ON AN EIGENVALUE AND EIGENFUNCTION PROBLEM OF THE EQUATION $\Delta u + \lambda u = 0$

BY IMSIK HONG

First of all, we shall observe a phenomenon on a two-dimensional eigenvalue problem of the equation $\Delta u + \lambda u = 0$ about the fixed boundary condition for a special domain whose boundary consists of a circumference of a circle and its centre. Let us denote by D such a domain, by C its circular boundary with radius R and by C^* its centre which is also a boundary point of D.

Next we take a sequence of annuli

$$D_1 \subset D_2 \subset \cdots \subset D_n \subset \cdots$$

exhausting the domain D. The boundary of the annulus D_n consists of two circular components. Let one of them be C which is the circular boundary of D, and another be C_n of radius R_n where $\lim_{n\to\infty} R_n = 0$.

Now consider an eigenvalue problem for the domain D_n:

$$\Delta u + \lambda u = 0 \quad \text{in} \quad D_n,$$
$$u = 0 \quad \text{on} \quad C + C_n,$$
$$n = 1, 2, \cdots.$$

Let the first eigenvalue and the first eigenfunction be λ_n and u_n, respectively. Then we can readily show that the following phenomenon occurs:

When n tends to ∞, the sequence $\{u_n\}$ satisfying a suitable normalization converges to the first eigenfunction of the whole circular domain together with its centre, namely $D + C^*$ and the same is true for the eigenvalue, i. e. λ_n tends to the first eigenvalue of $D + C^*$.

In fact, let the polar coordinates be denoted by (r, θ). Then $u = a_1 J_0(\sqrt{\lambda}\, r) + a_2 Y_0(\sqrt{\lambda}\, r)$ is a general solution for $\Delta u + \lambda u = 0$ which is independent of θ, where J_0 and Y_0 denote the Bessel functions of the zero-th order, and a_1 and a_2 are any constants. As $u = 0$ on C_n and C, so we have the relations

$$a_1 J_0(\sqrt{\lambda}\, R_n) + a_2 Y_0(\sqrt{\lambda}\, R_n) = 0,$$
$$a_1 J_0(\sqrt{\lambda}\, R) + a_2 Y_0(\sqrt{\lambda}\, R) = 0.$$

Since $J_0(\sqrt{\lambda}\, R_n) \to 1$ and $Y_0(\sqrt{\lambda}\, R_n) \to -\infty$ for $R_n \to 0$, the first equation implies that a_2 must tend to zero, and hence the limit function of u_n becomes

Received September 30, 1957.

$a_1 J_0(\sqrt{\lambda}\, r)$ where a_1 is to be determined by a certain normalization. From the second equation there follows $J_0(\sqrt{\lambda}\, R) = 0$, so that the limit function satisfies the fixed boundary condition on the circular boundary C.

Since we see that $J_0(0) = 1 \neq 0$, the first eigenfunction of the same problem for a circle never vanishes at its centre. Therefore, C^* can not be regarded as a boundary point of our limit function. But, we may explain this phenomenon in the following way:

Our limit function satisfies the boundary condition except for a single isolated boundary point C^* (of course, of capacity zero).

Thus this suggests us to investigate the following more general problem: Let C be a smooth closed curve, D' be the bounded domain surrounded by C, and C^* be a closed set lying entirely in the interior of D'. We now consider the domain D whose boundary consists of C and C^*. We take a sequence of domains

$$D_1 \subset D_2 \subset \cdots \subset D_n \subset \cdots$$

exhausting D, i. e. a sequence $\{D_n\}$ such that $\lim_{n\to\infty} D_n = D$. Let the boundary of the domain D_n consist of C and C_n where C_n consists of a finite number of smooth curves tending to C^* as $n \to \infty$.

Now consider the eigenvalue problem:

$$\Delta u + \lambda u = 0 \quad \text{in} \quad D_n,$$
$$u = 0 \quad \text{on} \quad C + C_n.$$

Denote by λ_n the first eigenvalue and by u_n the first eigenfunction normalized by

$$\iint_{D_n} u_n^2 d\sigma = 1 \quad \text{and} \quad u_n > 0,$$

where $d\sigma$ denotes the area element. Then, what would be the behavior of u_n and λ_n as n tends to ∞? It was Prof. M. Tsuji who has kindly recommended this problem for study.

In the present paper, investigating this problem, we obtain the following result:

THEOREM. *$\lim_{n\to\infty} u_n = v$ and $\lim_{n\to\infty} \lambda_n = \rho$ exist and are determined independently of the choice of exhausting sequence. Moreover, the limit function v and the limit value ρ satisfy the equation $\Delta v + \rho v = 0$ in D together with the condition $v = 0$ on the boundary of D except for a set of capacity zero, where the exceptional points are identical with those of Green's function for the same domain D.*[1]

1) In particular, if C^* consists of a finite number of smooth curves, then the limit function v and the limit value ρ are the first eigenfunction and the first eigenvalue, respectively, for the domain D itself.

Especially if C^* is of capacity zero, the limit function v and the limit value ρ coincide with the first eigenfunction and the first eigenvalue for the domain D', i.e. $D + C^*$, respectively.

The proof of this theorem will be given in the several steps as follows.

§ **1.** Since our eigenvalue is a monotone domain function decreasing in the strict sense, so we have

$$\lambda_n > \lambda_{n+1} \qquad \text{for every } n.$$

Moreover

$$\lambda_n > \mu;$$

here μ denotes the first eigenvalue of the domain D'. Hence, there exists a limit value of λ_n as $n \to \infty$.

Put

$$\lim_{n\to\infty} \lambda_n = \rho,$$

then $\rho \geqq \mu$. This limit value ρ is uniquely determined no matter how the sequence of exhausting domains is chosen. In fact, take another sequence of exhausting domains

$$\overline{D}_1 \subset \overline{D}_2 \subset \cdots \subset \overline{D}_n \subset \cdots.$$

Let the boundary of $\overline{D}_n$ be C and $\overline{C}_n$ where $\overline{C}_n$ consists of a finite number of smooth closed curves. Let corresponding eigenvalues be

$$\overline{\lambda}_1, \overline{\lambda}_2, \cdots, \overline{\lambda}_n, \cdots$$

and set

$$\lim_{n\to\infty} \overline{\lambda}_n = \overline{\rho}.$$

Since D_n tends to D, there exists a D_N such that $\overline{D}_k \subset D_N$ for a fixed $\overline{D}_k$, hence $\overline{\lambda}_k > \lambda_N$. Therefore $\overline{\rho} \geqq \rho$. On the other hand, fixing D_h we get $\rho \geqq \overline{\rho}$, just in the same way. Thus $\rho = \overline{\rho}$, so that $\lim_{n\to\infty}\lambda_n = \rho$ is determined independently of the choice of exhausting sequence.

§ **2.** In order to investigate the behavior of the function u_n, we take in D a domain A bounded by Γ and C_ε where Γ consists of a finite number of closed smooth curves which enclose C^* and have the distance 2ε from C^*, while C_ε is a curve with the distance ε from C. Moreover, we take in the domain which is surrounded by Γ and includes C^*, a finite number of curves Γ' with the distance ε from Γ. Then C_ε and Γ' bound another domain A'. If we take an integer m large enough, then all of the C_n with $n > m$ become to lie outside of the domain A'.

Therefore

$$A' \subset D_n \qquad \text{for } n \geqq m.$$

Now, we will show that the function u_n is uniformly bounded in A with respect to n.

To prove it, let us take an arbitrary point p in A and a circle K of radius ε about p. Since evidently K is contained in A', and also in D_n, we get, by Schwarz's inequality

$$(1) \qquad \iint_K |u_n| d\sigma \leqq \left(\iint_K u_n^2 d\sigma\right)^{1/2} \left(\iint_K d\sigma\right)^{1/2} \leqq \sqrt{\pi\varepsilon^2} = \sqrt{a}$$

where a denotes the area of the circle K.

On the other hand, we have an equality

$$(2) \qquad J_0(\sqrt{\lambda_n}\, r)\, u_n(p) = \frac{1}{2\pi}\int_0^{2\pi} u_n d\theta$$

where J_0 is the Bessel function of the zero-th order and the integral in the right hand member is taken over the circumference of the circle about p with radius r. Multiplying both sides of (2) by r and integrating, we have

$$(3) \qquad u_n(p)\int_0^{\varepsilon} J_0(\sqrt{\lambda_n}\, r)\, r dr = \frac{1}{2\pi}\int_0^{\varepsilon}\int_0^{2\pi} u_n r d\theta dr.$$

From the beginning, let ε be small enough such that $0 < \varepsilon < j_0/\sqrt{\lambda_1}$ where j_0 is the first positive zero of the function J_0. From the property of J_0 we have, for $\varepsilon \geqq r \geqq 0$,

$$J_0(\sqrt{\lambda_n}\, r) \geqq J_0(\sqrt{\lambda_1}\, r) \geqq J_0(\sqrt{\lambda_1}\, \varepsilon) \equiv k > 0.$$

Then from (3)

$$|u_n(p)| \int_0^{\varepsilon} k r dr \leqq \frac{1}{2\pi}\iint_K |u_n| d\sigma,$$

and therefore

$$(4) \qquad |u_n(p)| \frac{k\varepsilon^2}{2} \leqq \frac{1}{2\pi}\iint_K |u_n| d\sigma,$$

which, together with (1), implies

$$|u_n(p)|\, k \leqq \frac{1}{\pi\varepsilon^2}\iint_K |u_n| d\sigma < \frac{1}{a}\sqrt{a}.$$

Hence

$$(5) \qquad |u_n(p)| < \frac{1}{k\sqrt{a}} \qquad \text{for every} \quad n \geqq m.$$

Thus $\{u_n(p)\}$ is uniformly bounded in A.

§ **3.** $\{\partial u_n(p)/\partial x\}$ and $\{\partial u_n(p)/\partial y\}$ are also uniformly bounded in A. In fact, $\partial u_n/\partial x$ also satisfies our differential equation, i.e.

$$\Delta\left(\frac{\partial u_n}{\partial x}\right) + \lambda_n\left(\frac{\partial u_n}{\partial x}\right) = 0 \qquad \text{in } D$$

where (x, y) denote the coordinates of p, i.e. $p(x, y)$.

From the same reasoning as in the preceding paragraph, we have the same kind of inequality for $n \geqq m$, as (4) in § 2,

$$\left|\frac{\partial u_n}{\partial x}\right| \frac{k\varepsilon^2}{2} \leqq \frac{1}{2\pi} \iint_K \left|\frac{\partial u_n}{\partial x}\right| d\sigma.$$

On the other hand, we have the relation

$$\iint_{D_n} \left\{\left(\frac{\partial u_n}{\partial x}\right)^2 + \left(\frac{\partial u_n}{\partial y}\right)^2\right\} d\sigma = \lambda_n < \lambda_1,$$

so that

$$\left(\iint_K \left|\frac{\partial u_n}{\partial x}\right| d\sigma\right)^2 \leqq \iint_K \left(\frac{\partial u_n}{\partial x}\right)^2 d\sigma \cdot \iint_K d\sigma < a \iint_{D_n} \left(\frac{\partial u_n}{\partial x}\right)^2 d\sigma < a\lambda_1.$$

This implies

$$(6) \qquad \left|\frac{\partial u_n(p)}{\partial x}\right| < \frac{1}{k}\sqrt{\frac{\lambda_1}{a}} \qquad \text{for} \quad n \geqq m.$$

Thus $\{\partial u_n(p)/\partial x\}$ is also uniformly bounded in A.

The same is true for $\{\partial u_n(p)/\partial y\}$.

After all, by the theorem of Ascoli-Arzelà, we can select a uniformly convergent subsequence $\{u_{n'}\}$ from $\{u_n\}$. Let us denote its limit function by v, i.e.

$$\lim_{n \to \infty} u_{n'} = v \qquad \text{in} \quad A.$$

§ **4.** Here we study furthermore about the uniform boundedness of the sequence $\{u_n\}$ in D. Let $\tilde{D}_n$ be a domain whose boundary consists of C_n and C' where C_n represents the boundary of D_n as we defined already, while C' does a closed smooth curve lying in the interior of the domain D_n and enclosing C_n.

Since $\tilde{D}_n \subsetneqq D_n$, all of the eigenvalues of the same problem for $\tilde{D}_n$ are greater than the first eigenvalue λ_n for D_n. Therefore the Green's function $\Gamma_n(p, q)$ of the equation $\Delta u + \lambda_n u = 0$ for $\tilde{D}_n$ is uniquely determined; here

$$\Gamma_n(p, q) = \frac{1}{2\pi} \log \frac{1}{r} + H_n(p, q)$$

where H_n is the regular solution of $\Delta u + \lambda_n u = 0$ for $\tilde{D}_n$.

By Green's formula

$$\int_{C'+C_n+\kappa} \left(u_n(q) \frac{\partial \Gamma_n}{\partial \nu} - \Gamma_n(q) \frac{\partial u_n}{\partial \nu}\right) ds = -\iint_{\tilde{D}_n - E} (u_n \Delta \Gamma_n - \Gamma_n \Delta u_n)\, d\sigma(q),$$

where ν denotes the inner normal of the boundary of $\tilde{D}_n - E$, and E is a small circular domain around p and κ is the boundary of E. By making the radius of the circle E tend to zero, we get

$$u_n(p) = \int_{C'} u_n \frac{\partial \Gamma_n}{\partial \nu} ds \qquad \text{for} \quad p \in \tilde{D}_n,$$

as $u_n = 0$ on C_n and $\Gamma_n = 0$ on $C_n + C'$.

In order to obtain an estimation for u_n in the domain $\tilde{D}_n$, we introduce an auxiliary harmonic function φ_n such that

$$(7) \qquad \begin{aligned} \Delta\varphi_n &= 0 \quad &&\text{in} \quad \tilde{D}_n, \\ \varphi_n &= u_n \quad &&\text{on} \quad C', \\ \varphi_n &= 0 \quad &&\text{on} \quad C_n. \end{aligned}$$

By Green's formula

$$(8) \qquad \int_{C'+C_n+\kappa} \left\{ \varphi_n \frac{\partial \Gamma_n}{\partial \nu} - \Gamma_n \frac{\partial \varphi_n}{\partial \nu} \right\} ds = -\iint_{\tilde{D}_n - E} \{ \varphi_n \Delta \Gamma_n - \Gamma_n \Delta \varphi_n \} d\sigma(q).$$

By making the radius of E tend to zero and from the boundary condition for φ_n and Γ_n, the left hand side of (8) becomes

$$\int_{C'} \varphi_n \frac{\partial \Gamma_n}{\partial \nu} ds (= u_n(p)).$$

As $\Delta\varphi_n = 0$, $\Delta\Gamma_n + \lambda_n \Gamma_n = 0$ in $\tilde{D}_n - E$, so the right hand side of (8) is equal to $\lambda_n \iint \varphi_n \Gamma_n d\sigma(q)$. Therefore by making the radius of E tend to zero, we get

$$u_n(p) = \lambda_n \iint_{\tilde{D}_n} \varphi_n \Gamma_n d\sigma.$$

From this and the maximum principle for harmonic functions, we can obtain an inequality

$$(9) \qquad |u_n(p)| \leqq \lambda_n \max_{q \in C'} |\varphi_n(q)| \int_{\tilde{D}_n} \Gamma_n(p, q) \, d\sigma(q).$$

Next we shall estimate $\Gamma_n(p, q)$ which is represented as

$$\Gamma_n(p, q) = -\frac{1}{4} Y_0(\sqrt{\lambda_n} r_{pq}) + B - \gamma_n(p, q),$$

where B represents the first maximum value of $Y_0/4$ and $\gamma_n(p, q)$ a function which satisfies the conditions

$$\Delta u + \lambda_n u = 0 \qquad \text{in} \quad \tilde{D}_n$$

$$u = B - \frac{1}{4} Y_0(\sqrt{\lambda_n} r_{pq}) \qquad \text{on the boundary of } \tilde{D}_n.$$

$\tilde{D}_n \subsetneqq D_n$ assures the unique determination of γ_n.

Since γ_n is a superharmanic function, its minimum is attained on the boundary and hence $\gamma_n(p, q) \geqq 0$.

Therefore we get

$$0 \leqq \Gamma_n(p, q) \leqq B - \frac{1}{4} Y_0(\sqrt{\lambda_n}\, r_{pq}) \tag{10}$$

which has $-\log r_{pq}/2\pi$ as the main term. Then we have

$$\iint_{\tilde{D}} \Gamma_n d\sigma \leqq \iint_{\tilde{D}_n} \left\{B - \frac{1}{4} Y_0(\sqrt{\lambda_n}\, r_{pq})\right\} d\sigma.$$

Finally we will give an estimation for the right hand member. If we take a circle d which contains the domain D' in its interior, then on account of the property of Y_0, we have

$$\iint_{\tilde{D}_n} \left\{B - \frac{1}{4} Y_0(\sqrt{\lambda_n}\, r_{pq})\right\} d\sigma \leqq \iint_d \left\{B - \frac{1}{4} Y_0(\sqrt{\lambda_n}\, r_{pq})\right\} d\sigma$$

$$\leqq \iint_d \left\{B - \frac{1}{4} Y_0(\sqrt{\lambda_1}\, r_{pq})\right\} d\sigma \equiv k' < \infty$$

where k' is independent of n. So we get

$$u_n(p) \leqq \lambda_n k' \max_{q \in C'} u_n(q).$$

Moreover, from § 2, C' can be taken in A such that

$$\max_{q \in C'} u_n(q) < M, \qquad \text{and} \qquad \lambda_1 > \lambda_n.$$

Thus we obtain an estimation

$$u_n(p) \leqq \lambda_1 k' M \qquad \text{in } \tilde{D}_n \text{ for any } n, \tag{11}$$

where the right hand side is independent of n.

Now the domain defined by

$$\tilde{D} = D_n - \tilde{D}_n$$

is surrounded by C and C' only. Then the uniform boundedness of $\{u_n\}$ in $\tilde{D}$ can be shown just in the same way as the above. Thus, combining the results of § 2 and this paragraph, the uniform boundedness of $\{u_n\}$ in D has been established.

§ **5.** For our limit function v the normalization condition $\iint_D v^2 d\sigma = 1$ holds as shown below. By setting

$$U_n(p) = \begin{cases} u_n(p) & \text{in} \quad D_n, \\ 0 & \text{in} \quad D - D_n \end{cases}$$

the definition of u_n in D_n is extended into the whole domain D.

Because U_n is uniformly bounded in D by § 4,

$$\lim_{n\to\infty}\iint_D U_n^2 d\sigma = \iint_D \lim_{n\to\infty} U_n^2 d\sigma = \iint_D v^2 d\sigma.$$

But

$$\lim_{n\to\infty}\iint_D U_n^2 d\sigma = \lim_{n\to\infty}\iint_{D_n} u_n^2 d\sigma = 1.$$

Hence

$$\iint_D v^2 d\sigma = 1. \tag{12}$$

§**6**. Next we shall show that the limit function v and the limit value ρ satisfy the integral equation

$$u(p) = \lambda \iint_D G(p,q)\, u(q)\, d\sigma(q) \tag{13}$$

where $2\pi G(p,q)$ denotes the ordinary Green's function for the domain D, and hence

$$G(p,q) = \frac{1}{2\pi}\log\frac{1}{r} + H(p,q),$$

where H denotes a regular harmonic function in D.

In the first step, let p be a fixed interior point of D, then $p \in D_n$ for sufficiently large n. It is known that (13) holds true for u_n, λ_n and D_n, namely

$$u_n(p) = \lambda_n \iint_{D_n} G_n(p,q)\, u_n(q)\, d\sigma(q);$$

here $2\pi G_n(p,q)$ denotes the ordinary Green's function for the domain D_n.

Now by setting

$$\mathfrak{G}_n(p,q) = \begin{cases} G_n(p,q) & \text{in} \quad D_n, \\ 0 & \text{in} \quad D - D_n, \end{cases}$$

we have

$$U_n = \lambda_n \iint_D \mathfrak{G}_n(p,q)\, U_n(q)\, d\sigma(q),$$

and

$$\mathfrak{G}_n(p,q)\, U_n(q) \leqq M\mathfrak{G}_n(p,q) \leqq MG(p,q),$$

by using the uniform boundedness of U_n in D.

Because $G(p,q)$ is integrable, so by Lebesgue's bounded convergence theorem

$$\lim_{n\to\infty}\iint_D \mathfrak{G}_n(p,q)\,U_n(q)\,d\sigma(q) = \iint_D \lim_{n\to\infty}\mathfrak{G}(p,q)\,U_n(q)\,d\sigma(q)$$
$$= \iint_D G(p,q)\,u(q)\,d\sigma(q).$$

Thus we have

$$v(p) = \rho\iint_D G(p,q)\,v(q)\,d\sigma(q).$$

In the second step, it will be seen that the relation still remains to hold even if p tends to a boundary point. What is to be shown is:

$$\begin{aligned}\lim_{p\to p_0} u(p) &= \lambda \lim_{p\to p_0}\iint_D G(p,q)\,u(q)\,d\sigma(q)\\ &= \lambda\iint_D \lim_{p\to p_0} G(p,q)\,u(q)\,d\sigma(q) = \lambda\iint_D G(p_0,q)\,u(q)\,d\sigma(q)\end{aligned} \tag{14}$$

where p_0 is a boundary point of D and

$$G(p_0,q) \equiv \lim_{p\to p_0} G(p,q).$$

In fact

$$G(p,q) \leqq \text{const}\cdot\log\frac{1}{r}, \qquad r = \overline{pq},$$

$$u(q) \leqq M \qquad \text{in } D,$$

$$G(p,q)\,u(q) \leqq \text{const}\cdot\log\frac{1}{r} \qquad \text{in } D.$$

By Lebesgue's bounded convergence theorem, we get the result required. Thus for any point in D and on the boundary point of D,

$$v(p) = \rho\iint_D G(p,q)\,v(q)\,d\sigma(q) \tag{15}$$

does hold.

§ **7.** For any interior point p of the given domain D, our limit function and limit value satisfy the equation

$$\Delta u + \lambda u = 0.$$

In fact, from § 6, for v and ρ

$$v(p) = \rho \iint_D G(p, q)\, v(q)\, d\sigma(q).$$

Let $p(x, y)$ be an arbitrary interior point of D and K a circle about p small enough to be contained in D. Then

$$\rho \iint_D G(p, q)\, v(q)\, d\sigma(q) = \rho \iint_{D-K} G(p, q)\, v(q)\, d\sigma(q) + \rho \iint_K G(p, q)\, v(q)\, d\sigma(q).$$

Denote the first and the second integrals in the right hand member by I_1 and I_2, respectively. Then

$$\frac{\partial I_1}{\partial x} = \iint_{D-K} \frac{\partial G}{\partial x} v(q)\, d\sigma(q), \qquad \frac{\partial^2 I_1}{\partial x^2} = \iint_{D-K} \frac{\partial^2 G}{\partial x^2} v(q)\, d\sigma(q).$$

The same is valid for $\partial^2 I_1/\partial y^2$, i. e.

$$\frac{\partial^2 I_1}{\partial y^2} = \iint_{D-K} \frac{\partial^2 G}{\partial y^2} v(q)\, d\sigma(q).$$

Therefore

$$\Delta I_1 = \frac{\partial^2 I_1}{\partial x^2} + \frac{\partial^2 I_1}{\partial y^2} = \iint_{D-K} \Delta G\, v(q)\, d\sigma(q) = 0,$$

as $\Delta G = 0$ in $D - K$.

As for I_2, it is well known

$$\Delta \left(\rho \iint_K G(p, q)\, v(q)\, d\sigma(q) \right) = -\rho v(p)$$

so that

$$\Delta v(p) + \rho v(p) = 0 \qquad \text{in } D.$$

Moreover, when p tends to a boundary point, $v(p)$ tends to zero except for a set of capacity zero. In fact, from § 6

$$\lim_{p \to p_0} v(p) = \rho \iint_D G(p_0, q)\, v(q)\, d\sigma(q)$$

where $\lim_{p\to p_0} G(p, q) = G(p_0, q)$, since we know that $G(p_0, q)$ becomes zero except for a set of capacity zero.

For the special case where the closed set C^* is of capacity zero, the limit function v is identical with the first eigenfunction of $D' = D + C^*$.

§ 8. What is left to be proved is the uniqueness for the limit function v. As in the last part of § 1 where the uniqueness of ρ was proved, taking another sequence of exhausting domains

$$\bar{D}_1 \subset \bar{D}_2 \subset \cdots \subset \bar{D}_n \subset \cdots.$$

Let the corresponding first eigenfunctions be

$$\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_n, \cdots$$

and $\bar{v}$ be a limit function of it. We shall prove $v = \bar{v}$.

First consider the case where C^* consists of a finite number of smooth closed curves. In this case, because the boundary curve is smooth everywhere, $G(p_0, q) = 0$ for all p_0 on the boundary. Therefore from (15), v satisfies the fixed boundary condition. Hence v and ρ are the first eigenfunction and the first eigenvalue of D. From the well known property of the first eigenvalue of such a domain, ρ must be simple, and the first eigenfunction must be unique, i. e. $v = \bar{v}$.

The above fact shows us that the first eigenfunction has the continuity relation on the domain provided the boundary of the domain consists of smooth curves.

Now we return to the general case, where the boundary does not need to consist of smooth curves only. Suppose that $v \neq \bar{v}$ in D. Then there would be a point p such that

$$|v(p) - \bar{v}(p)| = a > 0$$

and for sufficiently large integers m, n and a small positive number ε,

$$|u_m(p) - v(p)| < \frac{\varepsilon}{2},$$

$$|\bar{u}_n(p) - \bar{v}(p)| < \frac{\varepsilon}{2}.$$

So we would get

$$|u_m(p) - \bar{u}_n(p)| > a - \varepsilon > 0. \tag{16}$$

But, on the other hand, we have

$$|u_m(p) - \bar{u}_n(p)| < \eta, \tag{17}$$

where η can be any small positive number making m and n large enough, by the above mentioned continuity relation between the first eigenfunction and the domain, as the boundaries of D_m and D_n consist of smooth curves. Then (16) contradicts (17).

Therefore $v = \bar{v}$, which proves that our limit function v is determined independently of the choice of exhausting sequence. Thus our theorem has been proved.

At the conclusion of this paper, the author wishes to express her gratitude to Prof. M. Tsuji for his kind suggestions and continuous encouragement and also to thank Prof. Y. Komatu for his considerate advice at all times.

DEPARTMENT OF MATHEMATICS,
TOKYO INSTITUTE OF TECHNOLOGY.