ON SOME ANALYTIC FUNCTIONS IN AN ANNULUS

By Hitoshi Abe

1. Introduction.

Recently Goodman [2] has studied the class of analytic functions $f(z)$ typically-real in the unit circle $|z|<1$, i.e., satisfying the condition

$$\Im f(z)\Im z \geqq 0 \quad \text{for} \quad |z|<1,$$

and further having a zero or pole of order 1 at the origin. He then has derived some sharp distortion theorems and many other results.

In the present paper we shall first deal with the class of functions which are single-valued and meromorphic and satisfy the condition $\Im f(z)\Im z \geqq 0$ in the annulus

$$q<|z|<1 \qquad (q>0).$$

Hereafter we denote this annulus by D. Under the condition $\Im f(z)\Im z\geqq 0$ we can easily show that all the poles of these functions lie on the real axis and are of order 1, as it has been remarked by Goodman in the case of the unit circle [2].

On the other hand, Komatu [3] has investigated conformal mapping of an annulus onto a ring domain bounded by convex curves or a star-like ring domain and established several distortion theorems. We shall secondly deal with the class of single-valued functions with positive real part in the annulus D by means of Robinson's results [5] and then derive distortion theorems related to Komatu's.

2. Typically-real function.

Let $f(z)$ be single-valued, meromorphic and typically-real in the annulus D. We begin with evaluating the absolute value of $f(a)$ $(q<|a|<1)$. In this section we assume $\Im a>0$. It will be understood later that the case of $\Im a<0$ is quite similar. In order to map the upper semi-annulus of D conformally onto the unit circle we consider the following three steps:

$$u=\frac{\pi}{2}+i\log\frac{z}{q}, \qquad z=iq\exp(-iu);$$

$$v=\operatorname{sn}(u,k), \qquad u=\operatorname{sn}^{-1}(v)=\int_0^v \frac{dv}{\sqrt{(1-v^2)(1-k^2v^2)}},$$

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$$k^2=16q^2\prod_{n=1}^{\infty}\left(\frac{1+q^{4n}}{1+q^{4n-2}}\right)^8;$$

$$t=\frac{v-v(a)}{v-\overline{v(a)}},\qquad\qquad v(a)=\mathrm{sn}\left(\frac{\pi}{2}+\mathrm{i}\log\frac{a}{q}\right).$$

In the first step the upper semi-annulus of D is mapped onto the rectangle D_1: $-\pi/2<\Re u<\pi/2$, $0<\Im u<\log(1/q)$, where an appropriate branch of $\log(z/q)$ is considered. In the second step D_1 is mapped onto the upper half plane D_2: $\Im v>0$, where an appropriate branch of the inverse function of the elliptic function $\mathrm{sn}\, u$ is considered. (See Nehari [4, p. 280].) In the last step D_2 is mapped onto the unit circle $|t|<1$. In this mapping $z=a$ corresponds to $t=0$. Now we put

$$g(t)=f(z(u(v(t)))).$$

Then $\Im g(t)>0$ in the unit circle $|t|<1$, and therefore $\Re(-ig(t))>0$. Hence by means of Carathéodory's theorem

$$|-ig'(0)|\leqq 2\Re(-ig(0))=2\Im f(a),$$

that is,

$$|f'(a)z'(u(a))u'(v(a))v'(0)|\leqq 2\Im f(a)\leqq 2|f(a)|.$$

Since

$$\left(\frac{dz}{du}\right)^{u=u(a)}=-ia,\quad \left(\frac{du}{dv}\right)^{v=v(a)}=\frac{1}{\sqrt{(1-v^2(a))(1-k^2v^2(a))}},\quad \left(\frac{dv}{dt}\right)^{t=0}=2i\Im v(a),$$

we have the following theorem:

THEOREM 1. *Let $f(z)$ be single-valued, meromorphic and typically-real in the annulus D. For each non-real a ($q<|a|<1$, $\Im a>0$),*

$$|f'(a)|\leqq\frac{|(1-v^2(a))(1-k^2v^2(a))|^{1/2}}{|a|\Im v(a)}|f(a)|,$$

where

$$v(a)=\mathrm{sn}\left(\frac{\pi}{2}+i\log\frac{a}{q}\right).$$

This estimate is sharp as is shown by

$$f_0(z)=i\frac{1+t(z)}{1-t(z)}=\frac{\mathrm{sn}\,(\pi/2+i\log(z/q))-\Re\,\mathrm{sn}\,(\pi/2+i\log(a/q))}{\Im\,\mathrm{sn}\,(\pi/2+i\log(a/q))}.$$

REMARK. The extremal function $w=f_0(z)$ maps the upper semi-annulus onto the upper half-plane $\Im w>0$ in such a way that the segment $-1<z<-q$ or $q<z<1$ correspond to the segment $-1/k<w<-1$ or $1<w<1/k$, respectively. It is sufficient to make analytic continuation of $f_0(z)$ along each segment by the principle of reflection in order to understand $f_0(z)$ as a single-valued analytic function in the annulus D.

Next we shall estimate the absolute value of $f(a)$ $(q<|a|<1)$. Let $a_0=i\cdot(1+q)/2$. Without loss of generality we can assume $f(a_0)=i$, because if the condition is not satisfied, it is only necessary to consider a linear transformation of $f(z)$. Moreover we assume $\Im a>0$ as in theorem 1. We consider

$$s(z)=\frac{v(z)-v(a_0)}{v(z)-\overline{v(a_0)}},$$

where $v(z)=\mathrm{sn}\,(\pi/2+\log(z/q))$. It is clear that the upper semi-annulus of D is mapped conformally onto the unit circle $|s|<1$ by $s(z)$ in such a way that $z=a_0$ corresponds to $s=0$. We put

$$g(s)=f(z(s)).$$

Then $\Re(-ig(s))>0$ and $-ig(0)=1$. Therefore, we get

$$\frac{1-|s|}{1+|s|}\leqq|g(s)|\leqq\frac{1+|s|}{1-|s|}$$

Thus we have the following theorem:

THEOREM 2. *Let $f(z)$ satisfy the same conditions as in theorem* 1 *and moreover $f(a_0)=i$, $(a_0=i(1+q)/2)$. For non-real a $(q<|a|<1, \Im a>0)$,*

$$\frac{1-|s(a)|}{1+|s(a)|}\leqq|f(a)|\leqq\frac{1+|s(a)|}{1-|s(a)|},$$

where

$$s(a)=\frac{\mathrm{sn}\,(\pi/2+i\log(a/q))-\mathrm{sn}\,(\pi/2+i\log(a_0/q))}{\mathrm{sn}\,(\pi/2+i\log(a/q))-\overline{\mathrm{sn}\,(\pi/2+i\log(a_0/q))}}.$$

This estimate is sharp as is shown by

$$f_0(z)=\frac{i(1+s(z))}{1-s(z)}=\frac{\mathrm{sn}\,(\pi/2+i\log(z/q))-\Re\,\mathrm{sn}\,(\pi/2+i\log(a_0/q))}{\Im\,\mathrm{sn}\,(\pi/2+i\log(a_0/q))}.$$

Now we can derive the following theorem immediately from theorems 1 and 2.

THEOREM 3. *Let $f(z)$ satisfy the same condition as in theorem* 2. *For each non-real a $(q<|a|<1, \Im a>0)$,*

$$|f'(a)|\leqq\frac{|(1-v^2(a))(1-k^2v^2(a)|^{1/2}}{|a|\Im v(a)}\cdot\frac{1+|s(a)|}{1-|s(a)|},$$

where $v(z)$ and $s(z)$ are the functions already defined.

In the following lines we shall estimate the absolute value of $f(a)$ $(q<|a|<1)$ by means of the Bergman kernel function. As before, we assume $\Im a>0$. Let $t=g(z)$ map the upper semi-annulus of D conformally onto the unit circle $|t|<1$ under the conditions $g(a)=0$ and $g'(a)>0$, and $K(z, a)$ denote the

Bergman kernel function of the upper semi-annulus of D. Then it is known (See Nehari [4, p. 252]) that

$$g'(z)=\sqrt{\frac{\pi}{K(a, a)}}K(z, a).$$

If we put

$$h(t)=f(g^{-1}(t))=f(z),$$

then $\Re(-ih(t))>0$ in $|t|<1$. Hence $|h'(0)|\leqq 2\Im f(a)$, that is,

$$|f'(a)|\leqq 2\sqrt{\pi K(a, a)}\,|f(a)|.$$

Now let δ denote the shortest distance from a to the boundary of the upper semi-annulus of D. Then $K(a, a)$ is estimated in terms of δ as follows (See Bergman [1, p. 6]):

$$K(a, a)<\frac{1}{\pi\delta^2}.$$

Hence we have the following theorem.

THEOREM 4. *Let $f(z)$ satisfy the same conditions as in theorem* 1. *For each non-real a $(q<|a|<1,\ \Im a>0)$,*

$$|f'(a)|\leqq 2\sqrt{\pi K(a, a)}|f(a)|<\frac{2|f(a)|}{\delta},$$

where δ denotes the shortest distance from a to the boundary of the upper semi-annulus of D.

3. Functions with positive real part and related ones.

We shall begin with proving the following two lemmas, which are slight extensions of Robinson's results [5].

LEMMA 1. *Let $f(z)$ be single-valued and regular in the annulus D except perhaps for a simple pole a $(q<|a|<1)$ and satisfy, for given $\varepsilon>0$ and any point ζ on the boundary of D,*

$$|f(z)|<1+\varepsilon \quad for \quad z\in D\cap V(\zeta),$$

where $V(\zeta)$ is a neighborhood of ζ determined appropriately for ε. If $q<|z|<1$ and $z/a<0$, then

$$|f(z)|\leqq 1.$$

Equality occurs only when $f(z)$ is a constant.

Proof. We may assume that $a<0$ and therefore $q<z<1$. Otherwise, putting $z'=\alpha z$ $(\alpha a<0,\ |\alpha|=1)$, $g(z')=f(\alpha^{-1}z')=f(z)$, we have only to consider $g(z')$ for $f(z)$. Without loss of generality we can assume $f(z_0)>0$ for a given z_0 with $q<z_0<1$. We put

$$2F(z)=f(z)+\overline{f(\bar{z})}.$$

Then $F(z)$ satisfies the same conditions as $f(z)$, and moreover $F(z)$ is real for real z and $F(z_0)=f(z_0)$. If $F(z)$ is regular in D, lemma 1 is clear. If $F(z)$ has a simple pole at a, $w=F(z)$ takes all real values from $w\geqq 1+\varepsilon$ or $w\leqq -(1+\varepsilon)$ in the interval $-1<z<-q$, because $F(a)=\infty$ and $|F(z)|<1+\varepsilon$ in the appropriate neighborhood of $\zeta=-1$ and $\zeta=-q$. Suppose $F(z_0)\geqq 1+\varepsilon$. Then this is incompatible with the fact that the values $w\geqq 1+\varepsilon$ are taken only once by $F(z)$. In fact, we consider the integral

$$\frac{1}{2\pi i}\int\frac{F'(z)}{F(z)-w}dz$$

along a simply closed curve which is sufficiently near the boundary of D, and then we see that if we make w tend to ∞, this integral tend to zero and therefore $F(z)$ takes every value $|w|\geqq 1+\varepsilon$ once by the principle of argument together with the continuity of this integral in w, because $F(z)$ has only one simple pole at a. Hence $F(z_0)<1+\varepsilon$, that is, $f(z_0)=F(z_0)\leqq 1$. If the equality occurs, $F(z)=1$ and hence $\Re f(z)=1$ for real z. Therefore on the positive real axis

$$\lim_{z\to q+0}f(z)=\lim_{z\to 1-0}f(z)=1,$$

because $|f(z)|<1+\varepsilon$ in the neighborhood of $\zeta=q$ and $\zeta=1$. On the other hand, $f(z)$ can take no value in $|w|\geqq 1+\varepsilon$ more than once on the segment $q<z<1$, where $\Re f(z)=1$. Hence $f(z)=1$ in D identically. This completes the proof.

Let a_0 and a_1 be two points in D, where $|a_0||a_1|=q$. And suppose that $H(z,1;\ a_0,a_1)$ is single-valued and regular, and has only two zero points of order 1 at a_0 and a_0, and its absolute value is equal to 1 on the boundary of D. Indeed such a function exists and it is expressed explicitly by means of the theta function as follows:

$$H(z,1;\ a_0,a_1)=\frac{z\,\theta(qz'/|a_0|)\theta(qz''/|a_1|)}{\theta(|a_0|z'/q)\,\theta(|a_1|z''/q)},$$

$$z'=\alpha_1 z\quad (|\alpha_1|=1,\quad \arg\alpha_1=\pi-\arg a_0),$$

$$z''=\alpha_2 z\quad (|\alpha_2|=1,\quad \arg\alpha_2=\pi-\arg a_1),$$

$$\theta(z)=\prod_{n=1}^{\infty}[(1-q^{2n})(1+q^{2n-1}z)(1+q^{2n-1}z^{-1})].$$

LEMMA 2. *Let $f(z)$ be single-valued and regular in the annulus D. Moreover $|f(z)|<1$ and $f(a_0)=0$, $(a_0=i(1+q)/2)$. Then*

$$|f(z)|\leqq|H(z,1;\ a_0,a_1)|\qquad (q<|z|<1),$$

where a_1 is determined by the relation $z/a_1<0$, and $|a_0||a_1|=q$. Equality occurs only when $f(z)=\alpha H(z,1;\ a_0,a_1)$ $(|\alpha|=1)$.

Proof. We denote $H(z, 1; a_0, a_1)$ simply by $H(z)$. We consider

$$F(z)=\frac{f(z)}{H(z)}.$$

Then $F(z)$ is single-valued and regular in D except perhaps for a simple pole a_1. Moreover, for given $\varepsilon>0$ and any point ζ on the boundary of D, there exists always such a neighborhood of ζ, denoted by $V(\zeta)$, that we have

$$|F(z)|<1+\varepsilon \qquad z\in V(\zeta)\cap D$$

Here by means of lemma 1 we have the relation

$$|F(z)\leqq 1, \quad \text{that is,} \quad |f(z)|\leqq|H(z)|.$$

The final part of the present lemma is also obtained similarly.

THEOREM 5. *Let $f(z)$ be single-valued and regular in the annulus D. Moreover* $\Re f(z)>0$ and $f(a_0)=1$ $(a_0=i(1+q)/2)$. *Then*

$$\frac{1-|H(z)|}{1+|H(z)|}\leqq|f(z)|\leqq\frac{1+|H(z)|}{1-|H(z)|}.$$

This estimate is sharp as is shown by

$$f_0(z)=\frac{1+\alpha H(z)}{1-\alpha H(z)} \qquad (|\alpha|=1).$$

Proof. The assumption $f(a_0)=1$ means no restriction on the generality. We put

$$g(z)=\frac{f(z)-1}{f(z)+1}.$$

Then $|g(z)|<1$ in D and $g(a_0)=0$. Hence by means of lemma 2

$$|g(z)|\leqq|H(z)|.$$

On the other hand

$$f(z)=\frac{1+g(z)}{1-g(z)}.$$

Therefore

$$\frac{1-|H(z)|}{1+|H(z)|}\leqq\left|\frac{1+g(z)}{1-g(z)}\right|\leqq\frac{1+|H(z)|}{1-|H(z)|}.$$

THEOREM 6. *Under the same conditions as in theorem* 5,

$$|f'(z)|\leqq\frac{4\pi\widehat{K}(z,\bar{z})}{(1-|H(z)|)^2},$$

where $\widehat{K}(z, z)$ is the Szegö kernel function of the annulus D.

Proof. We remember that $f(z)=(1+g(z))/(1-g(z))$ with $|g(z)|<1$ in D. Hence (See Bergman [4, p. 87] and Nehari [4, p. 391])

$$|g'(z)|\leqq 2\pi\hat{K}(z,\bar{z})=\sum_{n=-\infty}^{\infty}\frac{|z|^{2n}}{1-q^{2n+1}}.$$

Therefore

$$|f'(z)|=\frac{2|g'(z)|}{|1-g(z)|^2}\leqq\frac{4\pi K(z,\bar{z})}{(1-|H(z)1|)^2}.$$

Finally by means of theorem 5 we shall prove two theorems. For this purpose, we define star-like ring domain and convex ring domain.

DEFINITION 1. Let D_s be a ring domain with the boundary consisting of two simply closed analytic curves which are both star-like with respect to the origin. Then we call D_s a *star-like domain.*

DEFINITION 2. Let D_c be a finite ring domain with the boundary consisting of two simply closed analytic convex curves. Then we call D_c a *convex ring domain.*

THEOREM 7. *Let $f(z)$ map conformally the annulus D onto a star-like ring domain D_s, where the circle $|z|=q$ correponds to the inner boundary of D_s and the circle $|z|=1$ to the outer one. Moreover let $f(z)$ be normalized by the condition $a_0f'(a_0)/f(a_0)=1$ $(a_0=i(1+q)/2)$. Then*

$$\frac{1-|H(z)|}{1+|H(z)|}\leqq\left|z\frac{f'(z)}{f(z)}\right|\leqq\frac{1+|H(z)|}{1-|H(z)|}.$$

Proof. The assumption $a_0f'(a_0)/f(a_0)=1$ means no restriction on the generality. Otherwise, it is only necessary to consider a linear transformation of $zf'(z)/f(z)$. Now $f(z)$ is regular on the boundary of D by the principle of analytic continuation, because the both boundary curves of D are analytic. Further we have $\Re(zf'(z)/f(z))>0$ on the boundary of D, because the boundary of D_s is star-like with respect to the origin. Hence by the principle of maximum on harmonic functions we get $\Re(zf'(z)/f(z))>0$ in D. Therefore we have the theorem 7.

THEOREM 8. *Let $f(z)$ map conformally the annulus D onto a convex ring domain D_c, where the circle $|z|=q$ corresponds to the inner boundary of D_c and the circle $|z|=1$ to the outer one. Moreover let $f''(a_0)=0$. Then*

$$\frac{1-|H(z)|}{1+|H(z)|}\leqq\left|1+z\frac{f''(z)}{f'(z)}\right|\leqq\frac{1+|H(z)|}{1-|H(z)|}.$$

Proof. The assumption $f''(a_0)=0$ means no restriction on the generality as in theorem 7. And by a similar discussion as in theorem 7, we see easily that $\Re(1+zf''(z)/f'(z))>0$ in D because of convexity of the boundary of D_c. This completes the proof.

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