A DISTORTION THEOREM ON SCHLICHT FUNCTIONS

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1. Let $\mathfrak{S}$ be the family of schlicht functions regular in $|z|<1$ with local expansion $f(z)=\sum_{n=1}^{\infty}a_n z^n$, $a_1=1$, at the origin. The estimation on the third coefficient is well known and its proof has been performed by quite different three ways, that is, the Löwner's differential equation method or its variants, the Schiffer's variational method and the symmetrization method.

Löwner's theorem. If $f(z)\in\mathfrak{S}$, then $|a_3|\leqq 3$. Equality sign occurs only for the Koebe's extremal function.

Suppose that $|f'(r)+f'(-r)|/|f(r)-f(-r)|\geqq(1+6r^2)/r$ for an infinite sequence $\{r_n\}$, $r_n\to 0$, then $(1+2\Re a_3 r^2+O(r^4))/r\geqq(1+6r^2)/r$ whence follows $\Re a_3\geqq 3$. Then the Löwner's theorem implies that there are only two functions $f(z)=z/(1\pm z)^2$ such that $\Re a_3\geqq 3$. For these functions, we have

$$\frac{|f'(r)+f'(-r)|}{|f(r)-f(-r)|}=\frac{1+6r^2+r^4}{r(1-r^4)}.$$

Consequently there follows that *any function* $f(z)\in\mathfrak{S}$ *satisfies*

$$\text{(A)}\qquad \frac{|f'(r)+f'(-r)|}{|f(r)-f(-r)|}\leqq\frac{1+6r^2+r^4}{r(1-r^4)}$$

for all sufficiently small r.

In the present paper we shall prove (A) without making use of the Löwner's theorem. Hence $|a_3|\leqq 3$ is an immediate consequence of our result. Our mode of proof is based upon the variational method due to Schiffer [2] and is somewhat complicated and tedious as compared with the other ways arriving at the Löwner's theorem. Our result is weak as it is not established for all r. To obtain its full result is a remaining problem.

2. We shall consider an extremal problem to obtain $\text{Max}_{f\in\mathfrak{S}}\{|f'(r)+f'(-r)|\div|f(r)-f(-r)|\}$, r being a positive number less than 1. Evidently, extremal functions $w=f(z)$ exist. For any extremal function $w=f(z)$ we shall obtain the following fact by the Schiffer's lemma [2]:

The complementary continuum C_f of the image domain $f(|z|<1)$ consists of a finite number of analytic arcs satisfying the following differential equation for a suitably chosen real parameter t:

$$\left(\frac{dw}{dt}\right)^2\frac{(R_2-R_1)((S_1-S_2)w-S_1R_2+S_2R_1)}{(S_1+S_2)(R_1-w)^2(R_2-w)^2}=1.$$

Here we put $R_1=f(r)$, $R_2=f(-r)$, $S_1=f'(r)$ and $S_2=f'(-r)$.

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If $S_1 = S_2$, then we say that C_f belongs to the case (α). Otherwise, we say that C_f belongs to the case (β). Let w_0 be the zero point of an expression $(S_1 - S_2)w - S_1R_2 + S_2R_1$. We must classify the case (β) into three subcases.

(I) C_f does not contain w_0.

(II) C_f contains w_0 as its finite end point.

(III) C_f contains w_0 as an inner point.

In the case (α), C_f is a single analytic arc which contains the point at infinity as an inner point. Then we have by the method due to Schiffer [2] that $w = f(z)$ satisfies in $|z| < 1$ a differential equation

$$G(z) \equiv z^2 w'(z)^2 \frac{(R_1 - R_2)^2}{(R_1 - w)^2 (R_2 - w)^2} = \mathfrak{A} \frac{z^2 (z - b)^2 (z - c)^2}{(z - r)^2 \left(z - \frac{1}{r}\right)^2 (z + r)^2 \left(z + \frac{1}{r}\right)^2},$$

b and c being two points on $|z| = 1$ such that $w(b)$ and $w(c)$ are two end points of C_f and $\mathfrak{A}$ is a constant factor to be determined.

In the case (β), the point at infinity is an end point of C_f. If $C_f \in (\mathrm{I})$, C_f consists of a single analytic arc and does not contain w_0. Then $w(z)$ satisfies in $|z| < 1$ a differential equation

$$G(z) \equiv z^2 w'(z)^2 \frac{(R_2 - R_1)((S_1 - S_2)w - S_1R_2 + S_2R_1)}{(S_1 + S_2)(R_1 - w)^2 (R_2 - w)^2}$$
$$= \mathfrak{A} \frac{z^2 (z - b)^2 (z - z_0)\left(z - \frac{1}{\bar{z}_0}\right)}{(z - r)^2 \left(z - \frac{1}{r}\right)^2 (z + r)^2 \left(z + \frac{1}{r}\right)^2}.$$

Here b and z_0 are determined by $|b| = 1$ and $|z_0| < 1$ together with the condition that $w(b)$ and $w(z_0)$ are the finite end point of C_f and w_0, respectively. $\mathfrak{A}$ is a constant to be determined.

If $C_f \in (\mathrm{II})$, C_f consists of a single analytic arc whose finite end point coincides with w_0, and $w(z)$ satisfies in $|z| < 1$ a differential equation

$$G(z) \equiv z^2 w'(z)^2 \frac{(R_2 - R_1)((S_1 - S_2)w - S_1R_2 + S_2R_1)}{(S_1 + S_2)(R_1 - w)^2 (R_2 - w)^2}$$
$$= \mathfrak{A} \frac{z^2 (z - b)^4}{(z - r)^2 \left(z - \frac{1}{r}\right)^2 (z + r)^2 \left(z + \frac{1}{r}\right)^2},$$

b being a point on $|z| = 1$ corresponding to the finite end point of C_f, that is, w_0. $\mathfrak{A}$ is a constant to be determined.

If $C_f \in (\mathrm{III})$, C_f consists of two or three analytic arcs all of which have w_0 as a common end point and C_f forks at w_0 with asymptotic angle 120°, one branch being eventually vacuous, and $w(z)$ satisfies in $|z| < 1$ a differential equation

$$G(z) \equiv z^2 w'(z)^2 \frac{(R_2 - R_1)((S_1 - S_2)w - S_1R_2 + S_2R_1)}{(S_1 + S_2)(R_1 - w)^2 (R_2 - w)^2}$$
$$= \mathfrak{A} \frac{z^2 (z - b)^2 (z - c)^2}{(z - r)^2 \left(z - \frac{1}{r}\right)^2 (z + r)^2 \left(z + \frac{1}{r}\right)^2},$$

b and c being two points on $|z|=1$ which, in general case, correspond to two finite end points of C_f. (In degenerate case, one of $w(b)$ and $w(c)$ may coincide with w_0 and the remaining one corresponds to an end point of C_f.) $\mathfrak{A}$ is a constant to be determined.

3. Case (β), (II). In this case, $G(z)$ satisfies a functional relation $G(z) = \overline{G(1/\bar{z})}$ in $|z| \leqq 1$ and $\arg G(z) \equiv \pi$, mod 2π, on $|z|=1$. Now we shall determine the constant factor $\mathfrak{A}$. Comparing the residues at $z=r$(resp. $z=-r$) in both sides of the differential equation, we have

$$\frac{-S_1}{S_1+S_2} = \mathfrak{A}\frac{r^2(r-b)^4}{4(1-r^2)^2(1+r^2)^2}$$

$$\left(\text{resp.}\quad \frac{-S_2}{S_1+S_2} = \mathfrak{A}\frac{r^2(r+b)^4}{4(1-r^2)^2(1+r^2)^2}\right).$$

Hence there holds

$$\mathfrak{A} = -2\frac{(1+r^2)^2(1-r^2)^2}{r^2(r^4+6b^2r^2+b^4)}.$$

On the other hand, we see that there holds $\mathfrak{A}b^4 = \overline{\mathfrak{A}}$ by virtue of $G(z) = \overline{G(1/\bar{z})}$. From these we can easily see that $b^4=1$ and $\mathfrak{A}$ is real.

If $b=1$, then $S_1/S_2 = (1-r)^4/(1+r)^4$. On the other hand, by the well-known distortion theorem on schlicht functions, there holds a distortion inequality

$$\frac{|S_1|}{|S_2|} \geqq \frac{(1-r)/(1+r)^3}{(1+r)/(1-r)^3} = \frac{(1-r)^4}{(1+r)^4}.$$

Here the equality sign occurs only for $w=f(z)=z/(1+z)^2$, for which we see

$$\frac{|f'(r)+f'(-r)|}{|f(r)-f(-r)|} = \frac{1+6r^2+r^4}{r(1-r^4)}.$$

If $b=-1$, then the procedure is similar as in the above and we obtain a fact that the extremal function is $w(z)=z/(1-z)^2$, for which we again get

$$\frac{|f'(r)+f'(-r)|}{|f(r)-f(-r)|} = \frac{1+6r^2+r^4}{r(1-r^4)}.$$

Let $b^2=-1$. then $\pi \equiv \arg G(1) \equiv \arg\mathfrak{A} + \arg(1\pm i)^4 \equiv \arg\mathfrak{A}+\pi$, mod 2π. Thus $\mathfrak{A}$ is real positive. On the other hand $\mathfrak{A}$ has been determined already and is equal to

$$-2\frac{(1-r^2)^2(1+r^2)^2}{r^2(r^4-6r^2+1)}.$$

Therefore we see that r has to satisfy an inequality $r^4-6r^2+1>0$, from which we obtain an inequality $\sqrt{2}-1<r$. We may omit the case $b^2=-1$, since our final aim is to obtain a distortion inequality for any sufficiently small r.

4. Case (α). In this case, $G(z)$ satisfies a functional relation $G(z) = \overline{G(1/\bar{z})}$ in $|z| \leqq 1$ and $\arg G(z) \equiv 0$, mod 2π, on $|z|=1$.

Comparing the residues at $z=r$ in both sides of the differential equation, we have

$$1 = \mathfrak{A} \frac{r^2(r-b)^2(r-c)^2}{4(1-r^2)^2(1+r^2)^2}.$$

Similarly, at $z = -r$, we have

$$1 = \mathfrak{A} \frac{r^2(r+b)^2(r+c)^2}{4(1-r^2)^2(1+r^2)^2}.$$

Thus $b + c = 0$ holds. Let b and c denote $e^{i\theta}$ and $e^{i\varphi}$, respectively, then $\theta - \varphi \equiv 0$, mod 2π. On the other hand, there holds $\overline{\mathfrak{A}} = \mathfrak{A} b^2 c^2$ by $G(z) = \overline{G(1/\bar{z})}$ and simultaneously we obtain

$$\mathfrak{A} = \frac{4(1+r^2)^2(1-r^2)^2}{r^2(r^2+bc)^2}.$$

Hence we have $b^2c^2 = 1$ or equivalently $\theta + \varphi \equiv 0$, mod π, from which we see that there are four cases to be considered, that is,

$$\begin{cases} b = 1 \\ c = -1 \end{cases}, \quad \begin{cases} b = -1 \\ c = 1 \end{cases}, \quad \begin{cases} b = i \\ c = -i \end{cases} \quad \text{and} \quad \begin{cases} b = -i \\ c = i \end{cases}.$$

If $bc = -1$, then $0 \equiv \arg G(i) \equiv \arg \mathfrak{A} + \arg i^2(i-1)^2(i+1)^2/\{(i-r)^2(i+r)^2 \cdot (i-1/r)^2(i+1/r)^2\} \equiv \arg \mathfrak{A} + \pi$, mod 2π. Thus $\arg \mathfrak{A} \equiv \pi$, mod 2π, that is, $\mathfrak{A}$ is negative. On the other hand $\mathfrak{A} = 4(1+r^2)^2/r^2$ is positive which is absurd. Therefore, we have only to consider the case $bc = 1$. Without loss of generality, we may consider the case $b = i$ and $c = -i$. In this case $\mathfrak{A}$ is equal to $4(1-r^2)^2/r^2$ and hence we have in $|z| < 1$ a differential equation

$$w'(z) \frac{R_1 - R_2}{(R_1 - w)(R_2 - w)} = \pm 2 \frac{1-r^2}{r} \cdot \frac{(z-i)(z+i)}{(z-r)\left(z - \frac{1}{r}\right)(z+r)\left(z + \frac{1}{r}\right)}.$$

In the right hand member of the above equation, the minus sign should be adopted. This is seen by comparison of the residues at $z = r$. Thus we have

$$\frac{R_1 - R_2}{(R_1 - w)(R_2 - w)} dw = \left(\frac{1}{z-r} - \frac{1}{z+r} - \frac{1}{z - \frac{1}{r}} + \frac{1}{z + \frac{1}{r}} \right) dz.$$

Integrating this and denoting the integration constant by C we have

$$\log \frac{w - R_1}{w - R_2} = \log \frac{(z-r)\left(z + \frac{1}{r}\right)}{\left(z - \frac{1}{r}\right)(z+r)} + C.$$

Here we shall fix a branch of logarithm of right hand side in the last equation such that it reduces to zero at $z = 0$. Let z tend to zero, then w tends to zero, and hence we see

$$C = \log \frac{R_1}{R_2}.$$

Therefore we have

$$\frac{R_2}{R_1} \cdot \frac{w - R_1}{w - R_2} = \frac{(z-r)\left(z + \frac{1}{r}\right)}{(z+r)\left(z - \frac{1}{r}\right)}.$$

Let z tend to r, then we have

$$\frac{R_2}{R_1}\cdot\frac{S_1}{R_1 - R_2} = \frac{-(1+r^2)}{2r(1-r^2)}.$$

Similarly let z tend to $-r$, then we have

$$\frac{R_2}{R_1}\cdot\frac{R_2 - R_1}{S_2} = \frac{2r(1-r^2)}{1+r^2}.$$

Our assumption $S_1 = S_2$ and two relations listed above imply that $R_1 = -R_2$. Thus we have

$$\frac{|S_1 + S_2|}{|R_1 - R_2|} = \frac{2|S_1|}{|R_1 - R_2|} = \frac{1+r^2}{r(1-r^2)}.$$

5. Case (β), (III). In this case, $G(z) = \overline{G(1/\bar{z})}$ in $|z| \leqq 1$ and $\arg G(z) \equiv \pi$, mod 2π. Let z tend to r, then we have

$$\frac{-S_1}{S_1 + S_2} = \mathfrak{A}\frac{r^2(r-b)^2(r-c)^2}{(1-r^2)^2(1+r^2)^2}.$$

Similarly, let z tend to $-r$, then we have

$$\frac{-S_2}{S_1 + S_2} = \mathfrak{A}\frac{r^2(r+b)^2(r+c)^2}{(1-r^2)^2(1+r^2)^2}.$$

Moreover $\mathfrak{A}b^2c^2 = \overline{\mathfrak{A}}$ holds since $G(z) = \overline{G(1/\bar{z})}$. From these three relations we see easily that $b^2c^2 = 1$ and that $\mathfrak{A}$ is real and is equal to

$$-2\frac{(1-r^2)^2(1+r^2)^2}{r^2(r^4 + (b^2 + c^2 + 4bc)r^2 + 1)}.$$

If $bc = -1$ and if b and c denote $e^{i\theta}$ and $e^{i\varphi}$, respectively, then $\theta + \varphi \equiv \pi$, mod 2π, and the denominator in the expression of $\mathfrak{A}$ is equal to $r^2(r^4 - 2 \cdot (2 - \cos 2\theta)r^2 + 1)$. Let h be a root of $x^2 - 2(2 - \cos 2\theta)x + 1 = 0$ in $(0, 1]$, that is, $h = 2 - \cos 2\theta - \sqrt{(2 - \cos 2\theta)^2 - 1}$. If h is not equal to 1, then $\mathfrak{A}$ is positive for $h < r^2 < 1$, and negative for $0 < r^2 < h$. If h is equal to 1, then $\cos 2\theta = 1$ and hence the denominator of $\mathfrak{A}$ is positive, whence follows that $\mathfrak{A}$ is negative for any r. On the other hand, there holds that $\pi \equiv \arg G(1) \equiv \arg \mathfrak{A} + \arg (1 - e^{i\theta})^2(1 - e^{i\varphi})^2 \equiv \arg \mathfrak{A} + \arg(-2 + \cos 2\theta + \cos 2\varphi)$, mod 2π, from which we can easily see that $\mathfrak{A}$ is positive if $\cos 2\theta + \cos 2\varphi \neq 2$. If $\cos 2\theta + \cos 2\varphi = 2$, then $2\theta \equiv 2\varphi \equiv 0$, mod 2π, and hence either $b = 1$ and $c = -1$ or $b = -1$ and $c = 1$. In general, we may consider the case $b = 1$ $c = -1$. In this case we can calculate $\arg G(z)$ as follows: let $z = i$ then $\arg G(i) \equiv \arg \mathfrak{A} + \arg(-4/(1 + r^2)^2(1 + 1/r^2)^2) \equiv \arg \mathfrak{A} + \pi$, which shows that $\arg \mathfrak{A} \equiv 0$, mod 2π, that is, $\mathfrak{A}$ is positive. At any rate, positivity of $\mathfrak{A}$ leads to a contradiction, since it is evident from its representation that $\mathfrak{A}$ is negative in this case. Thus we have only to consider the case that $\cos 2\theta + \cos 2\varphi \neq 2$. In this case it can be seen that r must satisfy an inequality $r^2 > h$.

In view of our final aim, that is, to obtain a distortion inequality for all sufficiently small r, we may neglect the case that $bc = -1$. Thus we may consider only the case $bc = 1$. For this case, we can easily see that $\mathfrak{A}$ is negative.

We shall now integrate our differential equation

$$w'(z)\frac{\left(1+\frac{S_1-S_2}{S_2R_1-S_1R_2}w\right)^{1/2}}{\left(1-\frac{w}{R_1}\right)\left(1-\frac{w}{R_2}\right)}=\pm\frac{(z-b)(z-c)}{(z-r)\left(z-\frac{1}{r}\right)(z+r)\left(z+\frac{1}{r}\right)}.$$

Here we fix a branch of $(1+(S_1-S_2)w/(S_2R_1-S_1R_2))^{1/2}$ which reduces to 1 at $w=0$. Then, comparing the residues of both sides, the plus sign should be adopted. For the sake of brevity, we shall denote $(1+(S_1-S_2)w/(S_2R_1-S_1R_2))^{1/2}$ by u. Then we have

$$\frac{R_1R_2}{S_2R_1-S_1R_2}\left[\frac{S_1}{\left(\frac{S_1(R_1-R_2)}{S_2R_1-S_1R_2}\right)^{1/2}}\log\frac{u-\left(\frac{S_1(R_1-R_2)}{S_2R_1-S_1R_2}\right)^{1/2}}{u+\left(\frac{S_1(R_1-R_2)}{S_2R_1-S_1R_2}\right)^{1/2}}\right.$$
$$\left.-\frac{S_2}{\left(\frac{S_2(R_1-R_2)}{S_2R_1-S_1R_2}\right)^{1/2}}\log\frac{u-\left(\frac{S_2(R_1-R_2)}{S_2R_1-S_1R_2}\right)^{1/2}}{u+\left(\frac{S_2(R_1-R_2)}{S_2R_1-S_1R_2}\right)^{1/2}}\right]$$
$$=\alpha\log\frac{z-r}{z-\frac{1}{r}}+\gamma\log\frac{z+r}{z+\frac{1}{r}}+C,$$

with $\alpha=\dfrac{-r(r-b)(r-c)}{2(1-r^4)}$ and $\gamma=\dfrac{r(r+b)(r+c)}{2(1-r^4)}$.

By comparison of the residues at z_0 lying on $|z|=1$ for which $w(z_0)=w_0$, we see that

$$\frac{R_1R_2}{S_2R_1-S_1R_2}\left[\frac{S_1}{\left(\frac{S_1(R_1-R_2)}{S_2R_1-S_1R_2}\right)^{1/2}}\log(-1)-\frac{S_2}{\left(\frac{S_2(R_1-R_2)}{S_2R_1-S_1R_2}\right)^{1/2}}\log(-1)\right]$$
$$=\alpha\log\frac{z_0-r}{z_0-\frac{1}{r}}+\gamma\log\frac{z_0+r}{z_0+\frac{1}{r}}+C.$$

By our fixation of the branch of u, the points $z=r$ and $z=-r$ correspond to $(S_1(R_1-R_2)/(S_2R_1-S_1R_2))^{1/2}$ and $(S_2(R_1-R_2)/(S_2R_1-S_1R_2))^{1/2}$, respectively. We see that the relations

$$\alpha=\frac{R_1R_2}{S_2R_1-S_1R_2}\cdot\frac{S_1}{\left(\frac{S_1(R_1-R_2)}{S_2R_1-S_1R_2}\right)^{1/2}}$$

and

$$\gamma=\frac{-R_1R_2}{S_2R_1-S_1R_2}\cdot\frac{S_2}{\left(\frac{S_2(R_1-R_2)}{S_2R_1-S_1R_2}\right)^{1/2}}$$

must hold, and hence we obtain

$$S_1^{1/2}\log\frac{u-\left(\frac{S_1(R_1-R_2)}{S_2R_1-S_1R_2}\right)^{1/2}}{u+\left(\frac{S_1(R_1-R_2)}{S_2R_1-S_1R_2}\right)^{1/2}}\cdot\frac{z-\frac{1}{r}}{z-r}\cdot\frac{-(z_0-r)}{z_0-\frac{1}{r}}$$
$$=S_2^{1/2}\log\frac{u+\left(\frac{S_2(R_1-R_2)}{S_2R_1-S_1R_2}\right)^{1/2}}{u+\left(\frac{S_2(R_1-R_2)}{S_2R_1-S_1R_2}\right)^{1/2}}\cdot\frac{z+r}{z+\frac{1}{r}}\cdot\frac{z_0+\frac{1}{r}}{-(z_0+r)}.$$

When z tends to r, u tends to $(S_1(R_1-R_2)/(S_2R_1-S_1R_2))^{1/2}$ and we have

$$S_1^{1/2}\log\frac{S_1-S_2}{4(R_1-R_2)}\cdot\frac{(1-r^2)(z_0-r)}{(rz_0-1)}=S_2^{1/2}\log\frac{S_1^{1/2}+S_2^{1/2}}{S_1^{1/2}-S_2^{1/2}}\cdot\frac{2r}{1+r^2}\cdot\frac{1+z_0r}{-(z_0+r)}.$$

Similarly, let z tend to $-r$, then we have

$$S_1^{1/2}\log\frac{S_1^{1/2}-S_2^{1/2}}{S_1^{1/2}+S_2^{1/2}}\cdot\frac{1+r^2}{2r}\cdot\frac{z_0-r}{z_0r-1}=S_2^{1/2}\log\frac{4(R_1-R_2)}{S_1-S_2}\cdot\frac{-(1+z_0r)}{(1-r^2)(z_0+r)}.$$

Remembering the fact that $(S_1/S_2)^{1/2}$ is equal to $-(r-b)(r-c)/(r+b)(r+c)$ $=-(1-2r\cos\theta+r^2)/(1+2r\cos\theta+r^2)$ which is real negative and separating the real part of the above two equations, we have

$$\frac{S_1^{1/2}}{S_2^{1/2}}\log\frac{4|R_1-R_2|}{|S_1-S_2|}(1-r^2)=\log\frac{|S_1^{1/2}+S_2^{1/2}|}{|S_1^{1/2}-S_2^{1/2}|}\frac{2r}{1+r^2}$$

and

$$\frac{S_1^{1/2}}{S_2^{1/2}}\log\frac{|S_1^{1/2}-S_2^{1/2}|}{|S_1^{1/2}+S_2^{1/2}|}\frac{1+r^2}{2r}=\log\frac{4|R_1-R_2|}{|S_1-S_2|}\frac{1}{1-r^2}.$$

Here use is made of the facts $|z_0-r|/|z_0r-1|=1$ and $|z_0+r|/|1+z_0r|=1$. Therefore we have

$$\frac{S_1}{S_2}\log\frac{|S_1-S_2|}{4|R_1-R_2|}(1-r^2)=\log\frac{|S_1-S_2|}{4|R_1-R_2|}(1-r^2),$$

which leads to a relation $|S_1-S_2|(1-r^2)=4|R_1-R_2|$, since $S_1\neq S_2$. Then we can conclude that

$$\left|\frac{S_1^{1/2}+S_2^{1/2}}{S_1^{1/2}-S_2^{1/2}}\right|=\frac{1+r^2}{2r}.$$

On the other hand, by $(S_1/S_2)^{1/2}=-(r-b)(r-c)/(r+b)(r+c)$, we obtain

$$\left|\frac{S_1^{1/2}+S_2^{1/2}}{S_1^{1/2}-S_2^{1/2}}\right|=\frac{2r|\cos\theta|}{1+r^2}.$$

From the above two relations, there holds an equality $(1+r^2)^2=4r^2|\cos\theta|$. At any rate this is absurd. Hence we can conclude that the case $(\beta),(\text{III})$ does not occur.

6. Case $(\beta),(\text{I})$. This case constitutes the most tedious and difficult part in our discussions. Though the previous cases could be really treated throughout the whole range of r, we must be obliged in the present case to state our result only for sufficiently small r.

In this case $G(z)$ satisfies again a functional relation $G(z)=\overline{G(1/\bar{z})}$ in $|z|\leqq 1$ and $\arg G(z)\equiv\pi$, mod 2π, on $|z|=1$. Let z tend to r, then we have

$$\frac{-S_1}{S_1+S_2}=\mathfrak{A}\frac{r^2(r-b)^2(r-z_0)\left(r-\dfrac{1}{\bar{z}_0}\right)}{4(1-r^4)^2}.$$

Similarly, let z tend to $-r$, then we have

$$\frac{-S_2}{S_1+S_2}=\mathfrak{A}\frac{r^2(r+b)^2(r+z_0)\left(r+\dfrac{1}{\bar{z}_0}\right)}{4(1-r^4)^2}.$$

From these two relations, we obtain

$$\mathfrak{A} = \frac{-2(1-r^4)^2}{r^2\left(r^4 + r^2\left(b^2 + \dfrac{z_0}{\bar{z}_0} + 2bz_0 + 2b\dfrac{1}{\bar{z}_0}\right) + b^2\dfrac{z_0}{\bar{z}_0}\right)}.$$

By $G(z) = \overline{G(1/\bar{z})}$, we have $\overline{\mathfrak{A}} = \mathfrak{A}b^2z_0/\bar{z}_0$. Therefore we can conclude that $b^2z_0/\bar{z}_0 = 1$ and $\mathfrak{A}$ is real. Let $b = e^{i\theta}$ and $z_0 = xe^{i\varphi}$, then $\theta + \varphi \equiv 0$, mod π, by $b^2z_0/\bar{z}_0 = 1$.

If $\theta + \varphi \equiv \pi$, mod 2π, then $\mathfrak{A}$ is positive if $h < r^2 < 1$ and negative if $0 < r^2 < h$, h being a zero point of $x^2 + 2x(\cos 2\theta - x - 1/x) + 1$ such that $0 < h < 1$, that is, $h = x + 1/x - \cos 2\theta - \sqrt{(x + 1/x - \cos 2\theta)^2 - 1}$. On the other hand we have $\pi \equiv \arg G(1) \equiv \arg \mathfrak{A} + \arg(-(1-\cos\theta)(x + 1/x + 2\cos\theta)) \equiv \arg\mathfrak{A} + \pi$, mod 2π, if $b \neq 1$. If $b = 1$, then $\pi \equiv \arg G(-1) \equiv \arg\mathfrak{A} + \arg(1-x)(1-1/x) \equiv \arg\mathfrak{A} + \pi$, mod 2π. At any rate, $\mathfrak{A}$ is positive. Therefore $r^2 > h$ holds. In view of our final aim, we may again neglect the case $\theta + \varphi \equiv \pi$, mod 2π. Thus we have only to consider the case $\theta + \varphi \equiv 0$, mod 2π.

Let $u = (1 + (S_1 - S_2)w/(S_2R_1 - S_1R_2))^{1/2}$ whose branch is fixed by $u(0) = 1$, and let $t = ((z - z_0)/(z - 1/\bar{z}_0))^{1/2}$ whose branch is fixed by $t(0) = |z_0|$. Then we have a differential equation

$$w'(z)\frac{u}{\left(1 - \dfrac{w}{R_1}\right)\left(1 - \dfrac{w}{R_2}\right)} = \frac{(b - z)\left((z - z_0)\left(z - \dfrac{1}{\bar{z}_0}\right)\right)^{1/2}}{(z - r)(z + r)\left(z - \dfrac{1}{r}\right)\left(z + \dfrac{1}{r}\right)}.$$

Integrating this equation and denoting the integration constant by $C(-r) \div 2(1 - r^4)$, we have

$$\frac{R_1R_2}{S_2R_1 - S_1R_2}\left[\frac{S_1}{\left(\dfrac{S_1(R_1 - R_2)}{S_2R_1 - S_1R_2}\right)^{1/2}}\log\frac{u - \left(\dfrac{S_1(R_1 - R_2)}{S_2R_1 - S_1R_2}\right)^{1/2}}{u + \left(\dfrac{S_1(R_1 - R_2)}{S_2R_1 - S_1R_2}\right)^{1/2}}\right.$$

$$\left. - \frac{S_2}{\left(\dfrac{S_2(R_1 - R_2)}{S_2R_1 - S_1R_2}\right)^{1/2}}\log\frac{u - \left(\dfrac{S_2(R_1 - R_2)}{S_2R_1 - S_1R_2}\right)^{1/2}}{u + \left(\dfrac{S_2(R_1 - R_2)}{S_2R_1 - S_1R_2}\right)^{1/2}}\right]$$

$$= \frac{(r - z_0)(r - b)}{2r\left(r - \dfrac{1}{r}\right)\left(r + \dfrac{1}{r}\right)}\cdot\left(\frac{\dfrac{1}{\bar{z}_0} - r}{z_0 - r}\right)^{1/2}\log\frac{t + \left(\dfrac{z_0 - r}{\dfrac{1}{\bar{z}_0} - r}\right)^{1/2}}{t - \left(\dfrac{z_0 - r}{\dfrac{1}{\bar{z}_0} - r}\right)^{1/2}}$$

$$+ \frac{\left(\dfrac{1}{r} - z_0\right)\left(\dfrac{1}{r} - b\right)}{\dfrac{2}{r}\left(r - \dfrac{1}{r}\right)\left(r + \dfrac{1}{r}\right)}\cdot\left(\frac{\dfrac{1}{\bar{z}_0} - \dfrac{1}{r}}{z_0 - \dfrac{1}{r}}\right)^{1/2}\log\frac{t - \left(\dfrac{z_0 - \dfrac{1}{r}}{\dfrac{1}{\bar{z}_0} - \dfrac{1}{r}}\right)^{1/2}}{t + \left(\dfrac{z_0 - \dfrac{1}{r}}{\dfrac{1}{\bar{z}_0} - \dfrac{1}{r}}\right)^{1/2}}$$

$$-\frac{(r+z_0)(r+b)}{2r\left(r-\frac{1}{r}\right)\left(r+\frac{1}{r}\right)}\cdot\left(\frac{\frac{1}{\bar{z}_0}+r}{z_0+r}\right)^{1/2}\log\frac{t+\left(\frac{z_0+r}{\frac{1}{\bar{z}_0}+r}\right)^{1/2}}{t-\left(\frac{z_0+r}{\frac{1}{\bar{z}_0}+r}\right)^{1/2}}$$

$$-\frac{\left(z_0+\frac{1}{r}\right)\left(\frac{1}{r}+b\right)}{\frac{2}{r}\left(r-\frac{1}{r}\right)\left(r+\frac{1}{r}\right)}\cdot\left(\frac{\frac{1}{\bar{z}_0}+\frac{1}{r}}{z_0+\frac{1}{r}}\right)^{1/2}\log\frac{t-\left(\frac{z_0+\frac{1}{r}}{\frac{1}{\bar{z}_0}+\frac{1}{r}}\right)^{1/2}}{t+\left(\frac{z_0+\frac{1}{r}}{\frac{1}{\bar{z}_0}+\frac{1}{r}}\right)^{1/2}}$$

$$+C\cdot\frac{1}{2\left(r-\frac{1}{r}\right)\left(r+\frac{1}{r}\right)}.$$

By our fixation of the branches of u and t,

$$0,\quad \left(\frac{r-z_0}{r-\frac{1}{\bar{z}_0}}\right)^{1/2}\quad\text{and}\quad\left(\frac{r+z_0}{r+\frac{1}{\bar{z}_0}}\right)^{1/2}$$

correspond to

$$0,\quad\left(\frac{S_1(R_1-R_2)}{S_2R_1-S_1R_2}\right)^{1/2}\quad\text{and}\quad\left(\frac{S_2(R_1-R_2)}{S_2R_1-S_1R_2}\right)^{1/2},$$

by $u=u(t)$, respectively. Thus we see that the relations

$$\frac{R_1R_2S_1^{1/2}}{(S_2R_1-S_1R_2)^{1/2}(R_1-R_2)^{1/2}}=-\frac{r(r-b)\left((z_0-r)\left(\frac{1}{\bar{z}_0}-r\right)\right)^{1/2}}{2(1-r^2)(1+r^2)}$$

and

$$\frac{R_1R_2S_2^{1/2}}{(S_2R_1-S_1R_2)^{1/2}(R_1-R_2)^{1/2}}=\frac{r(r+b)\left((z_0+r)\left(\frac{1}{\bar{z}_0}+r\right)\right)^{1/2}}{2(1-r^2)(1+r^2)}$$

must be true. Let z tend to z_0, then t approaches zero and hence u also approaches zero, and we obtain

$$C=r^2\left(b-\frac{1}{r}\right)\left(\left(z_0-\frac{1}{r}\right)\left(\frac{1}{\bar{z}_0}-\frac{1}{r}\right)\right)^{1/2}\log(-1)$$
$$-r^2\left(b+\frac{1}{r}\right)\left(\left(z_0+\frac{1}{r}\right)\left(\frac{1}{\bar{z}_0}+\frac{1}{r}\right)\right)^{1/2}\log(-1),$$

whence follows

$$(r-b)\left((r-z_0)\left(r-\frac{1}{\bar{z}_0}\right)\right)^{1/2}\log\frac{u-\left(\frac{S_1(R_1-R_2)}{S_2R_1-S_1R_2}\right)^{1/2}}{u+\left(\frac{S_1(R_1-R_2)}{S_2R_1-S_1R_2}\right)^{1/2}}\cdot\frac{t+\left(\frac{z_0-r}{\frac{1}{\bar{z}_0}-r}\right)^{1/2}}{t-\left(\frac{z_0-r}{\frac{1}{\bar{z}_0}-r}\right)^{1/2}}$$

$$- (r+b)\Big((r+z_0)\Big(r+\frac{1}{\bar{z}_0}\Big)\Big)^{1/2} \log \frac{u - \left(\dfrac{S_2(R_1-R_2)}{S_2R_1-S_1R_2}\right)^{1/2}}{u + \left(\dfrac{S_2(R_1-R_2)}{S_2R_1-S_1R_2}\right)^{1/2}} \cdot \frac{t + \left(\dfrac{z_0+r}{\dfrac{1}{\bar{z}_0}+r}\right)^{1/2}}{t - \left(\dfrac{z_0+r}{\dfrac{1}{\bar{z}_0}+r}\right)^{1/2}}$$

$$= (1+rb)\Big((1+rz_0)\Big(1+\frac{r}{\bar{z}_0}\Big)\Big)^{1/2} \log \frac{-t + \left(\dfrac{1+rz_0}{1+\dfrac{r}{\bar{z}_0}}\right)^{1/2}}{t + \left(\dfrac{1+rz_0}{1+\dfrac{r}{\bar{z}_0}}\right)^{1/2}}$$

$$- (1-rb)\Big((1-rz_0)\Big(1-\frac{r}{\bar{z}_0}\Big)\Big)^{1/2} \log \frac{-t + \left(\dfrac{1-rz_0}{1-\dfrac{r}{\bar{z}_0}}\right)^{1/2}}{t + \left(\dfrac{1-rz_0}{1-\dfrac{r}{\bar{z}_0}}\right)^{1/2}}.$$

Let z tend to r, then t tends to $|z_0|((1-r/z_0)/(1-r\bar{z}_0))^{1/2}$, and we have

$$\Big(1-\frac{r}{b}\Big)\Big(\Big(1-\frac{r}{z_0}\Big)(1-\bar{z}_0r)\Big)^{1/2} \log \frac{S_1-S_2}{R_1-R_2} \frac{\Big(1-\dfrac{r}{z_0}\Big)(1-\bar{z}_0r)}{z_0-\dfrac{1}{\bar{z}_0}} \frac{z_0}{\bar{z}_0}$$

$$- \Big(1+\frac{r}{b}\Big)\Big(\Big(1+\frac{r}{z_0}\Big)(1+\bar{z}_0r)\Big)^{1/2} \log \frac{S_1^{1/2}-S_2^{1/2}}{S_1^{1/2}+S_2^{1/2}} \frac{\left(\dfrac{1-\dfrac{r}{z_0}}{1-\bar{z}_0r}\right)^{1/2} + \left(\dfrac{1+\dfrac{r}{z_0}}{1+\bar{z}_0r}\right)^{1/2}}{\left(\dfrac{1-\dfrac{r}{z_0}}{1-\bar{z}_0r}\right)^{1/2} - \left(\dfrac{1+\dfrac{r}{z_0}}{1+\bar{z}_0r}\right)^{1/2}}$$

$$= (1+br)\Big((1+z_0r)\Big(1+\frac{r}{\bar{z}_0}\Big)\Big)^{1/2} \log \frac{\left(\dfrac{1+z_0r}{1+\dfrac{r}{\bar{z}_0}}\right)^{1/2} - |z_0|\left(\dfrac{1-\dfrac{r}{z_0}}{1-\bar{z}_0r}\right)^{1/2}}{\left(\dfrac{1+z_0r}{1+\dfrac{r}{\bar{z}_0}}\right)^{1/2} + |z_0|\left(\dfrac{1-\dfrac{r}{z_0}}{1-\bar{z}_0r}\right)^{1/2}}$$

$$- (1-br)\Big((1-z_0r)\Big(1-\frac{r}{\bar{z}_0}\Big)\Big)^{1/2} \log \frac{\left(\dfrac{1-z_0r}{1-\dfrac{r}{\bar{z}_0}}\right)^{1/2} - |z_0|\left(\dfrac{1-\dfrac{r}{z_0}}{1-\bar{z}_0r}\right)^{1/2}}{\left(\dfrac{1-z_0r}{1-\dfrac{r}{\bar{z}_0}}\right)^{1/2} + |z_0|\left(\dfrac{1-\dfrac{r}{z_0}}{1-\bar{z}_0r}\right)^{1/2}}.$$

Similarly, let z tend to $-r$, then we have

$$\left(1-\frac{r}{b}\right)\left(\left(1-\frac{r}{z_0}\right)(1-\bar{z}_0 r)\right)^{1/2}\log\frac{S_1^{1/2}-S_2^{1/2}}{S_1^{1/2}+S_2^{1/2}}\,\frac{\left(\dfrac{1+\dfrac{r}{z_0}}{1+\bar{z}_0 r}\right)^{1/2}+\left(\dfrac{1-\dfrac{r}{z_0}}{1-\bar{z}_0 r}\right)^{1/2}}{\left(\dfrac{1-\dfrac{r}{z_0}}{1-\bar{z}_0 r}\right)^{1/2}-\left(\dfrac{1+\dfrac{r}{z_0}}{1+\bar{z}_0 r}\right)^{1/2}}$$

$$-\left(1+\frac{r}{b}\right)\left(\left(1+\frac{r}{z_0}\right)(1+\bar{z}_0 r)\right)^{1/2}\log\frac{S_1-S_2}{R_1-R_2}\,\frac{(1+\bar{z}_0 r)\left(1+\dfrac{r}{z_0}\right)}{z_0-\dfrac{1}{\bar{z}_0}}\,\frac{z_0}{\bar{z}_0}$$

$$=(1+br)\left((1+z_0 r)\left(1+\frac{r}{\bar{z}_0}\right)\right)^{1/2}\log\frac{\left(\dfrac{1+z_0 r}{1+\dfrac{r}{\bar{z}_0}}\right)^{1/2}-|z_0|\left(\dfrac{1+\dfrac{r}{z_0}}{1+\bar{z}_0 r}\right)^{1/2}}{\left(\dfrac{1+z_0 r}{1+\dfrac{r}{\bar{z}_0}}\right)^{1/2}+|z_0|\left(\dfrac{1+\dfrac{r}{z_0}}{1+\bar{z}_0 r}\right)^{1/2}}$$

$$-(1-br)\left((1-z_0 r)\left(1+\frac{r}{\bar{z}_0}\right)\right)^{1/2}\log\frac{\left(\dfrac{1-z_0 r}{1-\dfrac{r}{\bar{z}_0}}\right)^{1/2}-|z_0|\left(\dfrac{1+\dfrac{r}{z_0}}{1+\bar{z}_0 r}\right)^{1/2}}{\left(\dfrac{1-z_0 r}{1-\dfrac{r}{\bar{z}_0}}\right)^{1/2}+|z_0|\left(\dfrac{1+\dfrac{r}{z_0}}{1+\bar{z}_0 r}\right)^{1/2}}.$$

Eliminating the term $(S_1-S_2)/(R_1-R_2)$ from these two relations, we have

$$\left(1-\frac{r^2}{z_0^2}\right)^{1/2}(1-\bar{z}_0^2 r^2)^{1/2}\left(1-\frac{r^2}{b^2}\right)\log\frac{\left(1-\dfrac{r}{z_0}\right)(1-\bar{z}_0 r)}{\left(1+\dfrac{r}{z_0}\right)(1+\bar{z}_0 r)}$$

$$-\left[\left(1+\frac{r}{z_0}\right)(1+\bar{z}_0 r)\left(1+\frac{r}{b}\right)^2-\left(1-\frac{r}{z_0}\right)(1-\bar{z}_0 r)\left(1-\frac{r}{b}\right)^2\right]$$

$$\times\log\frac{S_1^{1/2}-S_2^{1/2}}{S_1^{1/2}+S_2^{1/2}}\,\frac{\left(\left(1-\dfrac{r}{z_0}\right)(1+\bar{z}_0 r)\right)^{1/2}+\left(\left(1+\dfrac{r}{z_0}\right)(1-\bar{z}_0 r)\right)^{1/2}}{\left(\left(1-\dfrac{r}{z_0}\right)(1+\bar{z}_0 r)\right)^{1/2}-\left(\left(1+\dfrac{r}{z_0}\right)(1-\bar{z}_0 r)\right)^{1/2}}$$

$$(\mathrm{Z})\qquad=\left(\left(1+\frac{r}{z_0}\right)(1+\bar{z}_0 r)\left(1+\frac{r}{\bar{z}_0}\right)(1+z_0 r)\right)^{1/2}\left(1+\frac{r}{b}\right)(1+br)$$

$$\times\log\frac{((1+z_0 r)(1-\bar{z}_0 r))^{1/2}-|z_0|\left(\left(1-\dfrac{r}{z_0}\right)\left(1+\dfrac{r}{\bar{z}_0}\right)\right)^{1/2}}{((1+z_0 r)(1-\bar{z}_0 r))^{1/2}+|z_0|\left(\left(1-\dfrac{r}{z_0}\right)\left(1+\dfrac{r}{\bar{z}_0}\right)\right)^{1/2}}$$

$$-\left(\left(1+\frac{r}{z_0}\right)(1+\bar{z}_0 r)\left(1-\frac{r}{\bar{z}_0}\right)(1-z_0 r)\right)^{1/2}\left(1+\frac{r}{b}\right)(1-br)$$

$$\times \log \frac{((1-z_0r)(1-\bar{z}_0r))^{1/2} - |z_0|\left(\left(1-\frac{r}{z_0}\right)\left(1-\frac{r}{\bar{z}_0}\right)\right)^{1/2}}{((1-z_0r)(1-\bar{z}_0r))^{1/2} + |z_0|\left(\left(1-\frac{r}{z_0}\right)\left(1-\frac{r}{\bar{z}_0}\right)\right)^{1/2}}$$

$$+\left((1+z_0r)\left(1+\frac{r}{\bar{z}_0}\right)(1-\bar{z}_0r)\left(1-\frac{r}{z_0}\right)\right)^{1/2}\left(1-\frac{r}{b}\right)(1+br)$$

$$\times \log \frac{((1+z_0r)(1+\bar{z}_0r))^{1/2} - |z_0|\left(\left(1+\frac{r}{z_0}\right)\left(1+\frac{r}{\bar{z}_0}\right)\right)^{1/2}}{((1+z_0r)(1+\bar{z}_0r))^{1/2} + |z_0|\left(\left(1+\frac{r}{z_0}\right)\left(1+\frac{r}{\bar{z}_0}\right)\right)^{1/2}}$$

$$-\left((1-z_0r)\left(1-\frac{r}{\bar{z}_0}\right)(1-\bar{z}_0r)\left(1-\frac{r}{z_0}\right)\right)^{1/2}\left(1-\frac{r}{b}\right)(1-br)$$

$$\times \log \frac{((1-z_0r)(1+\bar{z}_0r))^{1/2} - |z_0|\left(\left(1-\frac{r}{\bar{z}_0}\right)\left(1+\frac{r}{z_0}\right)\right)^{1/2}}{((1-z_0r)(1+\bar{z}_0r))^{1/2} + |z_0|\left(\left(1-\frac{r}{\bar{z}_0}\right)\left(1+\frac{r}{z_0}\right)\right)^{1/2}}.$$

We shall now enter into the local considerations. What we want to show is that the equation (Z) has no solution for any b, z_0 and r. Unfortunately we have failed to decide that this is true or not. Thus we are obliged to retract our original intention. In the present step we can show that (Z) has no solution for any b, z_0 and for all sufficiently small r. Suppose that (Z) is satisfied by a sufficiently small r for a suitable pair (b, z_0). Since both sides of (Z) are odd analytic functions of r, under somewhat tedious calculations the equation (Z) reduces for a sufficiently small r to an asymptotic identity

$$-\left(\frac{1}{z_0}+\bar{z}_0\right)-\left(\frac{2}{b}+\bar{z}_0+\frac{1}{z_0}\right)\log\frac{\frac{2}{b}+\bar{z}_0+\frac{1}{z_0}}{\frac{1}{z_0}-\bar{z}_0}$$

$$=\left(2b+z_0+\frac{1}{\bar{z}_0}\right)\log\frac{1-|z_0|}{1+|z_0|}-\frac{2z_0}{|z_0|}+O(r^2).$$

Neglecting the O-term, we shall denote this equation by (Y) and we shall remark that (Y) coincides with the equation (3.5) in Schaeffer–Spencer's paper [1]. Their procedure leading to a contradiction has been carried under the effect of the above equation (Y) and their supposition $|a_3|\geqq 3$.

Now we shall suppose that

$$\left|\frac{S_1-S_2}{R_1-R_2}\right|\geqq\frac{1+6r^2+r^4}{r(1-r^4)}$$

remains true for a sufficiently small r. Then we have an inequality

$$\text{(X)}\qquad \left|\frac{S_2R_1-S_1R_2}{R_1^2R_2^2}\right|=|\mathfrak{A}|\left|\frac{S_1+S_2}{R_1-R_2}\right|$$

$$\geqq\frac{1+6r^2+r^4}{r(1-r^4)}\cdot\frac{2(1-r^4)^2}{r^2\left(r^4+r^2\left(b^2+\frac{1}{b^2}+2bz_0+2b\frac{1}{\bar{z}_0}\right)+1\right)}$$

for the same r. Therefore we can conclude that

$$b^2 + \frac{1}{b^2} + 2b\left(z_0 + \frac{1}{\bar{z}_0}\right) \geqq 6,$$

since $|S_2R_1 - S_1R_2|/|R_1^2R_2^2| = 2|1 + O(r^4)|/r^3$ and the right hand member in (X) is asymptotically equal to

$$\frac{2}{r^3}\left(1 + \left(6 - b^2 - \frac{1}{b^2} - 2bz_0 - \frac{2b}{\bar{z}_0}\right)r^2 + O(r^4)\right).$$

Now all the necessary tools in order to lead to a contradiction has been obtained and we may repeat the reasonings in the Schaeffer–Spencer's paper [1] with some necessary modifications caused by O–term.

7. Now we have finished the discussions for all cases. And we can conclude that, for any $f \in \mathfrak{S}$,

$$\left|\frac{f'(r) + f'(-r)}{f(r) - f(-r)}\right| \leqq \frac{1 + 6r^2 + r^4}{r(1 - r^4)}$$

holds for all sufficiently small r and the equality sign occurs only for two Koebe's extremal functions $z/(1 \pm z)^2$.

Let r^* be the upper bound of r such that for $r < r^*$ our result surely remains valid. It seems to the author that r^* is really equal to 1. Consequently, to show this is now a remaining problem. Finally we should remark that r^* can be estimated from below if we perform our calculations with some cares in our local considerations and in the Schaeffer–Spencer's rejection processes.

Addendum in proof. It is regretted that a paper by M. Goulsin "On distortion theorems in the theory of conformal mapping. Rec. Math. **18** (1946), pp. 379—389" has not been taken into attention of the present author. In Goulsin's paper the problem has been completely solved by means of the Löwner's method, which is quite different from that availed here.

References

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