A DISTORTION THEOREM ON SCHLICHT FUNCTIONS

BY KÔTARO OIKAWA

A kind of distortion on schlicht functions meromorphic in the unit circle is discussed by Y. Komatu and H. Nishimiya [1]. In their paper, the fundamental role is played by an estimation of the spherical derivative

$$Df(z) = \frac{|f'(z)|}{1+|f(z)|^2}$$

of schlicht functions *regular* in the unit circle. Let $\mathfrak{S}$ be, as usual, the family of schlicht functions $w = f(z)$ regular in $|z| < 1$ and satisfying conditions $f(0) = 0$ and $f'(0) = 1$, and r^* be a positive root of the quadratic equation $r^2 + r - 1 = 0$, i.e. $r^* = (\sqrt{5} - 1)/2 = 0.618\cdots$. The above mentioned estimation is as follows:

THEOREM OF KOMATU-NISHIMIYA [1].[1] *For any $f(z) \in \mathfrak{S}$, the following holds on $|z| = r$:*

$$\frac{1-r^2}{r^2+(1+r)^4} \leqq Df(z) \leqq \frac{1-r^2}{r^2+(1-r)^4} \quad if \quad 0 \leqq r \leqq r^{*2} = 0.382\cdots,$$

$$\frac{1-r^2}{r^2+(1+r)^4} \leqq Df(z) < \frac{1}{2r}\cdot\frac{1+r}{1-r} \quad if \quad r^{*2} < r \leqq r^*,$$

$$\frac{(1-r)^3}{1+r}\cdot\frac{1}{r^2+(1-r)^4} < Df(z) < \frac{1}{2r}\cdot\frac{1+r}{1-r} \quad if \quad r^* < r < 1.$$

Every equality sign appearing in the above estimations is realized only by the Koebe function.

The purpose of the present note is to give the best possible estimation for all r and discuss the property of functions realizing the equality sign, which will be done by the well-known variational method due to M. Schiffer [3]. The author wishes to express his gratitude to Professors Y. Komatu and M. Ozawa for their valuable suggestions.

1. We begin with statement of our results. Let r_0 be the root of the equation $r^6 - 4r^5 + 7r^4 - 10r^3 + 7r^2 - 4r + 1 = 0$ contained in the interval $0 < r < 1$, i.e.,

---

Received June 3, 1957.

1) The original estimation in [1] involves a mistake. Inequalities given here are derived by their own method in repairing the mistake.

$$r_0 = \frac{1}{3}\Big[4 + \sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}} \tag{1}$$
$$- \sqrt{\Big\{4 + \sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}}\Big\}^2 - 36}\Big]$$
$$= 0.412\cdots.$$

THEOREM 1. *For any $f(z) \in \mathfrak{S}$, its spherical derivative on $|z| = r$ is estimated as follows:*

$$\frac{1 - r^2}{r^2 + (1 + r)^4} \leqq Df(z) \leqq \frac{1 - r^2}{r^2 + (1 - r)^4} \qquad \text{if } 0 \leqq r \leqq r_0, \tag{2}$$

$$\frac{1 - r^2}{r^2 + (1 + r)^4} \leqq Df(z) \leqq Q(r) \qquad \text{if } r_0 \leqq r < 1; \tag{3}$$

the function $Q(r)$ is defined by

$$\log Q(r) = \frac{\alpha^2 - 1}{\alpha^2 + 1} \log \frac{1}{2(\alpha^3 - \alpha)} + \log \frac{2\alpha}{1 - r^2}, \tag{4}$$

where $\alpha = \alpha(r)$ is the inverse function of

$$\log r = \sqrt{\frac{\alpha^2 - 1}{\alpha^2 + 1}} \log 2(\alpha^3 - \alpha) + \log \frac{\sqrt{\alpha^2 + 1} - \sqrt{\alpha^2 - 1}}{\sqrt{\alpha^2 + 1} + \sqrt{\alpha^2 - 1}}.\text{[2)]} \tag{5}$$

These estimations are exact.

Properties of functions which realize equality signs will be summarized as THEOREM 2 at the end of this note.

2. In order to prove the above theorem, we introduce two kinds of functionals on $\mathfrak{S}$,

$$I_r(f) = \operatorname*{Max}_{|z|=r} Df(z) \qquad \text{and} \qquad J_r(f) = \operatorname*{Min}_{|z|=r} Df(z)$$

for all $r\,(0 \leqq r < 1)$, and consider extremal problems to obtain $\mathrm{Max}_{f \in \mathfrak{S}} I_r(f)$ and $\mathrm{Min}_{f \in \mathfrak{S}} J_r(f)$.

Evidently, extremal functions $w = f(z)$ exist and we may assume without loss of generality that they satisfy the condition

$$f(a) = \alpha > 0 \quad \text{where} \quad Df(a) = I_r(f) \qquad (\text{resp. } = J_r(f)),\ |a| = r. \tag{6}$$

By making use of the Schiffer's method, [3], we see that the complementary continuum C_f of the image domain $f(|z| < 1)$ consists of a finite number of analytic arcs satisfying the following differential equation for a suitably chosen real parameter t:

$$\left(\frac{dw}{dt}\right)^2 \frac{\alpha - \alpha^3 - 2w}{w^2(w - \alpha)^2} = 1, \qquad \text{if } f \text{ is the maximizing function of } I_r, \tag{7}$$

2) The fact that the function $\alpha = \alpha(r)$ is actually defined by (5) will be shown in § 5.

(8) $$\left(\frac{dw}{dt}\right)^2 \frac{\alpha - \alpha^3 - 2w}{w^2(w-\alpha)^2} = -1, \quad \text{if } f \text{ is the maximizing function of } J_r.$$

3. First of all, we consider the minimizing function $w = f(z)$. For this purpose we observe the integral curve of (8) containing $w = \infty$, in other words the Γ-structure (see [2]) defined by

$$\Re \int \frac{\sqrt{\alpha - \alpha^3 - 2w}}{w(w-\alpha)}\, dw = \text{const.}$$

Using the general theory of Γ-structure [2], we can see that it contains $w = \infty$ as an end point and contains the interval $[\alpha, +\infty]$ (cf. $(\alpha - \alpha^3)/2 < \alpha$). Furthermore, no fork point is contained in $[\alpha, +\infty]$. Since $\alpha \notin C_f$, we immediately see that C_f coincides with the interval $[1/4, +\infty]$, $f(z) = z/(1+z)^2$ and $a = r$, which completes the proof of the left inequalities of (2) and (3).

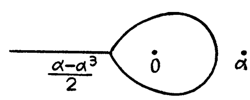
4. To consider the maximizing function $w = f(z)$ of I_r satisfying (6), we observe the Γ-structure of

(9) $$\Im \int \frac{\sqrt{\alpha - \alpha^3 - 2w}}{w(w-\alpha)}\, dw = \text{const.},$$

which is derived from (7).

If $(\alpha - \alpha^3)/2 \geqq -1/4$, the situation is similar to the last section, and we obtain analogously that $C_f = [-\infty, -1/4]$, $f(z) = z/(1-z)^2$ and $a = r$.

If $(\alpha - \alpha^3)/2 < -1/4$, we have to observe the Γ-structure more closely. By the above cited method [2] we see that it is, as is illustrated, symmetric to the real axis, and consists of the interval $[-\infty, (\alpha - \alpha^3)/2]$ and a loop which passes through $(\alpha - \alpha^3)/2$, surrounds 0 and separates α from 0. Then, C_f is a subcontinuum of it, which contains ∞ and at the same time $(\alpha - \alpha^3)/2$ as an (relative) inner point.



Now, denoting by α_0 the root of the equation $(\alpha - \alpha^3)/2 = -1/4$, we know that $(\alpha - \alpha^3)/2 \geqq -1/4$ is equivalent to $\alpha \leqq \alpha_0$. Then, $f(a) = \alpha = r/(1-r)^2$ because of the above result. The r satisfying $\alpha_0 = r/(1-r)^2$ is equal to the r_0 defined by (1), so that $\alpha \leqq \alpha_0$ implies $r \leqq r_0$. If $\alpha > \alpha_0$, the distortion theorem of Koebe shows that $\alpha = f(a) < r/(1-r)^2$, hence $r > r_0$. Therefore $r \leqq r_0$ and $(\alpha - \alpha^3)/2 \leqq -1/4$ are equivalent.

Consequently, for $0 \leqq r \leqq r_0$, we know that $C_f = [-\infty, -1/4]$, $f(z) = z/(1-z)^2$ and $a = r$, which implies the right inequality of (2).

5. The case of $r > r_0$ remains. From (7) we can see, by the method used in [3], that the maximizing function $w = f(z)$ for $r > r_0$, satisfies the following differential equation in $|z| < 1$:

(10) $$\frac{z^2(\alpha - \alpha^3 - 2w)}{w^2(w-\alpha)^2}\left(\frac{dw}{dz}\right)^2 = -\left(\alpha - \frac{1}{\alpha}\right)\frac{(z - e^{i\varphi})^2(z - e^{i\psi})^2}{(z-a)^2(z - 1/\bar{a})^2},$$

where φ and ψ are real and $e^{i\varphi} \neq e^{i\psi}$. Since the right hand side is nonnegative on $|z|=1$ because of (7), we can easily see

$$e^{i(\varphi+\psi)}=\frac{a}{\bar{a}}. \tag{11}$$

Taking the square root of (10), we get

$$\frac{\sqrt{\alpha-\alpha^3-2w}}{w(w-\alpha)}\frac{dw}{dz}=-i\sqrt{\alpha-\frac{1}{\alpha}}\frac{(z-e^{i\varphi})(z-e^{i\psi})}{z(z-a)(z-1/\bar{a})}, \tag{12}$$

(cf. $\alpha-1/\alpha>0$ for $\alpha>\alpha_0>1$), where the branch of $\sqrt{\alpha-\alpha^3-2w}$ is chosen such a one that takes $i\sqrt{\alpha-\alpha^3}$ at $z=0$. Comparing the residues of both sides at $z=a$ we get

$$\sqrt{\frac{\alpha^2+1}{\alpha^2-1}}=\frac{1+r^2-\bar{a}(e^{i\varphi}+e^{i\psi})}{1-r^2}. \tag{13}$$

By (11) and (13), the equation (12) is changed into

$$\frac{\sqrt{\alpha-\alpha^3-2w}}{w(w-\alpha)}\frac{dw}{dz}=-i\sqrt{\alpha-\frac{1}{\alpha}}\cdot\frac{1}{z} \tag{14}$$
$$+i\sqrt{\alpha+\frac{1}{\alpha}}\left(\frac{1}{z-a}+\frac{1}{z-1/\bar{a}}\right).$$

Putting $u=\sqrt{\alpha-\alpha^3-2w}$ and integrating, we have

$$\sqrt{\alpha^2-1}\log(u+i\sqrt{\alpha^3-\alpha})-\sqrt{\alpha^2+1}\log(u+i\sqrt{\alpha^3+\alpha})$$
$$-\sqrt{\alpha^2-1}\log\left(\frac{u-i\sqrt{\alpha^3-\alpha}}{z}\right)+\sqrt{\alpha^2+1}\log\left(\frac{u-i\sqrt{\alpha^3+\alpha}}{z-a}\right) \tag{15}$$
$$=-\sqrt{\alpha^2+1}\log\left(z-\frac{1}{\bar{a}}\right)+k,$$

where k is the integration constant and branch of every logarithm is taken suitably. Now we observe the values of *real part* of both sides of this equality at $z=0$, $z=a$ and $z=c$ respectively, where c is defined by $f(c)=(\alpha-\alpha^3)/2$. From the values at $z=0$ we can determine $\Re k$ and, using it, we can see from the values at $z=c$ that the equality (5)

$$\log r=\sqrt{\frac{\alpha^2-1}{\alpha^2+1}}\log 2(\alpha^3-\alpha)+\log\frac{\sqrt{\alpha^2+1}-\sqrt{\alpha^2-1}}{\sqrt{\alpha^2+1}+\sqrt{\alpha^2-1}}$$

holds. The values at $z=a$ show

$$\log|f'(a)|=\sqrt{\frac{\alpha^2-1}{\alpha^2+1}}\log\left\{\frac{\sqrt{\alpha^2+1}-\sqrt{\alpha^2-1}}{\sqrt{\alpha^2+1}+\sqrt{\alpha^2-1}}\cdot\frac{2(\alpha^3-\alpha)}{r}\right\} \tag{16}$$
$$+\log\left\{\frac{1}{r}\cdot\frac{\sqrt{\alpha^2+1}-\sqrt{\alpha^2-1}}{\sqrt{\alpha^2+1}+\sqrt{\alpha^2-1}}\right\}+\log\frac{2\alpha}{1-r^2},$$

therefore

$$\log I_r(f)=\log Df(a)=\log|f'(a)|-\log(1+\alpha^2)$$

$$= \frac{\alpha^2-1}{\alpha^2+1}\log\frac{1}{2(\alpha^3-\alpha)} + \log\frac{2\alpha}{1-r^2},$$

which implies the formula (4).

Now, concerning (5), we know

$$\frac{d\log r}{d\alpha} = \frac{2\alpha\log 2(\alpha^3-\alpha)}{(\alpha^2+1)\sqrt{\alpha^4-1}} + \frac{1}{\alpha}\sqrt{\frac{\alpha^2-1}{\alpha^2+1}} > 0, \quad \text{for} \quad \alpha \geqq \alpha_0 (>1),$$

$$(\log r)_{\alpha=\alpha_0} = \log r_0, \qquad \lim_{\alpha\to\infty}\log r = \infty.$$

So that the inverse function $\alpha = \alpha(r)$ of (5) is determined for $r_0 \leqq r < 1$. The theorem is hereby proved completely.

6. For $r > r_0$, we have shown that C_f of the maximizing function $w = f(z)$ is a subcontinuum of the Γ-structure of (9) which contains $w = \infty$ and whose mapping radius (with respect to $w = 0$) is equal to one. Conversely, any function $f(z) \in \mathfrak{S}$ whose C_f has this property is the maximizing function of I_r for r corresponding to α by (5). In fact, defining a by $f(a) = \alpha$, the preceding consideration shows that $f(z)$ satisfies the differential equation (12). Consequently we know that $\log|a|$ and $\log|f'(a)|$ satisfy (5) and (16) respectively, which implies $|a| = r$ and $Df(a) = Q(r)$, i.e. $f(z)$ is maximizing. Summing up all, we have

THEOREM 2. *Equality signs in* (2) *and* (3) *are realized, except the trivial transformation* $f(z) \mid e^{-i\lambda}f(e^{i\lambda}z)$,

(i) *for the left inequalities of* (2) *and* (3), *by and only by* $f(z) = z/(1-z)^2$ *at* $z = -r$,

(ii) *for the right inequality of* (2), *by and only by* $f(z) = z/(1-z)^2$ *at* $z = r$,

(iii) *for the right inequality of* (3), *by any function* $f(z) \in \mathfrak{S}$ *whose* C_f *is a subcontinuum of the* Γ*-structure of* (9) (*figure in* § 3) *with mapping radius equal to one and containing* $w = \infty$, *and only by them, at* $z = a$ *such that* $f(a) = \alpha = \alpha(r)$.

It is to be noted that, in case (iii), maximizing functions form, even disregarding the trivial transformations, a family generated by one real parameter.

We remark finally that $\lim_{r\to 1}\alpha(r) < \infty$.

REFERENCES

[1] KOMATU, Y., AND H. NISHIMIYA, On distortion in schlicht mappings. Kōdai Math. Sem. Rep., 2 (1950), 47—50.

[2] SCHAEFFER, A.C., AND D.C. SPENCER, Coefficient regions for schlicht functions. New York, 1950.

[3] SCHIFFER, M., A method of variation within the family of simple functions. Proc. London Math. Soc. (2), 44 (1938), 432—449.

DEPARTMENT OF MATHEMATICS,
TOKYO INSTITUTE OF TECHNOLOGY.