ON ANALYTIC FUNCTIONS WITH POSITIVE REAL PART IN A CIRCLE

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1. Introduction.

Let $\mathfrak{R}=\{\Phi(z)\}$ be the class of analytic functions which are regular and of positive real part in the unit circle $|z|<1$ and normalized by $\Phi(0)=1$. Rogosinski [6] has developed a systematic study on this class. It relates closely to the papers of Carathéodory [1, 2] on the variability region of the coefficients of Taylor expansion of $\Phi(z)\in\mathfrak{R}$.

Among several problems Rogosinski has, in particular, dealt with the estimation on the length of image-curve of $|z|=r\ (0<r<1)$ by a (not necessarily univalent) mapping $\Phi(z)\in\mathfrak{R}$. He has reported in the paper cited above two different proofs for the result. One due to himself depends on a generalization of Schwarz lemma on bounded functions and another on the Poisson integral which has been, so he says, really informed him by G. Szegö. Though the latter proof is straightforward and simple, a limiting process has to be taken into account so that the part on fully determining the extremal functions has been missed out.

The last-mentioned lack in Szegö's proof can, however, be removed by making use of the representation of Herglotz type instead of Poisson's. The Herglotz representation is really well known but it will be frequently referred to in subsequent arguments so that we re-formulate it here for the sake of convenience explicitly as a lemma; for instance, cf. [3].

LEMMA 1. *It is necessary and sufficient for $\Phi(z)\in\mathfrak{R}$ that $\Phi(z)$ is representable by means of Herglotz integral*

$$\Phi(z)=\int_{-\pi}^{\pi}\frac{e^{i\varphi}+z}{e^{i\varphi}-z}\,d\rho(\varphi)$$

where $\rho(\varphi)$ is a real-valued function defined for $-\pi<\varphi\leqq\pi$ which is increasing and has the total variation equal to unity, i.e.

$$d\rho(\varphi)\geqq 0 \qquad (-\pi<\varphi\leqq\pi), \qquad\qquad \int_{-\pi}^{\pi}d\rho(\varphi)=1.$$

Based on the integral representation stated in lemma 1, we shall first give an alternative proof of Rogosinski's theorem. It proceeds formally analogous as Szegö's but it is effective in determining the whole of extremal func-

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tions. It will further serve as a model for dealing with more general problems. Now, the theorem to be proved may be formulated as follows:

THEOREM 1. *Let $L(r) \equiv L(r; \Phi)$ be the length of the image-curve of $|z|=r$ by a mapping $\Phi(z) \in \mathfrak{R}$. Then we have*

$$L(r) \equiv r\int_{-\pi}^{\pi} |\Phi'(re^{i\theta})|\, d\theta \leqq \frac{4\pi r}{1-r^2}$$

and the equality sign in the estimation appears for any fixed r if and only if $\Phi(z)$ is a linear function which maps $|z|<1$ onto the right half-plane, i.e. $\Phi(z)$ is of the form $(1+\varepsilon z)/(1-\varepsilon z)$ with $|\varepsilon|=1$.

Proof. The representation for $\Phi(z)$ given in lemma 1 implies

$$\begin{aligned}\int_{-\pi}^{\pi} |\Phi'(re^{i\theta})|\, d\theta &= \int_{-\pi}^{\pi} d\theta \left| \int_{-\pi}^{\pi} \frac{2e^{i\varphi}}{(e^{i\varphi}-re^{i\theta})^2}\, d\rho(\varphi)\right| \\ &\leqq \int_{-\pi}^{\pi} d\theta \int_{-\pi}^{\pi} \left| \frac{2e^{i\varphi}}{(e^{i\varphi}-re^{i\theta})^2}\right| d\rho(\varphi) = \int_{-\pi}^{\pi} d\rho(\varphi) \int_{-\pi}^{\pi} \frac{2}{|1-re^{i(\theta-\varphi)}|^2}\, d\theta \\ &= \int_{-\pi}^{\pi} \frac{4\pi}{1-r^2}\, d\rho(\varphi) = \frac{4\pi}{1-r^2},\end{aligned}$$

whence readily follows the desired estimation. Any function $\rho(\varphi)$ associated to an extremal function $\Phi(z)$ is characterized by the relation

$$\int_{-\pi}^{\pi} \frac{e^{i\varphi}}{(e^{i\varphi}-re^{i\theta})^2}\, d\rho(\varphi) = \int_{-\pi}^{\pi} \left|\frac{e^{i\varphi}}{(e^{i\varphi}-re^{i\theta})^2}\right| d\rho(\varphi)$$

valid for every θ throughout $-\pi<\theta\leqq\pi$, since both members are continuous in θ. This condition is equivalent to the requirement that for every fixed θ the quantity $e^{i\varphi}/(e^{i\varphi}-re^{i\theta})^2$ has the same argument for every φ $(-\pi<\varphi\leqq\pi)$ with $d\rho(\varphi)>0$. But, we can show, moreover, that for any fixed θ this quantity has never the same argument for any distinct values of φ. In fact, as φ varies from $-\pi$ to π, the quantity

$$\frac{re^{i\theta}e^{i\varphi}}{(e^{i\varphi}-re^{i\theta})^2} = \frac{re^{i(\theta-\varphi)}}{(1-re^{i(\theta-\varphi)})^2}$$

describes the curve which is described by $re^{i\theta}/(1-re^{i\theta})^2$ as θ varies from $-\pi$ to π. This curve is nothing but the image-curve of $|z|=r$ by Koebe's function $z/(1-z)^2$. Since it is simple and strictly star-like with respect to the origin, any two distinct points on the curve have never the same argument. Consequently, in order that $\rho(\varphi)$ is associated to an extremal function, it is necessary and sufficient that $\rho(\varphi)$ remains unchanged except a single jump with the height necessarily equal to unity. Hence, the form of extremal function is given by

$$\Phi(z) = \frac{1+\varepsilon z}{1-\varepsilon z}, \qquad \varepsilon = e^{-i\varphi_0},$$

φ_0 being a real value with $-\pi < \varphi_0 \leqq \pi$ where the associated function $\rho(\varphi)$ jumps.

For the functional estimated in theorem 1, the range of integration with respect to θ corresponds to the whole circumference $-\pi < \theta \leqq \pi$. However, the length of the image-curve of any arc on $|z| = r$ can be also estimated in a similar manner. In fact, we can state more generally the following extension.

THEOREM 2. *Let E be any measurable set contained in $-\pi < \theta \leqq \pi$. Then, for any $\Phi(z) \in \mathfrak{R}$, we have*

$$r\int_E |\Phi'(re^{i\theta})|\, d\theta \leqq \frac{8r}{1-r^2} \arctan\left(\frac{1+r}{1-r}\tan\frac{mE}{4}\right),$$

where mE denotes the measure of E. In case $mE > 0$ the equality sign holds for any fixed r if and only if E differs from an interval $\alpha < \theta < \beta$ by a set of measure zero and $\Phi(z)$ is of the form $(1+\varepsilon z)/(1-\varepsilon z)$ with $\varepsilon = e^{-i(\alpha+\beta)/2}$ or $\varepsilon = e^{-i\varphi_0}$ for $\beta < \alpha + 2\pi$ or $\beta = \alpha + 2\pi$, respectively, φ_0 being an arbitrary real number.

Proof. We obtain quite similarly as in the proof of theorem 1 the estimation

$$\int_E |\Phi'(re^{i\theta})|\, d\theta \leqq 2\int_E \frac{d\theta}{|1-re^{i\theta}|^2}.$$

Now, the integrand of the last integral, i.e. $1/(1-2r\cos\theta + r^2)$ is an even function of θ which decreases strictly as θ increases from 0 to π. Hence we get

$$\int_E \frac{d\theta}{|1-re^{i\theta}|^2} \leqq 2\int_0^{mE/2} \frac{d\theta}{|1-re^{i\theta}|^2} = \frac{4}{1-r^2} \arctan\left(\frac{1+r}{1-r}\tan\frac{mE}{4}\right).$$

The assertion about extremal functions is also verified similarly as in theorem 1.

2. Lemmas.

The main purpose of the present paper is first to generalize Rogosinski's theorem 1 to some extent by means of the representation for $\Phi(z) \in \mathfrak{R}$ referred to in lemma 1 and then to establish several mean distortion theorems on the class $\mathfrak{R}$ in a systematic manner. Namely, we observe a linear operator $\mathcal{L}$ applied to $\Phi(z)$ and an increasing convex function $F(X)$. Our problem to be discussed is then to obtain the precise estimate for the functional defined by

$$\int_{-\pi}^{\pi} F(|\mathcal{L}[\Phi(re^{i\theta})]|)\, d\theta \qquad (0 < r < 1)$$

within the class $\mathfrak{R}$ in terms of a definite function of r and further to determine extremal functions for the estimation.

The first part of the problem can be really dealt with similarly as in theorem 1. It will be shown that the functional under consideration is majorized by its value attained by substituting $(1+\varepsilon z)/(1-\varepsilon z)$ with $|\varepsilon|=1$ for $\Phi(z)$, i.e. this linear function possesses always the extremal character. However, other extremal functions can appear for certain operators $\mathcal{L}$. Accordingly, the second part of the problem is not so simple if we attempt to determine the whole of extremal functions. It will be convenient to establish some preparatory lemmas which will serve for this purpose. We begin with a lemma on an elementary mapping which will play in the principal lemma an auxiliary role of cancelling the zero-points or poles of a function without altering its argument.

LEMMA 2. *Let $c \neq 0$ be a point in the interior or on the boundary of a circular disc $|z| \leqq r$. Then the function $\omega(z, c)$ which maps $|z|<r$ onto the whole plane cut along a slit lying on the positive real axis and is normalized by $\omega(0, c)=\infty$, $\omega(c, c)=0$ and $|\operatorname{Res}(0; \omega)|=r^2$ is uniquely determined and its explicit expression is given by*

$$\omega(z, c)=e^{-i \arg c}(c-z)\left(1-\frac{r^2}{\bar{c} z}\right).$$

Let further n be any positive integer. Then the function defined by

$$\Omega(z, c)=\prod_{k=0}^{n-1} \omega(z, c e^{2k\pi i/n})$$

is a single-valued function of z^n regular for $z \neq 0$ but not of any higher power of z.

Proof. The uniqueness assertion of the mapping function is readily verified, for instance, by means of a theorem on radial slit mapping. That it is expressed as written in the lemma may be shown directly. In fact, we have

$$\omega(re^{i\theta}, c)=\frac{1}{|c|}|c-re^{i\theta}|^2,$$

so that the image of $|z|=r$ by $\omega(z, c)$ is a segment on the positive real axis. Since $\omega(z, c)$ is regular except a simple pole at the origin, it is surely univalent in $|z|<r$. It is evident that the normalization conditions are also satisfied. Next, it is readily seen from its explicit expression that $\omega(z, c)$ satisfies for any real φ the relation

$$\omega(ze^{i\varphi}, c)=\omega(z, ce^{-i\varphi})$$

which implies

$$\Omega(ze^{2\pi i/n}, c) = \prod_{k=0}^{n-1} \omega(ze^{2\pi i/n}, ce^{2k\pi i/n})$$

$$= \prod_{k=0}^{n-1} \omega(z, ce^{2(k-1)\pi i/n}) = \Omega(z, c).$$

Thus, $\Omega(z, c)$ being invariant under the substitution $z\,|\,ze^{2\pi i/n}$, it is a single-valued function of z^n. Since it has a pole of order n at the origin, it cannot be a single-valued function of a higher power of z.

Now we state the principal lemma which is fundamental for our subsequent discussions.

LEMMA 3. *Let $\Psi(z)$ be an analytic function meromorphic in $|z| < r$ which is non-constant and regular along $|z| = r$. Let n denote the greatest integer such that $\Psi(z)$ is a single-valued function of z^n. Further let $\rho(\varphi)$ be a real-valued function defined for $-\pi < \varphi \leqq \pi$ which is increasing and has the total variation equal to $P > 0$. Then, in order that the relation*

$$\left| \int_{-\pi}^{\pi} \Psi(re^{i(\theta-\varphi)})\, d\rho(\varphi) \right| = \int_{-\pi}^{\pi} |\Psi(re^{i(\theta-\varphi)})|\, d\rho(\varphi)$$

holds identically with respect to θ throughout $-\pi < \theta \leqq \pi$, it is necessary and sufficient that $\rho(\varphi)$ remains unchanged except possibly at mn jump-points which are distributed equidistantly in $-\pi < \varphi \leqq \pi$. Here m is a positive integer defined as follows: Let the sets of zero-points and of poles of $\Psi(z)$ contained in $0 < |z| \leqq r$ which are irreducible with respect to their arguments taken by $\bmod 2\pi/n$ be $\{a_\lambda\}_{\lambda=1}^{\alpha}$ and $\{b_\mu\}_{\mu=1}^{\beta}$, respectively, which are counted according to respective multiplicities, and let N be the greatest integer such that the function defined by

$$Y(z) = \Psi(z) \prod_{\mu=1}^{\beta} \Omega(z, b_\mu) \Big/ \prod_{\lambda=1}^{\alpha} \Omega(z, a_\lambda)$$

is a single-valued function of z^N, $\Omega(z, c)$ being the function introduced in lemma 2. Then we put $N = mn$.

Proof. The sufficiency assertion is readily verified. In fact, let $\rho(\varphi)$ remain unchanged except at mn jumps

$$\varphi_0 + \varphi_{jk} = \varphi_0 + 2((j-1)n + k)\pi/mn \qquad (0 \leqq k \leqq n-1,\ 1 \leqq j \leqq m)$$

with the heights $\rho_{jk} \geqq 0$, respectively. Since $Y(z)$ is single-valued in z^{mn}, we then have

$$\Psi(re^{i(\theta-\varphi_0-\varphi_{jk})}) = Y(re^{i(\theta-\varphi_0)}) \prod_{\lambda=1}^{\alpha} \Omega(re^{i(\theta-\varphi_0-\varphi_{jk})}, a_\lambda) \Big/ \prod_{\mu=1}^{\beta} \Omega(re^{i(\theta-\varphi_0-\varphi_{jk})}, b_\mu)$$

so that, by virtue of $\arg \Omega(z, c) = 0$ valid along $|z| = r$, there follows

$$\left| \int_{-\pi}^{\pi} \Psi(re^{i(\theta-\varphi)})\, d\rho(\varphi) \right|$$
$$= | Y(re^{i(\theta-\varphi_0)}) | \sum_{j=1}^{m} \sum_{k=0}^{n-1} \rho_{jk} \prod_{\lambda=1}^{\alpha} \Omega(re^{i(\theta-\varphi_0-\ jk)}, a_\lambda) \Big/ \prod_{\mu=1}^{\beta} \Omega(re^{i(\theta-\varphi_0-\varphi_{jk})}, b_\mu)$$
$$= \int_{-\pi}^{\pi} | \Psi(re^{i(\theta-\varphi)} | \, d\rho(\varphi).$$

The necessity proof proceeds as follows. If the relation under consideration holds identically with respect to θ, the quantity $\arg \Psi(ze^{-i\varphi})$ with any fixed $z = re^{i\theta}$ must have the same value depending only on θ for every value of φ with $d\rho(\varphi) > 0$ provided $\Psi(ze^{-i\varphi})$ does not vanish. Hence, $\Omega(z, c)$ being real and positive along $|z| = r$, $\arg Y(ze^{-i\varphi})$ has also the same property. Let φ_0 and φ_1 be any values of φ with $d\rho(\varphi) > 0$, and we observe the function defined by

$$X(z) = \frac{Y(ze^{-i\varphi_0})}{Y(ze^{-i\varphi_1})}.$$

Since $Y(z)$ is regular and non-vanishing throughout $0 < |z| \leqq r$, $X(z)$ is regular for $|z| \leqq r$ even at $z = 0$. Further it remains real and positive along $|z| = r$. In general, any bounded set lying entirely on the real axis cannot be the boundary of the image of $|z| \leqq r$ by a (not necessarily univalent) mapping regular there unless it degenerates to a single point. Consequently, $X(z)$ must be a constant which is real and positive. Let the Laurent expansion of $Y(z)$ be

$$Y(z) = \sum c_\nu z^{n_\nu}$$

where $\{n_\nu\}$ is a strictly increasing (finite or infinite) sequence of integers for which the sequence of corresponding coefficients $\{c_\nu\}$ does not involve zero. We then get

$$Y(ze^{-i\varphi_0}) = X(0) Y(ze^{-i\varphi_1})$$

and obtain further, by comparing the coefficients of z^{n_ν},

$$e^{-in_\nu\varphi_0} = X(0) e^{-in_\nu\varphi_1}$$

for any ν. Since $X(0)$ is real positive, we get $X(0) = 1$, whence follows

$$n_\nu(\varphi_1 - \varphi_0) \equiv 0 \qquad (\mathrm{mod}\ 2\pi)$$

for any ν. Now since $N = mn$ is equal to the greatest common measure of the set $\{n_\nu\}$, there exist two members of the set, n_γ and n_δ say, such that N is just the greatest common measure of them. Hence, for some integers u and v, we have $un_\gamma - vn_\delta = N$. Consequently, the above relations applied to $n_\nu = n_\gamma$ and to $n_\nu = n_\delta$ imply

$$N(\varphi_1 - \varphi_0) \equiv 0 \qquad (\mathrm{mod}\ 2\pi).$$

Thus, φ_0 and φ_1 being arbitrary pair where $d\rho(\varphi)$ does not vanish, it has been shown that $\rho(\varphi)$ must possess the property stated in the lemma.

For subsequent purpose it will often become necessary to know the actual values of the integers n and m defined in lemma 3. Accordingly, as a supplement of this lemma, we pick out particular cases for which we formulate the following lemma.

LEMMA 4. *In lemma 3, if $\{a_\lambda\}$ and $\{b_\mu\}$ are, in particular, both vacuous, namely if $\Psi(z)$ is regular and non-vanishing throughout $0<|z|\leqq r$, then we have $m=1$. Further if $\Psi'(0)\neq 0$, then we have $n=1$.*

It may be noted by the way that, if $\Psi(z)$ has zero-points or poles, then m may be actually greater than unity. It may be illustrated by an example. Let $0<|a|<r$ and put, as before,

$$\Omega(z, a)=\prod_{k=0}^{n-1}\omega(z, ae^{2k\pi i/n}).$$

Then this function is a single-valued function of z^n but not of any higher power of z. Hence, the function defined by

$$\Psi(z)=z^{mn}\Omega(z, a)$$

with any positive integer m is also of the same nature. But, for any $\rho(\varphi)$ with the jumps alone at mn values $\varphi_{jk}=2((j-1)n+k)\pi/mn$ $(0\leqq k\leqq n-1,$ $1\leqq j\leqq m)$ with any respective heights ρ_{jk}, we get

$$\left|\int_{-\pi}^{\pi}\Psi(re^{i(\theta-\varphi)})\,d\rho(\varphi)\right|=\left|\sum_{j=1}^{m}\sum_{k=0}^{n-1}\Psi(re^{i(\theta-\varphi_{jk})})\rho_{jk}\right|=\sum_{j=1}^{m}\sum_{k=0}^{n-1}r^{mn}\Omega(re^{i(\theta-\varphi_{jk})})\rho_{jk}$$

$$=\sum_{j=1}^{m}\sum_{k=0}^{n-1}|\Psi(re^{i(\theta-\varphi_{jk})})|\rho_{jk}=\int_{-\pi}^{\pi}|\Psi(re^{i(\theta-\varphi)})|\,d\rho(\varphi).$$

3. Main results.

We are now in position to formulate our main theorems. We observe a linear operator $\mathcal{L}$ which has $\mathfrak{R}$ as its domain of argument function and produces by applying to any $\Phi(z)\in\mathfrak{R}$ an analytic function $\mathcal{L}[\Phi(z)]$ single-valued about the origin. It is supposed that the operator is homogeneous of degree zero, i.e., for any constant c, the function $\mathcal{L}[\Phi(z)]$ coincides after substitution $z\,|\,cz$ with $\mathcal{L}[\Phi(cz)]$. In particular, we have

$$\mathcal{L}_z\left[\frac{e^{i\varphi}+z}{e^{i\varphi}-z}\right]=\mathcal{L}_z\left[\frac{1+e^{-i\varphi}z}{1-e^{-i\varphi}z}\right]=\mathcal{L}_\zeta\left[\frac{1+\zeta}{1-\zeta}\right]^{\zeta=e^{-i\varphi}z}.$$

THEOREM 3. *Let $\mathcal{L}$ denote a linear operator defined for Φ and $\mathcal{L}[\Phi(z)]$ applied to $\Phi(z)\in\mathfrak{R}$ be an analytic function meromorphic in $|z|<r(<1)$ and regular along $|z|=r$. Let $\mathcal{L}$ be commutable with the integration with respect to $\rho(\varphi)$ in the representation for $\Phi(z)$ stated in lemma 1. Let* *further $F(X)$ be a bounded increasing convex (and hence necessarily continuous) function defined for the range of $|\mathcal{L}|$. Then, for any $\Phi(z)\in\mathfrak{R}$, we have*

$$\int_{-\pi}^{\pi} F(|\mathcal{L}[\Phi(re^{i\theta})]|)\,d\theta \leqq \int_{-\pi}^{\pi} F\left(\left|\mathcal{L}\left[\frac{1+re^{i\theta}}{1-re^{i\theta}}\right]\right|\right)d\theta.$$

The function $(1+\varepsilon z)/(1-\varepsilon z)$ is for any constant ε with $|\varepsilon|=1$ always an extremal function of this estimation.

Proof. The representation for $\Phi(z)$ given in lemma 1 implies

$$|\mathcal{L}[\Phi(re^{i\theta})]| = \left|\int_{-\pi}^{\pi} \mathcal{L}\left[\frac{e^{i\varphi}+re^{i\theta}}{e^{i\varphi}-re^{i\theta}}\right] d\rho(\varphi)\right| \leqq \int_{-\pi}^{\pi} \left|\mathcal{L}\left[\frac{e^{i\varphi}+re^{i\theta}}{e^{i\varphi}-re^{i\theta}}\right]\right| d\rho(\varphi),$$

since the operation $\mathcal{L}$ and the integration with respect to $\rho(\varphi)$ are supposed commutable. The increasing character and the convexity of $F(X)$ imply further

$$F(|\mathcal{L}[\Phi(re^{i\theta})]|) \leqq F\left(\int_{-\pi}^{\pi} \left|\mathcal{L}\left[\frac{e^{i\varphi}+re^{i\theta}}{e^{i\varphi}-re^{i\theta}}\right]\right| d\rho(\varphi)\right)$$
$$\leqq \int_{-\pi}^{\pi} F\left(\left|\mathcal{L}\left[\frac{e^{i\varphi}+re^{i\theta}}{e^{i\varphi}-re^{i\theta}}\right]\right|\right) d\rho(\varphi),$$

since $\rho(\varphi)$ is increasing for $-\pi<\varphi\leqq\pi$ and has the total variation equal to unity. Hence, integrating with respect to θ, we get

$$\int_{-\pi}^{\pi} F(|\mathcal{L}[\Phi(re^{i\theta})]|)\,d\theta \leqq \int_{-\pi}^{\pi} d\theta \int_{-\pi}^{\pi} F\left(\left|\mathcal{L}\left[\frac{e^{i\varphi}+re^{i\theta}}{e^{i\varphi}-re^{i\theta}}\right]\right|\right) d\rho(\varphi)$$
$$= \int_{-\pi}^{\pi} d\rho(\varphi) \int_{-\pi}^{\pi} F\left(\left|\mathcal{L}\left[\frac{e^{i\varphi}+re^{i\theta}}{e^{i\varphi}-re^{i\theta}}\right]\right|\right) d\theta = \int_{-\pi}^{\pi} F\left(\left|\mathcal{L}\left[\frac{1+re^{i\theta}}{1-re^{i\theta}}\right]\right|\right) d\theta.$$

For $\Phi(z)=(1+\varepsilon z)/(1-\varepsilon z)$ with $|\varepsilon|=1$, it is evident that the equality sign holds surely in the estimation.

In dealing entirely with extremal functions in the estimation stated in theorem 3, we first consider especially the case where $F(X)$ is linear. Then, by virtue of the homogeneity of the relation to be considered, we have only to observe the case $F(X)=X$.

THEOREM 4. *Under the conditions imposed on $\mathcal{L}$ in theorem 3, suppose further that $\mathcal{L}[(1+z)/(1-z)]$ is non-constant and let n denote the greatest integer such that it is a single-valued function of z^n. Then the relation*

$$\int_{-\pi}^{\pi} |\mathcal{L}[\Phi(re^{i\theta})]|\,d\theta = \int_{-\pi}^{\pi} \left|\mathcal{L}\left[\frac{1+re^{i\theta}}{1-re^{i\theta}}\right]\right| d\theta$$

holds if and only if $\Phi(z)$ is of the form

$$\Phi(z) = \sum_{j=1}^{m}\sum_{k=0}^{n-1} \rho_{jk}\frac{e^{2((j-1)n+k)\pi i/mn}+\varepsilon z}{e^{2((j-1)n+k)\pi i/mn}-\varepsilon z}$$

where ε *is a constant with* $|\varepsilon|=1$ *and* $\{\rho_{jk}\}$ *is a set of real numbers satisfying*

$$\rho_{jk}\geqq 0 \quad (0\leqq k\leqq n-1,\ 1\leqq j\leqq m), \qquad \sum_{j=1}^{m}\sum_{k=0}^{n-1}\rho_{jk}=1,$$

and m *is a positive integer defined in lemma* 3 *in which* $\Psi(z)$ *is replaced by* $\mathcal{L}[(1+z)/(1-z)]$.

Proof. Based on the proof given above for theorem 3, we see that the extremal character of $\Phi(z)$ is characterized in terms of its associated function $\rho(\varphi)$ by the requirement

$$\left|\int_{-\pi}^{\pi}\mathcal{L}\left[\frac{e^{i\varphi}+re^{i\theta}}{e^{i\varphi}-re^{i\theta}}\right]d\rho(\varphi)\right|=\int_{-\pi}^{\pi}\left|\mathcal{L}\left[\frac{e^{i\varphi}+re^{i\theta}}{e^{i\varphi}-re^{i\theta}}\right]\right|d\rho(\varphi)$$

to be valid identically with respect to θ. We now apply lemma 3 to the function defined by

$$\Psi(z)=\mathcal{L}\left[\frac{1+z}{1-z}\right],$$

whence follows that for any extremal function $\Phi(z)$ its associated function $\rho(\varphi)$ remains unchanged except possibly at $\varphi_0+2((j-1)n+k)\pi/mn$ $(0\leqq k\leqq n-1, 1\leqq j\leqq m)$ with the heights $\rho_{jk}\geqq 0$, respectively, say; here m, n and $\{\rho_{jk}\}$ satisfy the conditions mentioned in the theorem. Thus, any extremal function must have the form stated in the theorem with $\varepsilon=e^{-i\varphi_0}$. Conversely, for any function of this form the relation under consideration holds good, as seen in the proof of lemma 3.

In case $F(X)$ is non-linear and hence strictly convex, the condition in theorem 4 imposed on extremal function is to be modified by supplementing a further condition.

THEOREM 5. *Under the conditions imposed on* $\mathcal{L}$ *and* $F(X)$ *in theorem* 3, *let further the increasing function* $F(X)$ *be strictly convex. Then the equality sign in the estimation given in theorem* 3 *holds for any fixed* r *if and only if* $\Phi(z)$ *is of the form given in theorem* 4 *and all those members among* m *quantities*

$$\left|\mathcal{L}\left[\frac{1+re^{i(\theta-\varphi_0-2(j-1)\pi/m)}}{1-re^{i(\theta-\varphi_0-2(j-1)\pi/m)}}\right]\right| \qquad (j=1,\cdots,m)$$

which correspond to the non-vanishing $\sum_{k=0}^{n-1}\rho_{jk}$ *have the same value for any* θ *throughout* $-\pi<\theta\leqq\pi$.

Proof. The condition that the equality sign appears in the estimation given in theorem 3 is equivalent to the requirement that the function $\rho(\varphi)$ associated to an extremal function $\Phi(z)$ satisfies beside the relation

$$\left|\int_{-\pi}^{\pi} \mathcal{L}\left[\frac{e^{i\varphi}+re^{i\theta}}{e^{i\varphi}-re^{i\theta}}\right] d\rho(\varphi)\right| = \int_{-\pi}^{\pi}\left|\mathcal{L}\left[\frac{e^{i\varphi}+re^{i\theta}}{e^{i\varphi}-re^{i\theta}}\right]\right| d\rho(\varphi)$$

a further relation

$$F\left(\int_{-\pi}^{\pi}\left|\mathcal{L}\left[\frac{e^{i\varphi}+re^{i\theta}}{e^{i\varphi}-re^{i\theta}}\right]\right| d\rho(\varphi)\right) = \int_{-\pi}^{\pi} F\left(\left|\mathcal{L}\left[\frac{e^{i\varphi}+re^{i\theta}}{e^{i\varphi}-re^{i\theta}}\right]\right|\right) d\rho(\varphi)$$

also identically with respect to θ. The first relation implies the consequence stated in theorem 4. Now, for any function of the form written in theorem 4, we get

$$F\left(\int_{-\pi}^{\pi}\left|\mathcal{L}\left[\frac{e^{i\varphi}+re^{i\theta}}{e^{i\varphi}-re^{i\theta}}\right]\right| d\rho(\varphi)\right) = F\left(\sum_{j=1}^{m}\sum_{k=0}^{n-1}\rho_{jk}\left|\mathcal{L}\left[\frac{1+re^{i(\theta-\varphi_0-2(j-1)\pi/m)}}{1-re^{i(\theta-\varphi_0-2(j-1)\pi/m)}}\right]\right|\right)$$

and

$$\int_{-\pi}^{\pi} F\left(\left|\mathcal{L}\left[\frac{e^{i\varphi}+re^{i\theta}}{e^{i\varphi}-re^{i\theta}}\right]\right|\right) d\rho(\varphi) = \sum_{j=1}^{m}\sum_{k=0}^{n-1}\rho_{jk} F\left(\left|\mathcal{L}\left[\frac{1+re^{i(\theta-\varphi_0-2(j-1)\pi/m)}}{1-re^{i(\theta-\varphi_0-2(j-1)\pi/m)}}\right]\right|\right),$$

since $\mathcal{L}[(1+z)/(1-z)]$ is a single-valued function of z^n. $F(X)$ being supposed strictly convex, the right-hand members of the last two equations are equal for any r if and only if the last-stated condition of the theorem is satisfied.

A remark may be supplemented. Let d denote the greatest common measure of the set consisting of those integers among $\{j\}_{j=1}^{m}$ for which $\sum_{k=0}^{n-1}\rho_{jk}$ does not vanish. Then the last condition of theorem 5 is equivalent to the requirement that the quantity $|\mathcal{L}[(1+re^{i\theta})/(1-re^{i\theta})]|$, qua function of θ, has $2d\pi/m$ as a period. In particular, if $\sum_{k=0}^{n-1}\rho_{jk}$ does not vanish for an integer j relatively prime to m, the condition is satisfied with $d=1$.

In case $m=1$, the last condition of theorem 5 degenerates to the trivial one, whence follows the following corollary:

COROLLARY. *The equality sign in the estimation given in theorem* 3 (*and a fortiori that given in theorem* 4) *holds for any fixed r if and only if $\Phi(z)$ is of the form*

$$\Phi(z) = \sum_{k=0}^{n-1}\rho_k \frac{e^{2k\pi i/n}+\varepsilon z}{e^{2k\pi i/n}-\varepsilon z}$$

with

$$|\varepsilon| = 1, \qquad \rho_k \geqq 0 \ (0 \leqq k \leqq n-1), \qquad \sum_{k=0}^{n-1}\rho_k = 1,$$

provided $F(X)$ is strictly increasing and the value of the integer m defined in lemma 3 *in which $\Psi(z)$ is replaced by $\mathcal{L}[(1+z)/(1-z)]$ is equal to unity.*

Though the following result is an immediate consequence of the corollary of theorem 5 combined with lemma 4, we write down it here for its frequent use in the subsequent discussions.

THEOREM 6. *Under the conditions imposed on $\mathcal{L}$ and $F(X)$ in theorem 3, let $\mathcal{L}[(1+z)/(1-z)]$ be non-constant, regular and non-vanishing for $0<|z|\leqq r$ and $F(X)$ be strictly increasing. Then the equality sign in the estimation given in theorem 3 holds for any fixed r if and only if $\Phi(z)$ is of the form given in the corollary of theorem 5. If, moreover, $\mathcal{L}[(1+z)/(1-z)]$ satisfies an additional condition that its derivative does not vanish at $z=0$, then the extremal function must be of the form $(1+\varepsilon z)/(1-\varepsilon z)$ with $|\varepsilon|=1$.*

In theorems 4, 5 and 6, it has been supposed that $\mathcal{L}[(1+z)/(1-z)]$ does not reduce to a constant. However, it is to be noted that, if it reduces to a constant, the operator $\mathcal{L}$ degenerates identically to a constant, i.e. its range consists of a single constant function. In fact, we then get by means of lemma 1 $\mathcal{L}[\Phi(z)]=\mathcal{L}[(1+z)/(1-z)]$ for any $\Phi(z)\in\mathfrak{R}$.

Here we supplement a remark on the corollary of theorem 5. We now suppose for a while that the operator $\mathcal{L}$ is also applicable termwise to the Taylor series of argument function of which the bases z^ν $(\nu=0,1,2,\cdots)$ do not belong to $\mathfrak{R}$ except the first one. Then we get, in particular,

$$\mathcal{L}\left[\frac{1+z}{1-z}\right]=\mathcal{L}[1]+2\sum_{\nu=1}^{\infty}\mathcal{L}[z^\nu].$$

Consequently, the condition that this function is single-valued with respect to z^n is equivalent to the system of conditions

$$\mathcal{L}[z^\nu]=0 \qquad \text{for} \quad \nu\not\equiv 0 \pmod{n},$$

whence follows

$$\mathcal{L}\left[\frac{1+z}{1-z}\right]=\mathcal{L}\left[\frac{1+z^n}{1-z^n}\right].$$

In particular, the quantity $\mathcal{L}[\Phi(z)]$ is then for any $\Phi(z)$ also a single-valued function of z^n. Thus, if we put

$$\mathcal{L}[\Phi(z)]=\psi(z^n) \qquad \text{and} \quad \mathcal{L}\left[\frac{1+z}{1-z}\right]=\psi^*(z^n)=\mathcal{L}\left[\frac{1+z^n}{1-z^n}\right],$$

the inequality stated in theorem 3 becomes

$$\int_{-\pi}^{\pi}F(|\psi(r^ne^{in\theta})|)\,d\theta\leqq\int_{-\pi}^{\pi}F(|\psi^*(r^ne^{in\theta})|)\,d\theta$$

which is evidently equivalent to

$$\int_{-\pi}^{\pi}F(|\psi(te^{i\sigma})|)\,d\sigma\leqq\int_{-\pi}^{\pi}F(|\psi^*(te^{i\sigma})|)\,d\sigma, \qquad t=r^n.$$

The last relation coincides formally with that given in theorem 3 with t instead of r and applied to an operator which transforms $\Phi(z)$ into $\psi(z)$. The corollary of theorem 5 then asserts that for such an operator the equality sign in the last estimation is realized not only by a function of the form $(1+\eta z^n)/(1-\eta z^n)$ with $|\eta|=1$ but also by any function of the form mentioned there.

4. Consequences.

Rogosinski's original theorem 1 is, of course, a particular case of theorems 3 and 6. In fact, putting

$$\mathcal{L}=z\frac{d}{dz} \qquad \text{and} \qquad F(X)=X,$$

we get from theorem 3

$$r\int_{-\pi}^{\pi}|\Phi'(re^{i\theta})|\,d\theta \leqq \int_{-\pi}^{\pi} r\frac{2}{|1-re^{i\theta}|^2}\,d\theta = \frac{4\pi r}{1-r^2}.$$

Since then the function

$$\mathcal{L}\left[\frac{1+z}{1-z}\right]=\frac{2z}{(1-z)^2}$$

does not vanish in $0<|z|<1$ and has the non-vanishing derivative at $z=0$, theorem 6 implies that the extremal function is of the form $(1+\varepsilon z)/(1-\varepsilon z)$ with $|\varepsilon|=1$.

In theorem 3, several sorts of choice are possible for $\mathcal{L}$ as well as F. Thus, for instance, we may take as $\mathcal{L}$ any operator which transforms $\Phi(z)$ into a linear combination of $\Phi^{(\mu)}(\gamma_j z)$ ($\mu=0, 1, \cdots, M$; $j=1, \cdots, J$), γ_j being arbitrary constants with $|\gamma_j|<1/r$. Among such operators we choose here as illustrating examples two special ones and state the following theorem which involves in itself Rogosinski's theorem 1 as a particular case.

THEOREM 7. (i) *Let n be a positive integer, ν a non-negative integer and $p\geqq 1$ a real number. Then, for any $\Phi(z)\in\mathfrak{R}$, we have*

$$\frac{1}{n}\int_{-\pi}^{\pi}\left|\sum_{k=0}^{n-1} e^{-2\nu k\pi i/n}\Phi^{(\nu)}(re^{i(\theta-2k\pi/n)})\right|^p d\theta \leqq \int_{-\pi}^{\pi}\left|\frac{\partial^\nu}{\partial r^\nu}\frac{1+r^n e^{i\theta}}{1-r^n e^{i\theta}}\right|^p d\theta$$
$$(0<r<1).$$

The equality sign holds for any fixed r if and only if $\Phi(z)$ is of the form

$$\Phi(z)=\sum_{k=0}^{n-1}\rho_k\frac{e^{2k\pi i/n}+\varepsilon z}{e^{2k\pi i/n}-\varepsilon z}$$

where ε and $\{\rho_k\}_{k=0}^{n-1}$ are defined as in the corollary of theorem 5.

(ii) *Let n be an even positive integer, ν a non-negative integer and $p\geqq 1$ a real number. Then, for any $\Phi(z)\in\mathfrak{R}$, we have*

$$\frac{1}{n}\int_{-\pi}^{\pi}\left|\sum_{k=0}^{n-1}(-1)^k e^{-2\nu k\pi i/n}\Phi^{(\nu)}(re^{i(\theta-2k\pi/n)})\right|^p d\theta \leqq \int_{-\pi}^{\pi}\left|\frac{\partial^\nu}{\partial r^\nu}\frac{2r^{n/2}e^{i\theta}}{1-r^n e^{2i\theta}}\right|^p d\theta$$
$$(0<r<1).$$

The equality sign holds for any fixed r if and only if $\Phi(z)$ is of the form

$$\Phi(z) = \sum_{\kappa=0}^{n/2-1} \rho_\kappa \frac{e^{4\kappa\pi i/n} + \varepsilon z}{e^{4\kappa\pi i/n} - \varepsilon z}$$

with

$$|\varepsilon| = 1, \qquad \rho_\kappa \geqq 0 \left(0 \leqq \kappa \leqq \frac{n}{2} - 1\right), \qquad \sum_{\kappa=0}^{n/2-1} \rho_\kappa = 1.$$

Proof. (i) Applying theorems 3 and 6 to the pair

$$\mathcal{L}[\Phi(z)] = z^\nu \frac{d^\nu}{dz^\nu} \frac{1}{n} \sum_{k=0}^{n-1} \Phi(ze^{-2k\pi i/n}) \quad \text{and} \quad F(X) = X^p,$$

the desired result follows readily after reducing both members by $r^{\nu p}$. In fact, we have only to remember the identity

$$\frac{1}{n} \sum_{k=0}^{n-1} \frac{1 + ze^{-2k\pi i/n}}{1 - ze^{-2k\pi i/n}} = \frac{1}{n} \sum_{k=0}^{n-1} \left(1 + 2 \sum_{h=0}^{\infty} z^h e^{-2hk\pi i/n}\right)$$
$$= 1 + 2 \sum_{\mu=1}^{\infty} z^{\mu n} = \frac{1 + z^n}{1 - z^n}$$

in which the last member shows, in particular, that $\mathcal{L}[(1+z)/(1-z)]$ is single-valued in z^n but not in any higher power of z, and then to transform the estimate by changing the integration variable θ into θ/n, whence follows

$$\int_{-\pi}^{\pi} \left| \frac{d^\nu}{d(re^{i\theta})^\nu} \frac{1 + (re^{i\theta})^n}{1 - (re^{i\theta})^n} \right|^p d\theta = \int_{-\pi}^{\pi} \left| \frac{\partial^\nu}{\partial r^\nu} \frac{1 + r^n e^{i\theta}}{1 + r^n e^{i\theta}} \right|^p d\theta.$$

(ii) We can proceed similarly as above. We now take

$$\mathcal{L}[\Phi(z)] = z^\nu \frac{d^\nu}{dz^\nu} \frac{1}{n} \sum_{k=0}^{n-1} (-1)^k \Phi(ze^{-2k\pi i/n}) \quad \text{and} \quad F(X) = X^p$$

and remember the identity

$$\frac{1}{n} \sum_{k=0}^{n-1} (-1)^k \frac{1 + ze^{-2k\pi i/n}}{1 - ze^{-2k\pi i/n}} = \frac{1}{n} \sum_{k=0}^{n-1} (-1)^k \left(1 + 2 \sum_{h=1}^{\infty} z^h e^{-2hk\pi i/n}\right)$$
$$= 2 \sum_{\mu=0}^{\infty} z^{(\mu+1/2)n} = \frac{2z^{n/2}}{1 - z^n}$$

in which the last member shows, in particular, that $\mathcal{L}[(1+z)/(1-z)]$ is single-valued in $z^{n/2}$ but not in any higher power of z. The estimate then becomes after change of variable

$$\int_{-\pi}^{\pi} \left| \frac{d^\nu}{d(re^{i\theta})^\nu} \frac{2(re^{i\theta})^{n/2}}{1 - (re^{i\theta})^n} \right|^p d\theta = \int_{-\pi}^{\pi} \left| \frac{\partial^\nu}{\partial r^\nu} \frac{2r^{n/2} e^{i\theta}}{1 - r^n e^{2i\theta}} \right|^p d\theta.$$

For $F(X) = X^p$, the inequality on convex functions used in the proof of theorem 3 reduces to Hölder's. More particularly, it degenerates for $F(X) = X$ to a trivial one, as seen in the proof of theorem 1.

On the other hand, it may be noted that in theorem 7 the operation $z^\nu(d/dz)^\nu$ involved in $\mathcal{L}$ can be replaced, for instance, by $(z\,d/dz)^\nu = (d/d\lg z)^\nu$ whence follows an analogous theorem.

It will be readily verified that, corresponding to the formulation of theorem 2, the range of integration in the estimation may be replaced by any measurable set E instead of $-\pi<\theta\leqq\pi$. For instance, the estimation in theorem 7 (i) reduces for $n=1$ after this generalization to

$$\int_E |\Phi^{(\nu)}(re^{i\theta})|^p\,d\theta \leqq \begin{cases} 2\displaystyle\int_0^{mE/2}\left|\frac{1+re^{i\theta}}{1-re^{i\theta}}\right|^p d\theta & (\nu=0), \\ (\nu!\,2)^p\,2\displaystyle\int_0^{mE/2}\frac{d\theta}{|1-re^{i\theta}|^{(\nu+1)p}} & (\nu=1,2,\cdots), \end{cases}$$

and the equality sign holds in every case for any fixed r if and only if $\Phi(z)$ is of the form stated in theorem 2 provided mE is positive.

We remember here by the way that we could consider the functional $(d/dz)^\nu(\Phi(z)-1)$ instead of $(d/dz)^\nu\Phi(z)$. Then the above estimation for $\nu=0$ would be replaced by

$$\int_E |\Phi(re^{i\theta})-1|^p d\theta \leqq 2^{p+1}\int_0^{mE/2}\frac{d\theta}{|1-re^{i\theta}|^p}$$

together with the same extremal functions, in conformity with the result given above for $\nu>0$; cf. also a remark at the end of this section.

Here it is noted, by the way, that the integral of the form

$$S_\lambda(r,\sigma)=\int_0^\sigma\frac{d\theta}{|1-re^{i\theta}|^\lambda}=\int_0^\sigma\frac{d\theta}{(1-2r\cos\theta+r^2)^{\lambda/2}}$$

with any constant $\sigma\geqq 0$ can be evaluated in terms of elementary functions provided λ is an even integer. Here we are interested in the case $\lambda>0$. Actual calculation will show that there exists, in general, a recurrence formula

$$S_{\lambda+2}(r,\sigma)=\frac{2}{\lambda r^{\lambda/2-1}(1-r)^2}\frac{d}{dr}(r^{\lambda/2}S_\lambda(r,\sigma)).$$

We may suppose $0\leqq\sigma\leqq\pi$. Since we have, in particular,

$$S_2(r,\sigma)=\int_0^\sigma\frac{d\theta}{|1-re^{i\theta}|^2}=\frac{2}{1-r^2}\arctan\left(\frac{1+r}{1-r}\tan\frac{\sigma}{2}\right),$$

the quantity $S_\lambda(r,\sigma)$ with an even positive integer λ as suffix can be obtained from $S_2(r,\sigma)$ by means of repeated differentiation combined with elementary operations. We observe, for instance, the particular case $\sigma=\pi$. Then, since we have $S_2(r,\pi)=\pi/(1-r^2)$, it is evident that $S_\lambda(r,\pi)$ with such a λ is a rational function of r^2. Moreover, it will be verified, for instance, by induction with respect to $\lambda/2$ that $S_\lambda(r,\pi)$ is expressed by the formula

$$S_\lambda(r,\pi)=\frac{\pi}{(1-r^2)^{\lambda-1}}\sum_{j=0}^{\lambda/2-1}\binom{\lambda/2-1}{j}^2 r^{2j}$$

provided λ is an even positive integer. On the other hand, for such a λ the quantity defined by

$$T_\lambda(r,\sigma)=\int_0^\sigma\left|\frac{1+re^{i\theta}}{1-re^{i\theta}}\right|^\lambda d\theta=\int_0^\sigma\left(\frac{2(1+r^2)}{1-2r\cos\theta+r^2}-1\right)^{\lambda/2}d\theta$$

is connected with $S_\lambda(r,\sigma)$ by the relation

$$T_\lambda(r,\sigma)=\sum_{\kappa=1}^{\lambda/2}(-1)^{\lambda/2-\kappa}\binom{\lambda/2}{\kappa}2^\kappa(1+r^2)^\kappa S_{2\kappa}(r,\sigma)-(-1)^{\lambda/2}.$$

In particular, the quantity $T_\lambda(r,\pi)$ with an even positive integer λ as suffix is a rational function of r^2 which can be explicitly written down.

As the next example illustrating a consequence of theorems 3 and 6, we state here the following theorem:

THEOREM 8. *Let $p\geqq 1$. Then, for any $\Phi(z)\in\Re$, we have*

$$\int_{-\pi}^{\pi}\left|\int_0^r\Phi(te^{i\theta})\,dt\right|^p d\theta\leqq\int_{-\pi}^{\pi}\left|2\lg\frac{1}{1-re^{i\theta}}-re^{i\theta}\right|^p d\theta\qquad(0<r<1).$$

The equality sign holds for any fixed r if and only if $\Phi(z)$ is of the form $(1+\varepsilon z)/(1-\varepsilon z)$ with $|\varepsilon|=1$.

Proof. We may put in theorems 3 and 6

$$\mathcal{L}[\Phi(z)]=\frac{1}{z}\int_0^z\Phi(z)\,dz=\frac{1}{r}\int_0^r\Phi(te^{i\theta})\,dt\quad(z=re^{i\theta})\quad\text{and}\quad F(X)=X^p,$$

whence readily follows the desired estimation after reducing both members of that obtained from theorem 3 by $1/r^p$. In applying theorem 6, we have only to verify that $\mathcal{L}[(1+z)/(1-z)]=-z^{-1}(2\lg(1-z)+z)$ does not vanish in $0<|z|<1$. It can be shown moreover that $\mathcal{L}[\Phi(z)]$ for any $\Phi(z)\in\Re$ vanishes nowhere in $|z|<1$. In fact, the condition $\Phi(z)\in\Re$ implies by virtue of a theorem due to Noshiro [4] and Wolff [7] (see the remark below) that its integral is univalent in $|z|<1$, so that $\mathcal{L}[\Phi(z)]$ does not vanish in $|z|<1$.

REMARK. In this occasion, for the sake of convenience, a very brief proof will be given of the theorem due to Noshiro and Wolff in a slightly precise form:

Let $f(z)$ be regular and satisfy $\Re f'(z)>0$ in a convex domain D. Then, for any points z_1 and z_2 in D, we have

$$\Re\frac{f(z_2)-f(z_1)}{z_2-z_1}>0.$$

In particular, $f(z)$ is then univalent in D.

In fact, the point $z_1 + t(z_2 - z_1)$ describes, as the real parameter t varies from 0 to 1, the segment connecting z_1 and z_2 which lies in D. Hence, we get

$$\Re \frac{f(z_2) - f(z_1)}{z_2 - z_1} = \int_0^1 \Re f'(z_1 + t(z_2 - z_1))\, dt > 0.$$

Cf. Ozaki [5].

While the function $-\lg X$ is convex for $X > 0$, it is not increasing so that theorem 3 cannot be applied directly for $F(X) = -\lg X$. However, a supplementary inequality can be derived with reference to this function.

THEOREM 9. *Let $\mathcal{L}$ satisfy the condition mentioned in theorem* 3. *Then, for any $\Phi(z) \in \Re$, we have*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \lg |\mathcal{L}[\Phi(re^{i\theta})]|\, d\theta \leqq \lg \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \mathcal{L}\left[\frac{1 + re^{i\theta}}{1 - re^{i\theta}} \right] \right| d\theta \right) \qquad (0 < r < 1).$$

Under the conditions imposed on $\mathcal{L}$ in the corollary of theorem 5, *the equality sign appears for any fixed r if and only if $\Phi(z)$ is of the form stated there and further $|\mathcal{L}[(1 + re^{i\theta})/(1 - re^{i\theta})]|$ is independent of θ.*

Proof. By taking $F(X) = X/2\pi$ in the estimation given in theorem 3, we get

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |\mathcal{L}[\Phi(re^{i\theta})]|\, d\theta \leqq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \mathcal{L}\left[\frac{1 + re^{i\theta}}{1 - re^{i\theta}} \right] \right| d\theta.$$

This combined with the concavity of $\lg X$ then implies

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} \lg |\mathcal{L}[\Phi(re^{i\theta})]|\, d\theta &\leqq \lg \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |\mathcal{L}[\Phi(re^{i\theta})]|\, d\theta \right) \\ &\leqq \lg \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \mathcal{L}\left[\frac{1 + re^{i\theta}}{1 - re^{i\theta}} \right] \right| d\theta \right). \end{aligned}$$

In the last relation, the equality sign of the first inequality holds if and only if $|\mathcal{L}[\Phi(re^{i\theta})]|$ is independent of θ, while that of the second holds if and only if $\Phi(z)$ is of the form stated in the corollary of theorem 5, provided $\mathcal{L}$ satisfies the conditions imposed there. But, for any $\Phi(z)$ of the last-stated form, we have

$$\mathcal{L}[\Phi(z)] = \sum_{k=0}^{n-1} \rho_k\, \mathcal{L}\left[\frac{e^{2k\pi i/n} + \varepsilon z}{e^{2k\pi i/n} - \varepsilon z} \right] = \sum_{k=0}^{n-1} \rho_k\, \mathcal{L}\left[\frac{1 + \varepsilon z}{1 - \varepsilon z} \right] = \mathcal{L}\left[\frac{1 + \varepsilon z}{1 - \varepsilon z} \right]$$

and hence $|\mathcal{L}[\Phi(re^{i\theta})]|$ and $|\mathcal{L}[(1 + re^{i\theta})/(1 - re^{i\theta})]|$ are simultaneously independent of θ or not.

Finally, it is to be noticed that throughout the present paper the results obtained for $\Re$ can be suitably modified for several other classes similar to $\Re$. The most essential point in our discussion is that any member of the class admits an integral representation of the form

$$\Phi(z) = \int_{-\pi}^{\pi} K(e^{-i\varphi}z)d\rho(\varphi)$$

with a certain kernel $K(e^{-i\varphi}z)$ which is in itself a member of the class for every value of φ and corresponds to $(e^{i\varphi}+z)/(e^{i\varphi}-z)$ in case of $\mathfrak{R}$. It may be seen, however, that theorems mentioned in §3 involve substantially such extension. In fact, the transform $\mathcal{L}[(1+z)/(1-z)]$ may be of quite general nature so that $\mathcal{L}[\Phi(z)]$ and $\mathcal{L}[(1+z)/(1-z)]$ can then be replaced by $\mathcal{L}[\Phi(z)]$ $\mathcal{L}[K(e^{-i\varphi}z)]$, respectively.

5. Applications.

Corresponding to length-distortion stated in theorem 1, we can derive as a consequence of itself an analogous estimation on areal distortion.

THEOREM 10. *Let E be any measurable set contained in $-\pi<\theta\leqq\pi$. Then, for any $\Phi(z)\in\mathfrak{R}$, we have*

$$\int_{r_0}^{r} r\,dr\int_E |\Phi'(re^{i\theta})|^2\,d\theta \leqq \Bigg[\frac{8r^2}{(1-r^2)^2}\arctan\left(\tau\frac{1+r}{1-r}\right)+\frac{r}{1-r^2}\left(\tau(1-r)+\frac{1+r}{\tau}\right) - \frac{(1+\tau^2)^2}{2\tau^2}\arctan\frac{2\tau r}{\tau^2+1+(\tau^2-1)r}\Bigg]_{r_0}^{r}$$
$$(0\leqq r_0<r<1)$$

where we put $\tau=\tan(mE/4)$ and denote by mE the measure of E. In case $mE>0$ extremal functions are characterized by the same condition as stated in theorem 2.

Proof. As a particular case of theorem 7 (i) we have noticed subsequently to its proof an estimation

$$\int_E |\Phi'(re^{i\theta})|^2\,d\theta \leqq 4\int_0^{mE/2}\frac{d\theta}{|1-re^{i\theta}|^4} = \frac{4}{1-r^2}\frac{d}{dr}\left(\frac{2r}{1-r^2}\arctan\left(\tau\frac{1+r}{1-r}\right)\right).$$

Integration with respect to r, after multiplied by r, leads readily to

$$\int_{r_0}^{r} r\,dr\int_E |\Phi'(re^{i\theta})|^2\,d\theta \leqq 4\int_{r_0}^{r}\frac{r}{1-r^2}\frac{d}{dr}\left(\frac{2r}{1-r^2}\arctan\left(\tau\frac{1+r}{1-r}\right)\right)dr$$

and actual evaluation of the last integral gives the desired result. The assertion on extremal functions is also evident.

The estimate in theorem 10 expresses, of course, the area of the image-domain of the curvilinear quadrilateral $|\arg z|<mE/2$, $r_0<|z|<r$ by the mapping $(1+z)/(1-z)$. In theorem 10, the case where E covers the whole circumference and r_0 becomes 0 may be of practical interest so that it will be especially stated below.

COROLLARY. *Let $A(r)\equiv A(r,\Phi)$ be the area of the image-domain of $|z|<r$*

by a mapping $\Phi(z) \in \mathfrak{R}$ which is counted according to the multiplicity. Then we have

$$A(r) \equiv \int_0^r r\,dr \int_{-\pi}^{\pi} |\Phi'(re^{i\theta})|^2\,d\theta \leqq \frac{4\pi r^2}{(1-r^2)^2} \qquad (0 < r < 1),$$

and the equality sign appears for any fixed r if and only if $\Phi(z)$ is of the form $(1+\varepsilon z)/(1-\varepsilon z)$ with $|\varepsilon| = 1$.

The estimate in this corollary is naturally equal to the limit value of the estimate in theorem 10 as $r_0 \to 0$ and $mE \to 2\pi$, i.e. $\tau \to \infty$. If we would attempt to obtain this corollary alone, direct calculation would lead to the result in simpler way. In fact, we have

$$A(r) \leqq \int_0^r r\,dr \int_{-\pi}^{\pi} \frac{4\,d\theta}{|1-re^{i\theta}|^4} = 8\pi \int_0^r \frac{r(1+r^2)}{(1-r^2)^3}\,dr = \frac{4\pi r^2}{(1-r^2)^2}.$$

On the other hand, this corollary may be regarded also as a particular case of a more general theorem 11 which will be formulated below. However, it will be of some interest that this particular case is also a consequence of theorem 1 combined with classical isoperimetric inequality which can be shown to be valid also in case of multivalent mapping. In fact, we have simply

$$A(r) \leqq \frac{1}{4\pi} L(r)^2 \leqq \frac{1}{4\pi}\left(\frac{4\pi r}{1-r^2}\right)^2 = \frac{4\pi r^2}{(1-r^2)^2}.$$

Any extremal function for the second inequality maps $|z| < r$ univalently onto a circle and hence it possesses simultaneously the extremal character for the first inequality. On the other hand, the same result may be regarded alternatively as a consequence of a well-known estimation for Taylor coefficients of $\Phi(z) \in \mathfrak{R}$. In fact, let the Taylor expansion of $\Phi(z)$ be

$$\Phi(z) = 1 + \sum_{\nu=1}^{\infty} c_\nu z^\nu.$$

Then, by means of the representation stated in theorem 1, we get

$$|c_\nu| = \left| 2\int_{-\pi}^{\pi} e^{-i\nu\varphi}\,d\rho(\varphi) \right| \leqq 2 \qquad (\nu \geqq 1),$$

and hence

$$A(r) = \int_0^r r\,dr \int_{-\pi}^{\pi} |\Phi'(re^{i\theta})|^2\,d\theta = \int_0^r r\,dr\, 2\pi \sum_{\nu=1}^{\infty} \nu^2 |c_\nu|^2 r^{2\nu-2}$$

$$= \pi \sum_{\nu=1}^{\infty} \nu |c_\nu|^2 r^{2\nu} \leqq 4\pi \sum_{\nu=1}^{\infty} \nu r^{2\nu} = \frac{4\pi r^2}{(1-r^2)^2}.$$

The extremal functions are characterized here by $|c_\nu| = 2$ for any $\nu \geqq 1$ while even a single equation $|c_1| = 2$ implies $\Phi(z) = (1+\varepsilon z)/(1-\varepsilon z)$ with $|\varepsilon| = 1$.

From the integral representation given in theorem 1, we see that, for any z in $|z|<1$, the point $\Phi(z)\in\mathfrak{R}$ laid on the w-plane lies always in the interior of the circle described by $w=(e^{i\varphi}+z)/(e^{i\varphi}-z)$ as φ varies from $-\pi$ to π. Consequently, the range of $\Phi(z)$ originating from $|z|<r(<1)$ is contained in the range-circle of $(1+z)/(1-z)$ as z varies throughout $|z|<r$. But $\Phi(z)$ may be multivalent so that this inclusion does not necessarily take place in the sense of subordination. However, the areal distortion illustrated above shows that the area of the image of $|z|<r$ by $\Phi(z)$ which is counted according to multiplicity does not exceed the area of the image-circle of $|z|<r$ by the mapping $(1+z)/(1-z)$.

Now, theorem 10 is a particular case of theorem 11 formulated below which is in itself an immediate consequence of theorems 3 and 6.

THEOREM 11. *Under the same assumptions as in theorem* 3, *we have*

$$\int_{r_0}^{r} r\,dr\int_{-\pi}^{\pi} F(|\mathcal{L}[\Phi(re^{i\theta})]|)\,d\theta \leqq \int_{r_0}^{r} r\,dr\int_{-\pi}^{\pi} F\left(\left|\mathcal{L}\left[\frac{1+re^{i\theta}}{1-re^{i\theta}}\right]\right|\right)d\theta \qquad (0\leqq r_0<r<1),$$

and the function $(1+\varepsilon z)/(1-\varepsilon z)$ *is for any* ε *with* $|\varepsilon|=1$ *always an extremal function of this estimation. By supplementing further assumptions of theorem* 6, *we can conclude that the equality sign appears, for any fixed* r_0 *and* r, *if and only if* $\Phi(z)$ *is of the form stated in theorem* 6.

6. A supplement.

In conclusion, we supplement here a slight remark. Rogosinski [6] has also pointed out that the length of the image-curve of any radial segment $\arg z=\theta$, $0<|z|<r$ by a mapping $\Phi(z)\in\mathfrak{R}$ can be estimated by

$$\int_0^r |\Phi'(te^{i\theta})|\,dt\leqq\frac{2r}{1-r}.$$

But this result is a quite immediate consequence of a distortion theorem on the class $\mathfrak{R}$ which may be re-formulated here as follows:

LEMMA 5. *For any* $\Phi(z)\in\mathfrak{R}$, *we have*

$$|\Phi^{(\nu)}(z)|\leqq\begin{cases}\dfrac{1+|z|}{1-|z|} & (\nu=0),\\[2ex] \dfrac{\nu!\,2}{(1-|z|)^{\nu+1}} & (\nu=1,2,\cdots).\end{cases}$$

The equality sign holds at any assigned point $te^{i\varphi_0}$ *with* $0<t<1$ *if and only if* $\Phi(z)=(1+\varepsilon z)/(1-\varepsilon z)$ *with* $\varepsilon=e^{-i\varphi_0}$.

Proof. From lemma 1 we have, for any $\Phi(z)\in\mathfrak{R}$,

$$\Phi^{(\nu)}(z)=\begin{cases}\displaystyle\int_{-\pi}^{\pi}\frac{e^{i\varphi}+z}{e^{i\varphi}-z}d\rho(\varphi) & (\nu=0),\\ \displaystyle\int_{-\pi}^{\pi}\frac{\nu!\,2}{(e^{i\varphi}-z)^{\nu+1}}d\rho(\varphi) & (\nu=1,2,\cdots)\end{cases}$$

with

$$d\rho(\varphi)\geqq 0 \quad (-\pi<\varphi\leqq\pi), \qquad \int_{-\pi}^{\pi}d\rho(\varphi)=1,$$

whence readily follows the desired results.

By means of this lemma, Rogosinki's result stated above can be quite readily verified. In fact, we may formulate the following theorem:

THEOREM 12. *Let* $0\leqq r_0<r<1$. *Then, for any* $\Phi(z)\in\Re$, *we have*

$$\int_{r_0}^{r}|\Phi^{(\nu)}(te^{i\theta})|\,dt\leqq\begin{cases}2\lg\dfrac{1-r_0}{1-r}-(r-r_0) & (\nu=0),\\ (\nu-1)!\,2\left(\dfrac{1}{(1-r)^{\nu}}-\dfrac{1}{(1-r_0)^{\nu}}\right) & (\nu=1,2,\cdots).\end{cases}$$

The equality sign in every case holds for assigned values of r_0, r *and* θ *if and only if* $\Phi(z)$ *is given by* $(e^{i\theta}+z)/(e^{i\theta}-z)$.

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