A NOTE ON THE LINEAR DIFFERENTIAL EQUATION OF FUCHSIAN TYPE WITH ALGEBRAIC COEFFICIENTS

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1. Statement of the problem. The purpose of this note is to solve the following problem concerning the linear differential equation of the form

$$\frac{d^n v}{dx^n} + p_1 \frac{d^{n-1} v}{dx^{n-1}} + \cdots + p_n v = 0 \tag{1}$$

where the coefficients $p_1, \cdots, p_n$ are all supposed to be algebraic functions of x

PROBLEM. *Decide the number of independent (complex) parameters necessary and sufficient to specify an equation in the totality of the equations of the form* (1) *of Fuchsian type.*

We begin with the statement of our assumptions and notations.

Let $\mathfrak{F}$ be a Riemann surface of an algebraic function

$$y = \varphi(x)$$

with genus p and consisting of r sheets. Denote by

$$\mathfrak{q}_1 = (\alpha_1, \beta_1), \ \cdots, \ \mathfrak{q}_s = (\alpha_s, \beta_s)$$

the branch points of $\varphi(x)$, and by

$$\mathfrak{r}_1 = (\infty, \gamma_1), \ \cdots, \ \mathfrak{r}_r = (\infty, \gamma_r)$$

the points at infinity on $\mathfrak{F}$ where the notation (α, β) stands for the point of $\mathfrak{F}$ such that $x = \alpha$, $y = \beta$. For simplicity's sake, we assume that

$$\mathfrak{q}_j \neq \mathfrak{r}_k \qquad \text{for} \quad j = 1, \cdots, s \quad \text{and} \quad k = 1 \cdots, r.$$

(i.e. no branch point of $\varphi(x)$ is situated at infinity.)

The cofficients $p_1, \cdots, p_n$ are supposed to be all one-valued and meromorphic on $\mathfrak{F}$ and the singular points of the equation (1) are denoted by

$$\mathfrak{p}_1 = (a_1, b_1), \ \cdots, \ \mathfrak{p}_m = (a_m, b_m).$$

It may happen that some of these singular points coincide with the branch points of $\varphi(x)$. In order to make our discussion possible to include such cases, we suppose that

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$$(2)\qquad \begin{aligned} &\mathfrak{p}_j = \mathfrak{q}_j \qquad &&\text{for } j \leqq \rho, \\ &\mathfrak{p}_j \neq \mathfrak{q}_k \qquad &&\text{for } \rho < j \leqq m \text{ and } \rho < k \leqq s \end{aligned}$$

where ρ is a non-negative integer. Similarly, we suppose that

$$(3)\qquad \begin{aligned} &\mathfrak{p}_{j+\rho} = \mathfrak{r}_j \qquad &&\text{for } j \leqq \sigma, \\ &\mathfrak{p}_j \neq \mathfrak{r}_k \qquad &&\text{for } j \leqq \rho \text{ or } \rho+\sigma < j \leqq m \text{ and } \sigma < k \leqq r, \end{aligned}$$

σ being a non-negative integer.

2. Coefficients of the equation of Fuchsian type. For the equation (1) to be of Fuchsian type, its coefficients $p_1, \cdots, p_n$ must satisfy certain conditions.

First, every p_k must have at most a pole of order k at $\mathfrak{p}_j$, $\rho+\sigma < j \leqq m$. Accordingly its Laurent expansion should be of the form

$$p_k = (x-a_j)^{-k}[A_{kj,0} + A_{kj,1}(x-a_j) + A_{kj,2}(x-a_j)^2 + \cdots],$$
$$k = 1, \cdots, n;\ \rho+\sigma < j \leqq m.$$

Next, we consider the equation (1) in the neighbourhood of $\mathfrak{q}_j$. If it is a branch point of $\varphi(x)$ of order $m_j - 1$ $(m_j > 1)$, we can take

$$\tau = (x - \alpha_j)^{1/m_j}$$

as a local uniformizing variable at $\mathfrak{q}_j$. Then, by simple calculation, we have

$$\frac{d^k v}{dx^k} = \sum_{i=1}^{k} c_{k,i} \frac{d^i v}{d\tau^i} \tau^{i-km_j}, \qquad k = 1, \cdots, n,$$

where $c_{k,i}$ are all non-vanishing constants, and the equation (1) is transformed into

$$(4)\qquad \frac{d^n v}{d\tau^n} + Q_1 \frac{d^{n-1} v}{d\tau^{n-1}} + \cdots + Q_n v = 0,$$

$$(5)\qquad \begin{cases} Q_1 = \dfrac{1}{c_{n,n}}(c_{n,n-1}\tau^{-1} + c_{n-1,n-1}p_1\tau^{m_j-1}), \\ Q_2 = \dfrac{1}{c_{n,n}}(c_{n,n-2}\tau^{-2} + c_{n-1,n-2}p_1\tau^{m_j-2} + c_{n-2,n-2}p_2\tau^{2m_j-2}), \\ \cdots\cdots\cdots\cdots \\ Q_k = \dfrac{1}{c_{n,n}}(c_{n,n-k}\tau^{-k} + c_{n-1,n-k}p_1\tau^{m_j-k} + \cdots + c_{n-i,n-k}p_i\tau^{im_j-k} \\ \qquad\qquad + \cdots + c_{n-k,n-k}p_k\tau^{km_j-k}), \\ \cdots\cdots\cdots\cdots \\ Q_{n-1} = \dfrac{1}{c_{n,n}}(c_{n,1}\tau^{-(n-1)} + c_{n-1,1}p_1\tau^{m_j(n-1)} + \cdots + c_{1,1}p_{n-1}\tau^{(n-1)m_j-(n-1)}), \\ Q_n = \dfrac{1}{c_{n,n}}p_n\tau^{nm_j-n}. \end{cases}$$

From our assumption (2), $\mathfrak{q}_j$ must be a regular singular point of (4) for $j \leqq \rho$. Hence every Q_k must have at most a pole of order k at $\mathfrak{q}_j$, $j \leqq \rho$. As can easily be seen from (5), this condition is satisfied if and only if every p_k has at most a pole of order km_j at $\mathfrak{q}_j$, $j \leqq \rho$. Therefore the Laurent expansion of p_k at $\mathfrak{q}_j$ should be of the form

$$\begin{aligned} p_k &= \tau^{-km_j}(B_{kj,0} + B_{kj,1}\tau + B_{kj,2}\tau^2 + \cdots) \\ &= (x-\alpha_j)^{-k}[B_{kj,0} + B_{kj,1}(x-\alpha_j)^{1/m_j} + B_{kj,2}(x-\alpha_j)^{2/m_j} + \cdots], \\ & \qquad\qquad k=1,\cdots,n;\ j \leqq \rho. \end{aligned}$$

For $j > \rho$, every Q_k must be regular at $\mathfrak{q}_j$ since they are regular points of (4). Therefore, from the first formula of (5), we should have

$$\begin{aligned} & p_1 = \tau^{-m_j}(B'_{1j,0} + B'_{1j,1}\tau + B'_{1j,2}\tau^2 + \cdots), \\ & c_{n,n-1} + c_{n-1,n-1}B'_{1j,0} = 0, \end{aligned}$$

other $B'_{1j,l}$ being arbitrary. Similarly from the second formula of (5), we should have

$$\begin{aligned} & p_2 = \tau^{-2m_j}(B'_{2j,0} + B'_{2j,1}\tau + B'_{2j,2}\tau^2 + \cdots), \\ & c_{n,n-2} + c_{n-1,n-2}B'_{1j,0} + c_{n-2,n-2}B'_{2j,0} = 0, \\ & c_{n-1,n-2}B'_{1j,1} + c_{n-2,n-2}B'_{2j,1} = 0. \end{aligned}$$

From this $B'_{2j,0}$ and $B'_{2j,1}$ are determined as linear functions of $B'_{1j,0}$ and $B'_{1j,1}$, other $B'_{2j,l}$ being arbitrary. Repeating the same reasoning, we generally have

$$\begin{aligned} p_k &= \tau^{-km_j}(B'_{kj,0} + B'_{kj,1}\tau + B'_{kj,2}\tau^2 + \cdots) \\ &= (x-\alpha_j)^{-k}[B'_{kj,0} + B'_{kj,1}(x-\alpha_j)^{1/m_j} + B'_{kj,2}(x-\alpha_j)^{2/m_j} + \cdots], \end{aligned}$$

where $B'_{kj,0}, \cdots, B'_{kj,k-1}$ are determined as linear functions of $B'_{ij,l}$, $i<k$, $l=0,1,2,\cdots$ (especially, for $k=n$, $B'_{nj,0} = \cdots = B'_{nj,n-1} = 0$), other $B'_{kj,l}$ being arbitrary.

Finally, we must determine the behaviour of p_k at $\mathfrak{r}_j$. For that purpose, however, it suffices to replace m_j by -1 and $\tau = (x-\alpha_j)^{1/m_j}$ by $\tau = x^{-1}$ in above discussions. Thus we have obtained the conditions for the equation (1) to be of Fuchsian type which can be stated as follows:

1. *Except the points* $\mathfrak{p}_1, \cdots, \mathfrak{p}_m$; $\mathfrak{q}_1, \cdots, \mathfrak{q}_s$ *every* p_k *should be regular.*
2. *In the neighbourhood of* $\mathfrak{p}_j$, $\rho+\sigma < j \leqq m$, *every* p_k *should be expanded in the form*

$$p_k = (x-a_j)^{-k}[A_{kj,0} + A_{kj,1}(x-a_j) + A_{kj,2}(x-a_j)^2 + \cdots].$$

3. *In the neighbourhood of* $\mathfrak{q}_j(=\mathfrak{p}_j)$, $j \leqq \rho$, *evry* p_k *should be expanded in the form*

$$p_k=(x-\alpha_j)^{-k}[B_{kj,0}+B_{kj,1}(x-\alpha_j)^{1/m_j}+B_{kj,2}(x-\alpha_j)^{2/m_j}+\cdots].$$

4. *In the neighbourhood of* $\mathfrak{r}_j(=\mathfrak{p}_{j+\rho})$, $j\leqq\sigma$, *every* p_k *should be expanded in the form*

$$p_k=x^{-k}[C_{kj,0}+C_{kj,1}x^{-1}+C_{kj,2}x^{-2}+\cdots].$$

5. *In the neighbourhood of* $\mathfrak{q}_j$, $\rho<j\leqq s$, *every* p_k *should be expanded in the form*

$$p_k=(x-\alpha_j)^{-k}[B'_{kj,0}+B'_{kj,1}(x-\alpha_j)^{1/m_j}+B'_{kj,2}(x-\alpha_j)^{2/m_j}+\cdots],$$

where $B'_{1j,0}$ *is a definite constnat and* $B'_{kj,0}\cdots$, $B'_{kj,k-1}$ *are linear functions of* $B'_{ij,l}$, $i<k$, $l=0,1,2,\cdots$.

6. *In the neighbourhood of* $\mathfrak{r}_j$, $\sigma<j\leqq r$, *every* p_k *should be expanded in the form*

$$p_k=x^{-k}[C'_{kj,0}+C'_{kj,1}x^{-1}+C'_{kj,2}x^{-2}+\cdots],$$

where $C_{1j,0}$ *is a definite constant and* $C'_{kj,0},\cdots,C'_{kj,k-1}$ *are linear functions of* $C'_{ij,l}$, $i<k$, $l=0,1,2,\cdots$.

3. Number of arbitrary constants contained in p_k. Suppose that $p_1,\cdots,p_{k-1}$ have been so determined as to satisfy the conditions 1 to 6 just obtained, then p_k will be characterized by following conditions:

a. p_k *is regular on* $\mathfrak{F}$ *except the points* $\mathfrak{p}_1,\cdots,\mathfrak{p}_m$; $\mathfrak{q}_1,\cdots,\mathfrak{q}_s$.
b. p_k *has a pole of order* k *at* $\mathfrak{p}_j$, $\rho+\sigma<j\leqq m$.
c. p_k *has a pole of order* km_j *at* $\mathfrak{q}_j(=\mathfrak{p}_j)$, $j\leqq\rho$.
d. p_k *has a zero of order* k *at* $\mathfrak{r}_j(=\mathfrak{p}_{j+\rho})$, $j\leqq\sigma$.
e. p_k *has a pole of order* km_j *at* $\mathfrak{q}_j$, $\rho<j\leqq s$, *and the first* k *coefficients of its Laurent expansion have some specified values.*
f. p_k *has a zero of order* k *at* $\mathfrak{r}_j$, $\sigma<j\leqq r$, *and the first* k *coefficients of its Taylor expansion have some specified values.*

The difference $f(x, y)$ of any two such functions always satisfies following conditions:

a′. $f(x,y)$ *is regular on* $\mathfrak{F}$ *except the points* $\mathfrak{p}_1,\cdots,\mathfrak{p}_m$; $\mathfrak{q}_1,\cdots,\mathfrak{q}_s$.
b′. $f(x,y)$ *has a pole of order* k *at* $\mathfrak{p}_j$, $\rho+\sigma<j\leqq m$.
c′. $f(x,y)$ *has a pole of order* km_i *at* $\mathfrak{q}_j(=\mathfrak{p}_j)$, $j\leqq\rho$.
d′. $f(x,y)$ *has a zero of order* k *at* $\mathfrak{r}_j(=\mathfrak{p}_{j+\rho})$, $j\leqq\sigma$.
e′. $f(x,y)$ *has a pole of order* km_j-k *at* $\mathfrak{q}_j$, $\rho<j\leqq s$.
f′. $f(x,y)$ *has a zero of order* $2k$ *at* $\mathfrak{r}_j$, $\sigma<j\leqq r$.

It is therefore obvious that p_k contains the same number of arbitrary constants as contained in a function $f(x, y)$ satisfying above conditions.

The number of arbitrary constants ν_k contained in $f(x, y)$ will be given by well-known Riemann-Roch's theorem which asserts that, if the degree of a divisor

$$\mathfrak{d} = \prod_{j \leq \rho} \mathfrak{p}_j^{km_j} \prod_{j \leq \sigma} \mathfrak{p}_{j+\rho}^{-k} \prod_{\rho+\sigma < j \leq m} \mathfrak{p}_j^k \prod_{\rho < j \leq s} \mathfrak{q}_j^{km_j - k} \prod_{\sigma < j \leq r} \mathfrak{r}_j^{-2k}$$

is greater than $2p-2$,

$$\nu_k = \deg(\mathfrak{d}) + 1 - p,$$

where $\deg(\mathfrak{d})$ means the degree of $\mathfrak{d}$. Now, since

$$\begin{aligned} \deg(\mathfrak{d}) &= k\Big[\sum_{j \leq \rho} m_j - \sigma + m - (\rho+\sigma) + \sum_{\rho < j \leq s} m_j - (s-\rho) - 2(r-\sigma)\Big] \\ &= k\Big[m - 2r + \sum_{j=1}^{s}(m_j - 1)\Big], \end{aligned}$$

and, according to Hurwitz' formula,

$$\sum_{j=1}^{s}(m_j - 1) = 2(r + p - 1),$$

we have

$$\deg(\mathfrak{d}) = k(m + 2p - 2) > 2p - 2.$$

Therefore

$$\nu_k = k(m + 2p - 2) + 1 - p.$$

Here we must notice that ν_k depends neither on ρ nor on σ.

4. Solution of the problem. From what we have shown, the number of independent parameters contained in the equation (1) of Fuchsian type is equal to

$$\begin{aligned} \nu &= \sum_{k=1}^{n} \nu_k = (m + 2p - 2)\sum_{k=1}^{n} k + n(1-p) \\ &= \frac{1}{2}n^2(m + 2p - 2) + \frac{1}{2}mn, \end{aligned} \tag{6}$$

if the position of singular points $\mathfrak{p}_1, \cdots, \mathfrak{p}_m$ is given.

If only the number m of singular points is given, and their position is unspecified, the number of independent parameters is given by

$$\nu = \frac{1}{2}n^2(m + 2p - 2) + \frac{1}{2}mn + m. \tag{7}$$

In the case when $p = 0$, the group of automorphisms of $\mathfrak{F}$ contains three independent parameters. Therefore, if we regard the equations which can be transformed mutually by a birational mapping of the Riemann surface as equivalent, formula (7) must be replaced by

$$\nu = \frac{1}{2}n^2(m - 2) + \frac{1}{2}mn + m - 3. \tag{8}$$

In the case when $p = 1$, similarly, we have

$$\nu = \frac{1}{2}n^2m + \frac{1}{2}nm + m - 1 = \frac{1}{2}mn(n+1) + m - 1, \tag{9}$$

since the group of automorphisms of $\mathfrak{F}$ is a one-parameter group.

For $p \geqq 2$, the totality of automorphisms of $\mathfrak{F}$ being finite, the formula (7) holds without modification.

Thus we have obtained the following theorem which will respond our problem.

THEOREM. *The number of independent (complex) parameters necessary and sufficient to specify an equation in the totality of the equations of the form (1) of Fuchsian type is equal to*

$$\frac{1}{2}n^2(m-2) + \frac{1}{2}mn + m - 3 \qquad \textit{for} \quad p = 0,$$

$$\frac{1}{2}mn(n+1) + m - 1 \qquad \textit{for} \quad p = 1,$$

$$\frac{1}{2}n^2(m+2p-2) + \frac{1}{2}mn + m \qquad \textit{for} \quad p \geqq 2,$$

when the position of the singularities is unspecified and the birationally equivalent equations are identified.

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