$ by the univalent mapping $w=\Phi^*(z)$ is the right half-plane cut along a vertical rectilinear slit which lies on $\mathfrak{R}w=1$ and is bisected by the real axis. The point $z=1$ corresponds to $w=\infty$.

The relation $\mathfrak{R}\Phi^*(z)=1$ valid for $|z|=q$ has been remarked in the proof of theorem 2. But it can be shown alternatively in more direct way. In fact, we have, by taking Legendre's relation into account,

$$\begin{aligned}\mathfrak{R}\Phi^*(qe^{i\theta}) &= 2\mathfrak{J}\zeta(i \lg q - \theta) - \frac{2\eta_1}{\pi} \lg q \\ &= 2\mathfrak{J}(\zeta(-\theta) - \eta_3) - \frac{2\eta_1}{\pi} \lg q \\ &= -\frac{2\eta_3}{i} - \frac{2\eta_1}{\pi} \lg q = 1.\end{aligned}$$

We next consider the derivative of $\Phi^*(z)$ with respect to $\lg z$, i.e.

$$z\Phi^{*\prime}(z) = 2\left(-\wp(i \lg z) - \frac{\eta_1}{\pi}\right).$$

As $z=re^{i\theta}$ describes the circumference $|z|=1$ from 1 to -1 and then from -1 to 1 both in the positive sense with respect to $|z|<1$, the point $\Phi^*(z)$ describes the imaginary axis from $+i\infty$ to 0 and then from 0 to $-i\infty$ monotonously. Hence, the point

$$[z\Phi^{*\prime}(z)]^{z=e^{i\theta}} = \frac{1}{i}\frac{d}{d\theta}\Phi^*(e^{i\theta})$$

lies always on the negative real axis. On the other hand, as $z=re^{i\theta}$ describes the circumference $|z|=q$ in a definite sense, the point $w=\Phi^*(z)$ describes both banks of the slit lying on $\Re w=1$ monotonously. Hence, the point

$$[z\Phi^{*\prime}(z)]^{z=qe^{i\theta}}=\frac{1}{i}\frac{d}{d\theta}\Phi^*(qe^{i\theta})$$

lies always on the real axis. Now, the function $z\Phi^{*\prime}(z)$ is regular throughout the closed annulus $q\leqq|z|\leqq 1$ except at a single point $z=1$ where it has a double pole. Consequently, the function $z\Phi^{*\prime}(z)$ is univalent in $q<|z|<1$ and its image is the whole plane cut along an infinite half-line lying on the negative real axis as well as a finite slit containing the origin and lying on the real axis, these slits corresponding to $|z|=1$ and $|z|=q$, respectively.

The univalency of $z\Phi^{*\prime}(z)$ in $q<|z|<1$ can be verified alternatively. In fact, we have only to remember that

$$e^{i\theta}\Phi^{*\prime}(e^{i\theta})=2\left(-\wp(-\theta)-\frac{\eta_1}{\pi}\right)$$

is a real-valued even function in θ and, as θ moves from 0 to π, its value moves monotonously from $-\infty$ to a finite value. It may be noted by the way that this finite value representing the end-point of the infinite slit is equal to

$$2\left(-\wp(-\pi)-\frac{\eta_1}{\pi}\right)=-2\left(e_1+\frac{\eta_1}{\pi}\right)=-\left(\frac{1}{2}+2\sum_{\nu=1}^{\infty}\frac{q^{2\nu}}{(1+q^{2\nu})^2}\right)$$

which is evidently negative; cf. [1].

On the other hand, two end-points of the finite slit are the images of $z=-q$ and $z=q$ and have the affixes

$$2\left(-\wp(i\lg q-\pi)-\frac{\eta_1}{\pi}\right)=-2\left(e_2+\frac{\eta_1}{\pi}\right)\text{ and }2\left(-\wp(i\lg q)-\frac{\eta_1}{\pi}\right)=-2\left(e_3+\frac{\eta_1}{\pi}\right)$$

which are negative and positive, respectively. Two boundary elements lying on the origin are the images of the points which correspond to the end-points of the slit originating from $|z|=q$ by the mapping $\Phi^*(z)$. Their arguments are the roots of the equation $\Phi^{*\prime}(qe^{i\theta})=0$, i.e.

$$\wp(i\lg q-\theta)+\frac{\eta_1}{\pi}=0$$

of which the left-hand member is even with respect to θ.

Finally, it will be convenient for later purpose to use series forms for $\Phi^*(z)$ and $z\Phi^{*\prime}(z)$ which are given by

$$\Phi^*(z)=\frac{2}{i}\left(\zeta(i\lg z)-\frac{\eta_1}{\pi}i\lg z\right)=1+2\sum_{\nu=-\infty}^{\infty}{}'\frac{z^\nu}{1-q^{2\nu}}$$

and

$$z\Phi^{*\prime}(z)=2\left(-\wp(i\lg z)-\frac{\eta_1}{\pi}\right)=2\sum_{\nu=-\infty}^{\infty}{}' \frac{\nu z^\nu}{1-q^{2\nu}}=2\sum_{\nu=-\infty}^{\infty}\frac{q^{2\nu}z}{(1-q^{2\nu}z)^2}\,.$$

Here the prime means that the summation extends over all the integers except zero.

4. A lemma.

In simply-connected case we have established a principal lemma for determining the whole of extremal functions in mean distortion theorems discussed there. It has been of considerably general nature and hence mostly satisfactory for our problems considered. It seems, however, difficult to extend the lemma directly to doubly-connected case. Accordingly, we restrict ourselves in the present case to a weaker and somewhat trivial form as formulated below. It is desirable to establish a proposition of more general nature.

LEMMA 1. *Let $f(\theta)$ be a complex-valued continuous function of a real variable which has $2\pi/n$ as a period, n being a positive integer. Let further $w=f(\theta)$ describe for $0\leqq\theta<2\pi/n$ a (closed) Jordan curve. Let $\rho(\varphi)$ be a real-valued function defined for $-\pi<\varphi\leqq\pi$ which is increasing and has the total variation equal to $P>0$. Then, in order that the relation*

$$\left|\int_{-\pi}^{\pi}f(\theta-\varphi)\,d\rho(\varphi)\right|=\int_{-\pi}^{\pi}|f(\theta-\varphi)|\,d\rho(\varphi)$$

holds identically with respect to θ throughout $-\pi<\theta\leqq\pi$, it is necessary and sufficient that $\rho(\varphi)$ remains unchanged except possibly at n jump-points which are distributed equidistantly in $-\pi<\varphi\leqq\pi$.

Proof. The sufficiency assertion is readily verified.[2] In fact, let $\rho(\varphi)$ remain unchanged except possibly at n equidistant jump-points $\varphi_0+2k\pi/n$ $(0\leqq k\leqq n-1)$ with the heights $\rho_k\geqq 0$, respectively. Since $f(\theta)$ is supposed periodic with $2\pi/n$ as its period, we then have

$$\begin{aligned}\left|\int_{-\pi}^{\pi}f(\theta-\varphi)\,d\rho(\varphi)\right|&=\left|\sum_{k=0}^{n-1}f\left(\theta-\varphi_0-\frac{2k\pi}{n}\right)\rho_k\right|=\left|\sum_{k=0}^{n-1}f(\theta-\varphi_0)\,\rho_k\right|\\&=|f(\theta-\varphi_0)|\,P=\sum_{k=0}^{n-1}|f(\theta-\varphi_0)|\,\rho_k\\&=\sum_{k=0}^{n-1}\left|f\left(\theta-\varphi_0-\frac{2k\pi}{n}\right)\right|\rho_k=\int_{-\pi}^{\pi}|f(\theta-\varphi)|\,d\rho(\varphi).\end{aligned}$$

To prove the necessity of the condition, suppose that the relation holds identically with respect to θ. Then, for any fixed θ, the quantity $\arg f(\theta-\varphi)$ must have the same value depending only on θ for every value of φ with $d\rho(\varphi)>0$, provided $f(\theta-\varphi)$ does not vanish. Let φ_0 and φ_1 be any values

2) For the sufficiency proof, we need not assume any mapping character of $f(\theta)$.

of φ with $d\rho(\varphi)>0$. By assumption, the quantity $f(\theta-\varphi)$ describes for every φ a Jordan curve, as θ varies from 0 to $2\pi/n$. Hence, in order that the equation $\arg f(\theta-\varphi_0)=\arg f(\theta-\varphi_1)$ can hold identically with respect to θ, it is necessary that $\varphi_1-\varphi_0\equiv 0 \pmod{2\pi/n}$ as desired.

5. Main results.

We are now in position to enter in our main discourse which proceeds for the most part similarly as in the simply-connected case previously discussed. We observe namely a linear operator $\mathcal{L}$ which has $\mathfrak{R}$ as its domain of argument function and produces by applying to any $\Phi(z)\in\mathfrak{R}$ an analytic function $\mathcal{L}[\Phi(z)]$ single-valued in $q<|z|<1$. It is supposed further that the operator is homogeneous of degree zero, i.e., for any constant c, the function $\mathcal{L}[\Phi(z)]$ coincides after substitution $z|cz$ with $\mathcal{L}[\Phi(cz)]$.

We begin with a general mean distortion theorem. Though its proof proceeds quite as in the simply-connected case, we reproduce it here fully since this theorem is fundamental for subsequent discussions.

THEOREM 3. *Let $\mathcal{L}$ denote a linear operator defined for $\mathfrak{R}$ and $\mathcal{L}[\Phi(z)]$ applied to $\Phi(z)\in\mathfrak{R}$ be an analytic function single-valued and meromorphic in $q<|z|<r\,(<1)$ and regular along $|z|=q$. Let $\mathcal{L}$ be commutable with the integration with respect to $\rho(\varphi)$ in the representation for $\Phi(z)$ stated in theorem* 1. *Let further $F(X)$ be a bounded increasing convex (and hence necessarily continuous) function defined for the range of $|\mathcal{L}|$. Then, for any $\Phi(z)\in\mathfrak{R}$, we have*

$$\int_{-\pi}^{\pi} F(|\mathcal{L}[\Phi(re^{i\theta})]|)\,d\theta \leqq \int_{-\pi}^{\pi} F(|\mathcal{L}[\Phi^*(re^{i\theta})]|)\,d\theta,$$

where $\Phi^*(z)$ denotes the particular function

$$\Phi^*(z)=\frac{2}{i}\Big(\zeta(i\lg z)-\frac{\eta_1}{\pi}i\lg z\Big).$$

The function $\Phi^*(\varepsilon z)$ is for any constant ε with $|\varepsilon|=1$ always an extremal function of this estimation.

Proof. The representation formula for $\Phi(z)$ given in theorem 1 implies

$$|\mathcal{L}[\Phi(re^{i\theta})]|=\Big|\int_{-\pi}^{\pi}\mathcal{L}[\Phi^*(re^{i(\theta-\varphi)})]\,d\rho(\varphi)\Big|\leqq\int_{-\pi}^{\pi}|\mathcal{L}[\Phi^*(re^{i(\theta-\varphi)})]|\,d\rho(\varphi),$$

since the operation $\mathcal{L}$ and the integration with respect to $\rho(\varphi)$ are supposed commutable. The increasing character and the convexity of $F(X)$ imply further

$$F(|\mathcal{L}[\Phi(re^{i\theta})]|)\leqq F\Big(\int_{-\pi}^{\pi}|\mathcal{L}[\Phi^*(re^{i(\theta-\varphi)}]|\,d\rho(\varphi)\Big)$$
$$\leqq\int_{-\pi}^{\pi}F(|\mathcal{L}[\Phi^*(re^{i(\theta-\varphi)})]|)\,d\rho(\varphi),$$

since $\rho(\varphi)$ is increasing for $-\pi<\varphi\leqq\pi$ and has the total variation equal to unity. Hence, integrating with respect to θ, we get

$$\int_{-\pi}^{\pi} F(|\mathcal{L}[\Phi(re^{i\theta})]|)\,d\theta \leqq \int_{-\pi}^{\pi} d\theta \int_{-\pi}^{\pi} F(|\mathcal{L}[\Phi^*(re^{i(\theta-\varphi)})]|)\,d\rho(\varphi)$$

$$= \int_{-\pi}^{\pi} d\rho(\varphi) \int_{-\pi}^{\pi} F(|\mathcal{L}[\Phi^*(re^{i(\theta-\varphi)})]|)\,d\theta = \int_{-\pi}^{\pi} F(|\mathcal{L}[\Phi^*(re^{i\theta})]|)\,d\theta.$$

Finally, for any $\varepsilon=e^{-i\varphi_0}$ with real φ_0, we have

$$\int_{-\pi}^{\pi} F(|\mathcal{L}[\Phi^*(re^{i(\theta-\varphi_0)})]|)\,d\theta = \int_{-\pi-\varphi_0}^{\pi-\varphi_0} F(|\mathcal{L}[\Phi^*(re^{i\theta})]|)\,d\theta$$

$$= \int_{-\pi}^{\pi} F(|\mathcal{L}[\Phi^*(re^{i\theta})]|)\,d\theta,$$

since the integrand of the last member is periodic in θ with the period equal to 2π.

As a supplement to theorem 3, it must be natural to consider the problem of determining extremal functions in the estimation. But lemma 1 which serves for this purpose has been formulated in a weaker form than in the simply-connected case. Accordingly, to deal with the problem, we have here to impose an additional restriction on the operator $\mathcal{L}$ which seems somewhat superfluous.

THEOREM 4. *Under the conditions imposed on $\mathcal{L}$ in theorem* 3, *suppose further that $\mathcal{L}[\Phi^*(z)]$ is non-constant and that the function $F(X)$ is strictly increasing. Moreover, let $\mathcal{L}[\Phi^*(z)]$ be a single-valued function of z^n and the image-curve of the arc $|z|=r$, $0\leqq \arg z<2\pi/n$ by the mapping $w=\mathcal{L}[\Phi^*(z)]$ be a Jordan curve. Then, the equality sign in the estimation given in theorem* 3 *holds if and only if $\Phi(z)$ is of the form*

$$\Phi(z)=\sum_{k=0}^{n-1}\rho_k\,\Phi^*(e^{-2k\pi i/n}\varepsilon z)$$

where ε is a constant with $|\varepsilon|=1$ and $\{\rho_k\}$ is a set of n real numbers satisfying

$$\rho_k\geqq 0 \quad (k=0,1,\cdots,n-1), \qquad \sum_{k=0}^{n-1}\rho_k=1.$$

Proof. Based on the proof given above for theorem 3, we see that the extremal character of $\Phi(z)$ is characterized in terms of its associated function $\rho(\varphi)$ by the requirements

$$\left|\int_{-\pi}^{\pi}\mathcal{L}[\Phi^*(re^{i(\theta-\varphi)})]\,d\rho(\varphi)\right| = \int_{-\pi}^{\pi}|\mathcal{L}[\Phi^*(re^{i(\theta-\varphi)})]|\,d\rho(\varphi)$$

and

$$F\left(\int_{-\pi}^{\pi} |\mathcal{L}[\Phi^*(re^{i(\theta-\varphi)})]|\, d\rho(\varphi)\right) = \int_{-\pi}^{\pi} F(|\mathcal{L}[\Phi^*(re^{i(\theta-\varphi)})]|)\, d\rho(\varphi),$$

both to be valid identically with respect to θ. By applying lemma 1 to the function defined by $f(\theta) = \mathcal{L}[\Phi^*(re^{i\theta})]$, it follows from the first requirement that for any extremal function $\Phi(z)$ its associated function $\rho(\varphi)$ remains unchanged except possibly at n jump-points which are distributed equidistantly in $-\pi < \varphi \leqq \pi$. Let $\rho(\varphi)$ jump at $\varphi_0 + 2k\pi/n$ $(0 \leqq k \leqq n-1)$ with the heights $\rho_k \geqq 0$, respectively. Then, the extremal function $\Phi(z)$ must have the form

$$\Phi(z) = \int_{-\pi}^{\pi} \Phi^*(e^{-i\varphi}z)\, d\rho(\varphi) = \sum_{k=0}^{n-1} \rho_k \Phi^*(e^{-i\varphi_0 - 2k\pi i/n}z).$$

Conversely, for any function $\Phi(z)$ of this form its associated function $\rho(\varphi)$ satisfies the above requirements. In fact, by virtue of the assumption that $\mathcal{L}[\Phi^*(z)]$ is single-valued with respect to z^n, we obtain

$$\left|\int_{-\pi}^{\pi} \mathcal{L}[\Phi^*(re^{i(\theta-\varphi)})]\, d\rho(\varphi)\right| = \left|\sum_{k=0}^{n-1} \rho_k \mathcal{L}[\Phi^*(re^{i(\theta-\varphi_0-2k\pi/n)})]\right|$$

$$= \left|\sum_{k=0}^{n-1} \rho_k \mathcal{L}[\Phi^*(re^{i(\theta-\varphi_0)})]\right| = |\mathcal{L}[\Phi^*(re^{i(\theta-\varphi_0)})]| = \sum_{k=0}^{n-1} \rho_k |\mathcal{L}[\Phi^*(re^{i(\theta-\varphi_0)})]|$$

$$= \sum_{k=0}^{n-1} \rho_k |\mathcal{L}[\Phi^*(re^{i(\theta-\varphi_0-2k\pi/n)})]| = \int_{-\pi}^{\pi} |\mathcal{L}[\Phi^*(re^{i(\theta-\varphi)})]|\, d\rho(\varphi)$$

and hence further

$$F\left(\int_{-\pi}^{\pi} |\mathcal{L}[\Phi^*(re^{i(\theta-\varphi_0)})]|\, d\rho(\varphi)\right) = F(|\mathcal{L}[\Phi^*(re^{i(\theta-\varphi_0)})]|)$$

$$= \sum_{k=0}^{n-1} \rho_k F(|\mathcal{L}[\Phi^*(re^{i(\theta-\varphi_0)})]|) = \int_{-\pi}^{\pi} F(|\mathcal{L}[\Phi^*(re^{i(\theta-\varphi)})]|)\, d\rho(\varphi).$$

6. Illustrating examples.

In theorems 3 and 4, several sorts of choice are possible for $\mathcal{L}$ as well as F. We first give a simple example which is really an immediate consequence of these theorems but may be of some practical interest.

THEOREM 5. *Let $L(r) \equiv L(r; \Phi)$ denote the length of the image-curve of $|z| = r$ by a (not necessarily univalent) mapping $\Phi(z) \in \mathfrak{R}$. Then we have*

$$L(r; \Phi) \leqq L(r; \Phi^*) \equiv 2\int_{-\pi}^{\pi} \left| -\wp(i \lg (re^{i\theta})) - \frac{\eta_1}{\pi} \right| d\theta \quad (q < r < 1).$$

The equality sign holds here if and only if $\Phi(z)$ is of the form $\Phi^*(\varepsilon z)$ with $|\varepsilon| = 1$.

Proof. Put $\mathcal{L} = z\, d/dz$ and $F(X) = X$. Then it follows from theorem 3 that

$$L(r;\,\Phi)=r\int_{-\pi}^{\pi}|\Phi'(re^{i\theta})|\,d\theta\leqq r\int_{-\pi}^{\pi}|\Phi^{*\prime}(re^{i\theta})|\,d\theta=L(r;\,\Phi^*).$$

In order to verify the assertion on extremal functions by means of theorem 4, we have only to remember that the function $\mathcal{L}[\Phi^*(z)]=z\Phi'(z)$ is univalent in $q<|z|<1$, a fact which has been shown in §3 and assures the condition imposed on $\mathcal{L}$ in theorem 4 with $n=1$.

As further illustrating examples, we can state more generally the following results:

THEOREM 6. *Let n be a positive integer and $F(X)$ be an increasing function convex for $X\geqq 0$. Then, for any $\Phi(z)\in\mathfrak{R}$, we have*

$$\int_{-\pi}^{\pi}F\left(\frac{1}{n}\left|\sum_{k=0}^{n-1}\Phi(re^{i(\theta-2k\pi/n)})\right|\right)d\theta\leqq\int_{-\pi}^{\pi}F(|\Phi^*(r^ne^{i\theta},\,q^n)|)\,d\theta\qquad(q<r<1)$$

and

$$\int_{-\pi}^{\pi}F\left(\frac{r}{n}\left|\sum_{k=0}^{n-1}e^{-2k\pi i/n}\Phi'(re^{i(\theta-2k\pi/n)})\right|\right)d\theta\leqq\int_{-\pi}^{\pi}F(nr^n|\Phi^{*\prime}(r^ne^{i\theta},\,q^n)|)\,d\theta$$
$$(q<r<1);$$

here the parameter q^n associated to Φ^* and $\Phi^{*\prime}$ indicates that the ζ- and $\wp$-functions involved in the expressions for Φ^* and $\Phi^{*\prime}$ depend on the primitive periods 2π and $-2i\lg q^n$. Suppose $F(X)$ be strictly increasing. Then the equality in every estimation holds if and only if $\Phi(z)$ is of the same form as in theorem 4.

Proof. We may put

$$\mathcal{L}[\Phi(z)]=\frac{1}{n}\sum_{k=0}^{n-1}\Phi(ze^{-2k\pi i/n})\quad\text{and}\quad\mathcal{L}[\Phi(z)]=z\frac{d}{dz}\frac{1}{n}\sum_{k=0}^{n-1}\Phi(ze^{-2k\pi i/n}),$$

respectively, whence readily follow both estimations desired. In fact, we remember the identities

$$\frac{1}{n}\sum_{k=0}^{n-1}\Phi^*(ze^{-2\pi ki/n})=\frac{1}{n}\sum_{k=0}^{n-1}\left(1+2\sum_{h=-\infty}^{\infty}{}'\frac{z^he^{2hk\pi i/n}}{1-q^{2h}}\right)$$
$$=1+2\sum_{\nu=-\infty}^{\infty}{}'\frac{z^{n\nu}}{1-q^{2n\nu}}=\Phi^*(z^n,\,q^n)$$

and

$$z\frac{d}{dz}\frac{1}{n}\sum_{k=0}^{n-1}\Phi^*(ze^{-2k\pi i/n})=nz^n\Phi^{*\prime}(z^n,\,q^n);$$

cf. the series expansion of $\Phi^*(z)$ given at the end of §3. Thus, theorem 3 implies

$$\int_{-\pi}^{\pi}F\left(\frac{1}{n}\left|\sum_{k=0}^{n-1}\Phi(re^{i(\theta-2k\pi/n)})\right|\right)d\theta\leqq\int_{-\pi}^{\pi}F(|\Phi^*((re^{i\theta})^n,\,q^n)|)\,d\theta$$
$$=\int_{-\pi}^{\pi}F(|\Phi^*(r^ne^{i\theta},\,q^n)|)\,d\theta$$

and

$$\int_{-\pi}^{\pi} F\left(\frac{r}{n}\left|\sum_{k=0}^{n-1} e^{-2k\pi i/n}\Phi'(re^{i(\theta-2k\pi/n)})\right|\right)d\theta \leqq \int_{-\pi}^{\pi} F(nr^n\,|\,\Phi^{*\prime}((re^{i\theta})^n,\,q^n)\,|)\,d\theta$$

$$= \int_{-\pi}^{\pi} F(nr^n\,|\,\Phi^{*\prime}(r^n e^{i\theta},\,q^n)\,|)\,d\theta.$$

The mapping characters explained in §3 then show that both $\Phi^*(z^n, q^n)$ and $nz^n\Phi^{*\prime}(z^n, q^n)$ satisfy the conditions imposed on $\mathcal{L}[\Phi^*(z)]$ in theorem 4, whence follows the final part of the present theorem.

While $-\lg X$ is a function convex for $X>0$, it is not increasing so that theorem 3 cannot be applied directly with $F(X)=-\lg X$. However, a supplementary inequality can be derived with reference to this function.

THEOREM 7. *Let $\mathcal{L}$ satisfy the conditions mentioned in theorem 3. Then, for any $\Phi(z)\in\mathfrak{R}$, we have*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \lg|\mathcal{L}[\Phi(re^{i\theta})]|\,d\theta \leqq \lg\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|\mathcal{L}[\Phi^*(re^{i\theta})]|\,d\theta\right) \quad (q<r<1).$$

Let $\mathcal{L}$ further satisfy the additional conditions imposed in theorem 4. Then, the equality sign in the estimation appears for any fixed r if and only if $\Phi(z)$ is of the form stated in theorem 4 and further $w=\mathcal{L}[\Phi^*(z)]$ maps $|z|=r$ onto an n-ply covered circle with center at the origin.

Proof. The estimation can be derived readily by means of theorem 3 applied to $F(X)=X/2\pi$ together with the concavity of $\lg X$. In fact, we thus get

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \lg|\mathcal{L}[\Phi(re^{i\theta})]|\,d\theta \leqq \lg\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|\mathcal{L}[\Phi(re^{i\theta})]|\,d\theta\right)$$

$$\leqq \lg\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|\mathcal{L}[\Phi^*(re^{i\theta})]|\,d\theta\right).$$

In the last relation the equality sign of the first inequality holds if and only if $|\Phi(re^{i\theta})|$ is independent of θ, while that of the second holds if and only if $\Phi(z)$ is of the form stated in theorem 4. But, for any $\Phi(z)$ of the last-mentioned form, we have

$$\mathcal{L}[\Phi(z)] = \sum_{k=0}^{n-1}\rho_k\,\mathcal{L}[\Phi^*(e^{-2k\pi i/n}\varepsilon z)] = \sum_{k=0}^{n-1}\rho_k\,\mathcal{L}[\Phi^*(\varepsilon z)] = \mathcal{L}[\Phi^*(\varepsilon z)]$$

so that $|\mathcal{L}[\Phi(re^{i\theta})]|$ and $|\mathcal{L}[\Phi^*(re^{i\theta})]|$ are simultaneously independent of θ. The image-curve of $|z|=r$, $0\leqq\arg z<2\pi/n$ by the mapping $w=\mathcal{L}[\Phi^*(z)]$ is supposed to be a Jordan curve and hence the requirement of this independency is equivalent to the final assertion of the theorem.

7. Areal distortion.

Theorems 3 and 4 imply as an immediate consequence an areal distortion which is formulated as follows:

THEOREM 8. *Under the conditions imposed on* $\mathcal{L}$ *and* $F(X)$ *in theorem 3, we have*

$$\int_{r_0}^{r} r\,dr \int_{-\pi}^{\pi} F(|\mathcal{L}[\Phi(re^{i\theta})]|)\,d\theta \leqq \int_{r_0}^{r} r\,dr \int_{-\pi}^{\pi} F(|\mathcal{L}[\Phi^*(re^{i\theta})]|)\,d\theta \qquad (q \leqq r_0 < r < 1)$$

and the function $\Phi^*(\varepsilon z)$ *is for any* ε *with* $|\varepsilon| = 1$ *always an extremal function of this estimation. By supplementing further assumptions imposed on* $\mathcal{L}$ *and* F *in theorem 4, we can conclude that the equality sign appears for any fixed* r_0 *and* r *if and only if* $\Phi(z)$ *is of the form stated in theorem 4.*

Though the subsequent theorem may be regarded as a particular case of theorem 8, it will be of some practical interest, so that we formulate it here explicitly.

THEOREM 9. *Let* $A(r_0, r) \equiv A(r_0, r; \Phi)$ *denote the area of the image-domain of* $(q \leqq)\, r_0 < |z| < r\,(<1)$ *by a (not necessarily univalent) mapping* $\Phi(z) \in \mathfrak{R}$, *the area being counted according to multiplicity. Then we have*

$$A(r_0, r) \equiv \int_{r_0}^{r} r\,dr \int_{-\pi}^{\pi} |\Phi'(re^{i\theta})|^2\,d\theta$$

$$\leqq \int_{r_0}^{r} \frac{4}{r}\,dr \int_{-\pi}^{\pi} \left| -\wp(i \lg (re^{i\theta})) - \frac{\eta_1}{\pi} \right|^2 d\theta = 4\pi \sum_{\nu=-\infty}^{\infty}{}' \frac{\nu}{(1-q^{2\nu})^2}(r^{2\nu} - r_0^{2\nu}).$$

The equality sign in the estimation appears for any fixed r_0 *and* r *if and only if* $\Phi(z)$ *is of the form*

$$\Phi(z) = \Phi^*(\varepsilon z) \equiv \frac{2}{i}\left(\zeta(i \lg (\varepsilon z)) - \frac{\eta_1}{\pi}\, i \lg (\varepsilon z)\right), \qquad |\varepsilon| = 1.$$

Proof. The estimate for $A(r_0, r)$ becomes

$$\int_{r_0}^{r} r\,dr \int_{-\pi}^{\pi} |\Phi^{*\prime}(re^{i\theta})|^2\,d\theta$$

$$= \int_{r_0}^{r} r\,dr \int_{-\pi}^{\pi} \frac{4}{r^2} \left| -\wp(i \lg (re^{i\theta})) - \frac{\eta_1}{\pi} \right|^2 d\theta = \int_{r_0}^{r} \frac{4}{r}\,dr \int_{-\pi}^{\pi} \left| \sum_{\nu=-\infty}^{\infty}{}' \frac{\nu (re^{i\theta})^\nu}{1-q^{2\nu}} \right|^2 d\theta$$

$$= \int_{r_0}^{r} \frac{4}{r}\, 2\pi \sum_{\nu=-\infty}^{\infty}{}' \left(\frac{\nu}{1-q^{2\nu}}\right)^2 r^{2\nu} dr = 4\pi \sum_{\nu=-\infty}^{\infty}{}' \frac{\nu}{(1-q^{2\nu})^2}(r^{2\nu} - r_0^{2\nu}).$$

The mapping character of $z\Phi^{*\prime}(z)$ explained in §3 shows by virtue of theorem 4 that extremal functions are only those of the form $\Phi^*(\varepsilon z)$ with $|\varepsilon| = 1$.

Theorem 9 may be proved alternatively by means of the estimation for Laurent coefficients of $\Phi(z) \in \mathfrak{R}$. For this purpose, we prepare a lemma on an integral representation for Laurent coefficients.

LEEMA 2. *Let the Laurent expansion of* $\Phi(z) \in \mathfrak{R}$ *be*

$$\Phi(z) = 1 + \sum_{\nu=-\infty}^{\infty}{}' c_\nu z^\nu.$$ [3]

Then we have

$$c_\nu = \frac{2}{1-q^{2\nu}} \int_{-\pi}^{\pi} e^{-i\nu\varphi}\, d\rho(\varphi) \qquad (\nu = \pm 1, \pm 2, \cdots),$$

where $\rho(\varphi)$ is the function associated to $\Phi(z)$ in theorem 1.

Proof. Remembering theorem 1 together with the expansion of $\Phi^*(z)$ given at the end of §3, we get

$$\Phi(z) = \int_{-\pi}^{\pi} \frac{2}{i}\left(\zeta(i \lg z + \varphi) - \frac{\eta_1}{\pi}(i \lg z + \varphi)\right) d\rho(\varphi)$$

$$= \int_{-\pi}^{\pi}\left(1 + 2\sum_{\nu=-\infty}^{\infty}{}' \frac{z^\nu e^{-i\nu\varphi}}{1-q^{2\nu}}\right) d\rho(\varphi) = 1 + 2\sum_{\nu=-\infty}^{\infty}{}' z^\nu \frac{2}{1-q^{2\nu}} \int_{-\pi}^{\pi} e^{-i\nu\varphi}\, d\rho(\varphi),$$

whence follows the desired representation.

Now, we have really

$$A(r_0, r) \equiv \int_{r_0}^{r} r\,dr \int_{-\pi}^{\pi} |\Phi'(re^{i\theta})|^2\, d\theta = \int_{r_0}^{r} r\,dr\, 2\pi \sum_{\nu=-\infty}^{\infty}{}' \nu^2 |c_\nu|^2 r^{2\nu-2}$$

$$= \pi \sum_{\nu=-\infty}^{\infty}{}' \nu |c_\nu|^2 (r^{2\nu} - r_0^{2\nu}) \leqq 4\pi \sum_{\nu=-\infty}^{\infty}{}' \frac{\nu}{(1-q^{2\nu})^2}(r^{2\nu} - r_0^{2\nu}).$$

The extremal functions are characterized by $|c_\nu| = 2/|1-q^{2\nu}|$ for any $\nu \neq 0$ while even a single equation $|c_1| = 2/(1-q^2)$ or $|c_{-1}| = -2/(1-q^{-2})$ implies that $d\rho(\varphi)$ must vanish except at a single jump, φ_0 say, whence follows $\Phi(z) = \Phi^*(\varepsilon z)$ with $\varepsilon = e^{-\varphi_0}$.

8. Radial distortion.

Theorem 5 concerns the distortion on the image-curve of a concentric circumference by a mapping function belonging to $\mathfrak{R}$. As a supplement we deal here with an analogous distortion on the image-curve of a radial segment. It is an immediate consequence of a distortion theorem on derivative in the class $\mathfrak{R}$ which may be formulated as follows:

LEMMA 3. *For any $\Phi(z) \in \mathfrak{R}$, we have*

$$\left|\left(z\frac{d}{dz}\right)^{2\nu+1} \Phi(z)\right| \leqq \begin{cases} -2\left(\wp(i \lg |z|) + \dfrac{\eta_1}{\pi}\right) & (\nu = 0), \\ (-1)^{\nu+1} 2\wp^{(2\nu)}(i \lg |z|) & (\nu = 1, 2, \cdots). \end{cases}$$

The equality sign holds at any assigned point $z = te^{i\varphi_0}$ with $q < t < 1$ if and only if $\Phi(z) = \Phi^*(e^{-i\varphi_0} z)$.

Proof. From lemma 1 we have, for any $\Phi(z) \in \mathfrak{R}$,

3) Cf. the final remark in § 1.

$$\left(z\frac{d}{dz}\right)^{2\nu+1}\Phi(z)=\begin{cases}\displaystyle\int_{-\pi}^{\pi}\left(-2\left(\wp(i\lg z+\varphi)+\frac{\eta_1}{\pi}\right)\right)d\rho(\varphi) & (\nu=0),\\ \displaystyle\int_{-\pi}^{\pi}(-1)^{\nu+1}2\wp^{(2\nu)}(i\lg z+\varphi)\,d\rho(\varphi) & (\nu=1,2,\cdots)\end{cases}$$

with

$$d\rho(\varphi)\geqq 0\quad(-\pi<\varphi\leqq\pi),\qquad \int_{-\pi}^{\pi}d\rho(\varphi)=1.$$

The integrands in the right-hand members can be expanded in Laurent series

$$z\frac{d}{dz}\Phi^*(z)=-2\left(\wp(i\lg z)+\frac{\eta_1}{\pi}\right)=2\sum_{\mu=-\infty}^{\infty}{}'\frac{\mu z^\mu}{1-q^{2\mu}}$$

and

$$\left(z\frac{d}{dz}\right)^{2\nu+1}\Phi^*(z)=(-1)^{\nu+1}2\wp^{(2\nu)}(i\lg z)=2\sum_{\mu=-\infty}^{\infty}{}'\frac{\mu^{2\nu+1}z^\mu}{1-q^{2\mu}}\qquad(\nu=1,2,\cdots),$$

both containing positive real coefficients alone except the vanishing constant terms. Consequently, there follows readily the desired results.

Though the derivative of any odd order of $\Phi^*(z)$ with respect to $\lg z$ can be so simply estimated as shown above, it is not the case for that of even order. In the latter case, the best estimate depending on r alone will be not analytic in r throughout $q<r<1$.

THEOREM 10. *Let $q\leqq r_0<r<1$. Then, for any $\Phi(z)\in\mathfrak{R}$, we have*

$$\int_{r_0}^{r}\left|\left(\frac{\partial}{\partial\lg t}\right)^{2\nu+1}\Phi(te^{i\theta})\right|d\lg t$$

$$\leqq\begin{cases}\displaystyle\frac{2}{i}\left(\zeta(i\lg r)-\zeta(i\lg r_0)-\frac{\eta_1}{\pi}i\lg\frac{r}{r_0}\right)=2\sum_{\mu=-\infty}^{\infty}{}'\frac{r^\mu-r_0^{\,\mu}}{1-q^{2\mu}} & (\nu=0),\\ \displaystyle(-1)^{\nu+1}\frac{2}{i}(\wp^{(2\nu-1)}(i\lg r)-\wp^{(2\nu-1)}(i\lg r_0))=2\sum_{\mu=-\infty}^{\infty}{}'\frac{\mu^{2\nu}(r^\mu-r_0^{\,\mu})}{1-q^{2\mu}} & (\nu=1,2,\cdots).\end{cases}$$

The equality sign in every estimation holds for assigned values of r_0, r and θ if and only if $\Phi(z)$ is given by $\Phi^*(e^{-i\theta}z)$.

Proof. By virtue of lemma 3, we have

$$\left|\left(\frac{\partial}{\partial\lg t}\right)^{2\nu+1}\Phi(te^{i\theta})\right|\leqq\left(\frac{d}{d\lg t}\right)^{2\nu+1}\Phi^*(t).$$

Integration with respect to $\lg t$ leads to

$$\int_{r_0}^{r}\left|\left(\frac{\partial}{\partial\lg t}\right)^{2\nu+1}\Phi(te^{i\theta})\right|d\lg t\leqq\left[\left(\frac{d}{d\lg t}\right)^{2\nu}\Phi^*(t)\right]_{t=r_0}^{t=r},$$

whence follows the desired estimation by substituting the actual expression for $\Phi^*(z)$. The extremal function is characterized by the relation $|(\partial/\partial \lg t)^{2\nu+1}\Phi(te^{i\theta})| = (d/d \lg t)^{2\nu+1}\Phi^*(t)$ to be valid for any t with $r_0 < t < r$, whence follows the final assertion of the theorem.

The case $\nu = 0$ in theorem 10 may be of practical interest so that it will be separated especially as a corollary in the following lines.

COROLLARY. *Let* $\Lambda(r_0, r, \theta) \equiv \Lambda(r_0, r, \theta; \Phi)$ *denote the length of the image-curve of a radial segment* $\arg z = \theta$, $(q \leqq) r_0 < |z| < r (< 1)$ *by a mapping* $\Phi(z) \in \mathfrak{R}$. *Then we have*

$$\Lambda(r_0, r, \theta) \equiv \int_{r_0}^{r} |\Phi'(te^{i\theta})|\, dt$$

$$\leqq \frac{2}{i}\left(\zeta(i \lg r) - \zeta(i \lg r_0) - \frac{\eta_1}{\pi} i \lg \frac{r}{r_0}\right) = 2 \sum_{\mu=-\infty}^{\infty}{}' \frac{r^{\mu} - r_0^{\mu}}{1 - q^{2\mu}}.$$

The bound is attained only by the extremal function

$$\Phi^*(e^{-i\theta}z) \equiv \frac{2}{i}\left(\zeta(i \lg (e^{-i\theta}z)) - \frac{\eta_1}{\pi} i \lg (e^{-i\theta}z)\right).$$

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