ON CONFORMAL MAPPING OF A DOMAIN WITH CONVEX OR STAR-LIKE BOUNDARY

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1. Introduction.

A necessary and sufficient condition for an analytic function $f(z)$ to be regular in the unit circle $E: |z| < 1$ and to map E univalently onto a convex domain is that it satisfies

$$\Re\left(1 + z\frac{f''(z)}{f'(z)}\right) > 0 \qquad \text{for} \quad |z| < 1.$$

This is a well-known classical theorem originally due to Study [11]. Its sufficiency proof particularly with respect to the regularity and univalency of $f(z)$ has been later supplemented by Kobori [4]. Once these properties of $f(z)$ having been established, it is ready to show that the convexity of the image-domain $f(E)$ follows from the condition of the theorem. In fact, in view of the relation

$$\frac{d}{d\varphi}\arg df(re^{i\varphi}) = \Re\left(1 + re^{i\varphi}\frac{f''(re^{i\varphi})}{f'(re^{i\varphi})}\right) > 0$$

for any fixed r with $0 \leqq r < 1$, the image-domain of any concentric circular disc $|z| < r(< 1)$ by $f(z)$ is convex so that $f(E)$ is itself convex.

On the other hand, Carathéodory [3] has given a proof in which the necessity of the condition is shown by making use of a convergence theorem on variable domains established by himself [2]. Later Radó [10] has given a very elementary proof of the fact that if $f(z)$ maps the whole circle $|z| < 1$ univalently onto a convex domain then it maps every concentric circle also onto a convex domain. The necessity part of Study's theorem may be regarded as its immediate consequence. In fact, there then holds

$$\Re\left(1 + z\frac{f''(z)}{f'(z)}\right) = \frac{d}{d\varphi}\arg df(re^{i\varphi}) \geqq 0 \qquad (z = re^{i\varphi})$$

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along every concentric circumference $|z| = r(<1)$ and, since the left-hand member is a harmonic function of z not identically vanishing, this implies the condition of Study's theorem.

A function which maps a circle univalently onto a star-like domain is closely connected with a function which maps the same circle onto a convex domain. The explicit connection is really given by a classical theorem originally due to Alexander [1]. Its background consists in the identity

$$z\frac{f'(z)}{f(z)} = 1 + z\frac{d^2}{dz^2}\int_0^z \frac{f(z)}{z}\,dz \Big/ \frac{d}{dz}\int_0^z \frac{f(z)}{z}\,dz$$

valid for any function $f(z)$ analytic in a circle about the origin where it vanishes, together with the relation

$$\frac{d}{d\varphi}\arg f(re^{i\varphi}) = \Re\left(re^{i\varphi}\frac{f'(re^{i\varphi})}{f(re^{i\varphi})}\right)$$

valid for any fixed r with $0 \leqq r < 1$. By means of this theorem, any theorem concerning convex mapping will be transferred to a corresponding theorem concerning star-like mapping, and vice versa.

In the present paper we shall first again deal with the condition for convex mapping of a circle by establishing an integral representation for mapping function. Its leading idea originates essentially from Study [11]. It will be shown that Study's theorem stated at the beginning of the present paper may be regarded as an immediate consequence of this representation. It serves further to establish several properties of convex mapping systematically some of which will be illustrated below. We then re-prove Alexander's theorem by establishing independently an integral representation for function which maps a circle univalently onto a star-like domain.

We shall next consider corresponding problems in case of an annulus as a doubly-connected basic domain. In the simply-connected case, though Radó's argument on convex mapping is very ingenious, it depends explicitly on the particularity of the unit circle as a basic domain since Schwarz lemma plays there its leading role. It seems therefore difficult to modify this argument simply so as to fit in the doubly-connected case. Thus we shall here also establish an integral representation for a function mapping an annulus onto a ring domain bounded by two convex curves. Some results corresponding to those in the simply-connected case will be obtained by means of this representation. A theorem analogous to Alexander's will be also established. It will play the correponding role of transference between mappings of an annulus onto domains bounded by convex and star-like boundary components.

Finally it would be noted, by the way, that for any analytic function w

$= f(z)$ and its inverse function (a branch) $z = g(w)$, there exist the relations

$$\frac{zf'(z)}{f(z)} = 1 \Big/ \frac{wg'(w)}{g(w)} \qquad \text{or} \qquad \frac{wg'(w)}{g(w)} = 1 \Big/ \frac{zf'(z)}{f(z)}$$

and

$$1 + \frac{zf''(z)}{f'(z)} = 1 - \frac{wg''(z)}{g'(w)} \Big/ \frac{wg'(w)}{g(w)} \quad \text{or} \quad 1 + \frac{wg''(w)}{g'(w)} = 1 - \frac{zf''(z)}{f'(z)} \Big/ \frac{zf'(z)}{f(z)}.$$

2. Mapping of a circle onto a convex domain.

As a preparatory consideration we first remark that the necessity part of Study's theorem referred to at the beginning of the present paper is readily derivable provided the boundary of the image-domain is not of so complicated nature such that the mapping function $f(z)$ satisfies certain smoothness condition on the closed unit circle. For instance, it is enough to suppose that $f(z)$ is piecewise regular on $|z| \leqq 1$, i. e. $f(z)$ is regular on $|z| \leqq 1$ except at a finite number of boundary points where $f'(z)$ has discontinuities of the first kind along $|z| = 1$. In fact, the boundary value of the function $\Re(1 + zf''(z)/f'(z))$ harmonic in $|z| < 1$ is then equal to $(d/d\varphi) \arg df(e^{i\varphi})$ at $z = e^{i\varphi}$ except at the discontinuities of $f'(z)$. The assumption that the boundary curve defined by $f(e^{i\varphi})$ for $-\pi \leqq \varphi < \pi$ is convex implies that $\arg df(e^{i\varphi})$ is, qua function of φ, increasing and hence $\Re(1 + zf''(z)/f'(z))$ has non-negative boundary value everywhere except at a finite number of the boundary points where it is bounded from below. Consequently, it must be positive throughout $|z| < 1$, as desired.

Moreover, we then have an integral representation of Herglotz type

$$1 + z\frac{f''(z)}{f'(z)} = \int_{-\pi}^{\pi} \frac{e^{i\varphi} + z}{e^{i\varphi} - z}\, d\rho(\varphi)$$

where

$$\rho(\varphi) = \frac{1}{2\pi}\int_{-\pi}^{\varphi} d \arg df(e^{i\varphi})$$

is an increasing function with total variation equal to unity. The representation of this type remains valid also in general by taking $\rho(\varphi)$ as a suitable function with the assigned property. This fact will be proved in the following theorem by means of a method of approximation.

THEOREM 2.1. *Let $f(z)$ be an analytic function mapping the unit circle E: $|z| < 1$ univalently onto a convex domain. Then there exists an increasing function $\rho(\varphi)$ defined for $-\pi \leqq \varphi < \pi$ with total variation equal to unity such that an integral representation of Herglotz type*

$$1 + z\frac{f''(z)}{f'(z)} = \int_{-\pi}^{\pi} \frac{e^{i\varphi} + z}{e^{i\varphi} - z}\, d\rho(\varphi)$$

holds for $z \in E$.

Proof. We approximate the image-domain $f(E)$ by a sequence of convex domains $\{D_\nu\}$ in the sense of Carathéodory's domain-kernel. Let the mapping function $f_\nu(z)$ of E onto D_ν be normalized by

$$f_\nu(0) = f(0) \qquad \text{and} \qquad \arg f_\nu'(0) = \arg f'(0).$$

We may further suppose that for every ν the function $f_\nu(z)$ admits of a representation of the form under consideration. For this purpose it suffices to observe, for instance, a sequence of domains bounded by convex rectilinear polygons which exhausts $f(E)$. We then have

$$1 + z\frac{f_\nu''(z)}{f_\nu'(z)} = \int_{-\pi}^{\pi} \frac{e^{i\varphi}+z}{e^{i\varphi}-z} d\rho_\nu(\varphi)$$

where $\rho_\nu(\varphi)$ is defined by

$$\rho_\nu(\varphi) = \frac{1}{2\pi}\int_{-\pi}^{\varphi} d \arg df_\nu(e^{i\varphi});$$

cf. [6]. Since $\rho_\nu(\varphi)$ satisfies

$$d\rho_\nu(\varphi) \geqq 0 \qquad \text{and} \qquad \int_{-\pi}^{\pi} d\rho_\nu(\varphi) = 1,$$

we can choose by Helly's selection theorem a subsequence $\{\rho_{\kappa_\nu}(\varphi)\}$ which converges for all values of φ. Let its limit function be denoted by $\rho(\varphi)$, i. e.

$$\lim_{\nu\to\infty} \rho_{\kappa_\nu}(\varphi) = \rho(\varphi).$$

Evidently $\rho(\varphi)$ is an increasing function defined for $-\pi \leqq \varphi < \pi$ with total variation equal to unity. Since $(\partial/\partial\varphi)((e^{i\varphi}+z)/(e^{i\varphi}-z))$ is for any $z \in E$ continuous with respect to φ, Lebesgue's convergence theorem implies

$$\lim_{\nu\to\infty}\int_{-\pi}^{\pi} \frac{e^{i\varphi}+z}{e^{i\varphi}-z} d\rho_{\kappa_\nu}(\varphi) = \lim_{\nu\to\infty}\left\{\left[\frac{e^{i\varphi}+z}{e^{i\varphi}-z}\rho_{\kappa_\nu}(\varphi)\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\rho_{\kappa_\nu}(\varphi)\frac{\partial}{\partial\varphi}\frac{e^{i\varphi}+z}{e^{i\varphi}-z}d\varphi\right\}$$

$$= \left[\frac{e^{i\varphi}+z}{e^{i\varphi}-z}\rho(\varphi)\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\rho(\varphi)\frac{\partial}{\partial\varphi}\frac{e^{i\varphi}+z}{e^{i\varphi}-z}d\varphi = \int_{-\pi}^{\pi}\frac{e^{i\varphi}+z}{e^{i\varphi}-z}d\rho(\varphi).$$

On the other hand, the sequence $\{f_\nu(z)\}$ converges as $\nu \to \infty$ uniformly to $f(z)$ in the wider sense in $|z| < 1$ by Carathéodory's theorem (cf. [2]) and hence $\{1 + zf_\nu''(z)/f_\nu'(z)\}$ converges to $1 + zf''(z)/f'(z)$ by Weierstrass' double series theorem. Consequently, the desired representation is obtained.

If it is required to prove merely the necessity part of Study's theorem, the above proof is correspondingly shortened. In fact, the inequality $\Re(1 + zf_\nu''(z)/f_\nu'(z)) > 0$ leads readily to the limit inequality $\Re(1 + zf''(z)/f'(z)) \geqq 0$ where the equality sign may be rejected since $f(z)$ does not degenerate to a constant. This is indeed the procedure used by Carathéodory [3] in his proof for this own fact.

COROLLARY. *For $f(z)$ satisfying the condition in theorem 2.1, there holds*

$$f(z) = f'(0)\int_0^z \exp\left(2\int_{-\pi}^{\pi} \lg \frac{e^{i\varphi}}{e^{i\varphi}-z}\, d\rho(\varphi)\right) dz + f(0)$$

with the same $\rho(\varphi)$ as in theorem 2.1, where the logarithm involved in the integrand denotes the branch reducing to zero for $z=0$.

Proof. Integration of the representation in theorem 2.1 with respect to $\lg z$, followed by a further integration with respect to z.

We now show that the converse of theorem 2.1 (or rather its corollary) is also valid.

THEOREM 2.2. *Let $\rho(\varphi)$ defined for $-\pi \leqq \varphi < \pi$ be an increasing function with total variation equal to unity. Then the function defined by*

$$f(z) = A\int_0^z \exp\left(2\int_{-\pi}^{\pi} \lg \frac{e^{i\varphi}}{e^{i\varphi}-z}\, d\rho(\varphi)\right) dz + B$$

maps the unit circle univalently onto a convex domain, where $A(=f'(0) \neq 0)$ and $B(=f(0))$ are any preassignable constants.

Proof. It is evident that $f(z)$ is regular in $|z|<1$. We now observe the image-curve of $|z|=r$ for a fixed r with $0<r<1$. In view of the relation

$$\begin{aligned}\frac{d}{d\theta}\arg df(re^{i\theta}) &= \frac{d}{d\theta}\Im \lg\,(ire^{i\theta}f'(re^{i\theta}))\\ &= \int_{-\pi}^{\pi}\frac{\partial}{\partial\theta}\Im\lg\frac{re^{i\theta}}{(e^{i\varphi}-re^{i\theta})^2}\,d\rho(\varphi) = \int_{-\pi}^{\pi}\frac{1-r^2}{|e^{i\varphi}-re^{i\theta}|^2}\,d\rho(\varphi) > 0,\end{aligned}$$

the tangent vector of this image-curve turns steadily in the positive direction as z moves along $|z|=r$ in the positive sense. Further, since we have

$$\int_{-\pi}^{\pi} d\arg df(re^{i\theta}) = \int_{-\pi}^{\pi} d\rho(\varphi)\int_{-\pi}^{\pi}\frac{1-r^2}{|e^{i\varphi}-re^{i\theta}|^2}\,d\theta = 2\pi,$$

the image-curve must be a simple closed curve. Consequently, by virtue of Darboux's theorem $f(z)$ maps $|z|<r$ univalently onto a convex domain for any r with $0<r<1$ whence follows the result.

Based on theorem 2.1 (or its corollary) and theorem 2.2, we can conclude that the representation for $f(z)$ given in theorem 2.2 is characteristic to a function mapping $|z|<1$ univalently onto a convex domain. This integral representation having been once established, we can derive systematically several well-known classical theorems on convex mapping some among which will be now illustrated in the following lines. Similar argument has been once announced in a previous paper [6].

THEOREM 2.3. *Let $f(z)$ map $|z|<1$ univalently onto a convex domain. Then, for $|z|<1$,*

$$\frac{1-|z|}{1+|z|} \leqq 1+\Re z\frac{f''(z)}{f'(z)} \leqq \frac{1+|z|}{1-|z|},$$

$$\frac{1-|z|}{1+|z|} \leqq \left|1+z\frac{f''(z)}{f'(z)}\right| \leqq \frac{1+|z|}{1-|z|},$$

$$\frac{1}{(1+|z|)^2} \leqq \left|\frac{f'(z)}{f'(0)}\right| \leqq \frac{1}{(1-|z|)^2}.$$

For any assigned z_0 with $0<|z_0|<1$, the equality sign in the left and right inequalities in every estimation can appear only for functions of the form $af_*(z; z_0)+b$ and $af^*(z; z_0)+b$ with any constants $a(\neq 0)$ and b, respectively, where

$$f_*(z; z_0) = \frac{z}{z+z_0/|z_0|} \quad \textit{and} \quad f^*(z; z_0) = f_*(z; -z_0) = \frac{z}{z-z_0/|z_0|}.$$

The extremal functions $f_*(z; z_0)$ and $f^*(z; z_0) = f_*(-z; z_0)$ map $|z|<1$ onto the same half-plane bounded by the vertical line which cuts the real axis at $1/2|z_0|$.

THEOREM 2.4. *Let $f(z)$ map $|z|<1$ univalently onto a convex domain. Then, for $|z|<1$,*

$$\left|\arg\frac{f'(z)}{f'(0)}\right| \leqq 2\arcsin|z|.$$

For any assigned z_0 with $0<|z_0|<1$, the quantity $\arg(f'(z)/f'(0))$ (*the branch vanishing at the origin and continued harmonically*) *can attain the values* $-2\arcsin|z_0|$ *and* $+2\arcsin|z_0|$ *if and only if $f(z)$ is of the form $af_*(z; z_0)+b$ and $af^*(z; z_0)+b$ with any constants $a(\neq 0)$ and b, respectively, where*

$$f_*(z; z_0) = \frac{z}{z_0(1-i\sqrt{1/|z_0|^2-1})-z}$$

and $f^*(z; z_0) = \overline{f_*(\bar{z}; \bar{z}_0)}$, *i.e.*

$$f^*(z; z_0) = \frac{z}{z_0(1+i\sqrt{1/|z_0|^2-1})-z}.$$

The extremal functions $f_*(z; z_0)$ and $f^*(z; z_0)$ map $|z|<1$ onto the same half-plane bounded by the vertical line through $1/2$.

By the way, it is noted that the extremal functions in theorem 2.3 depend essentially only on $\arg z_0$ while those in theorem 2.4 depend on $\arg z_0$ as well as $|z_0|$.

3. Mapping of a circle onto a star-like domain.

A function mapping a circle univalently onto a star-like domain is, as previously noticed in the introduction, in closed connection with a function mapping the same circle univalently onto a convex domain. In order to arrive at Alexander's theorem which gives the explicit connection between these mappings, we first establish an integral representation analogous to that given in theorem 2.1 and characteristic to a function with a star-like image.

THEOREM 3.1. *Let $f(z)$ be an analytic function mapping the unit circle $E: |z| < 1$ univalently onto a domain star-like with respect to the point $f(0) = 0$. Then there exists an increasing function $\rho(\varphi)$ defined for $-\pi \leqq \varphi < \pi$ with total variation equal to unity such that an integral representation of Herglotz type*

$$z\frac{f'(z)}{f(z)} = \int_{-\pi}^{\pi} \frac{e^{i\varphi}+z}{e^{i\varphi}-z}\, d\rho(\varphi)$$

holds for $z \in E$.

Proof. Let the image-domain $f(E)$ be approximated by a sequence of domains $\{D_\nu\}$ in the sense of Carathéodory's domain-kernel, where every D_ν is star-like with respect to the origin. It is readily seen that we can suppose every D_ν be, for instance, bounded by a rectilinear polygon. Then the mapping function $f_\nu(z)$ of E onto D_ν satisfying the normalization $f_\nu(0) = f(0)$ as well as $\arg f_\nu'(0) = \arg f'(0)$ admits of a representation

$$z\frac{f_\nu'(z)}{f_\nu(z)} = \int_{-\pi}^{\pi} \frac{e^{i\varphi}+z}{e^{i\varphi}-z}\, d\rho_\nu(\varphi)$$

where $\rho_\nu(\varphi)$ is defined by

$$\rho_\nu(\varphi) = \frac{1}{2\pi}\int_{-\pi}^{\varphi} d\arg f_\nu(e^{i\varphi}).$$

In view of the star-likeness of $f_\nu(z)$, we have

$$d\rho_\nu(\varphi) \geqq 0 \qquad \text{and} \qquad \int_{-\pi}^{\pi} d\rho_\nu(\varphi) = 1.$$

The remaining part of the proof proceeds then quite similarly as in the proof of theorem 2.1.

COROLLARY. *For $f(z)$ satisfying the condition in theorem 3.1, there holds*

$$f(z) = f'(0)\, z \exp\left(2\int_{-\pi}^{\pi} \lg \frac{e^{i\varphi}}{e^{i\varphi}-z}\, d\rho(\varphi)\right)$$

with the same $\rho(\varphi)$ as in theorem 3.1, where the logarithm in the integrand denotes the branch which reduces to zero for $z = 0$.

THEOREM 3.3. *Let $\rho(\varphi)$ defined for $-\pi \leqq \varphi < \pi$ be an increasing function with total variation equal to unity. Then the function defined by*

$$f(z) = Az \exp\Big(2\int_{-\pi}^{\pi} \lg \frac{e^{i\varphi}}{e^{i\varphi}-z}\, d\rho(\varphi)\Big)$$

maps the unit circle univalently onto a domain star-like with respect to the origin, where $A(=f'(0))$ is any non-vanishing constant.

Proof. Similar as the proof of theorem 2.2. It will be only necessary to note the relations

$$\frac{d}{d\theta} \arg f(re^{i\theta}) = \int_{-\pi}^{\pi} \frac{1-r^2}{|e^{i\varphi}-re^{i\theta}|^2}\, d\rho(\varphi) > 0$$

and

$$\int_{-\pi}^{\pi} d \arg f(re^{i\theta}) = \int_{-\pi}^{\pi} d\rho(\varphi) \int_{-\pi}^{\pi} \frac{1-r^2}{|e^{i\varphi}-re^{i\theta}|^2}\, d\theta = 2\pi,$$

both valid for any r with $0 < r < 1$.

Theorem 3.1 (or its corollary) together with theorem 3.2 shows that the representation for $f(z)$ given in theorem 3.2 is characteristic to a function mapping $|z| < 1$ univalently onto a domain star-like with respect to $f(0) = 0$. Comparing this representation with that for convex mapping, we see that the former is obtained from the latter by merely substituting $zf'(z)/f(z)$ instead of $1 + zf''(z)/f'(z)$. This fact is just the content of Alexander's theorem which may be fully stated as follows.

THEOREM 3.3. *Let $\mathfrak{K} = \{f_c(z)\}$ be the class consisting of functions with the property mentioned in theorem 2.1 and $\mathfrak{St} = \{f_s(z)\}$ the class consisting of functions with the property mentioned in theorem 3.1. Then they are connected by the relation*

$$1 + z\frac{f_c''(z)}{f_c'(z)} = z\frac{f_s'(z)}{f_s(z)}.$$

More precisely, for any function $f_c(z) \in \mathfrak{K}$ the function defined by

$$f_s(z) = Czf_c'(z)$$

with any constant $C(\neq 0)$ belongs to $\mathfrak{St}$ and, conversely, for any $f_s(z) \in \mathfrak{St}$ the function defined by

$$f_c(z) = A\int_0^z f_s(z)\frac{dz}{z} + B$$

with any constants $A(\neq 0)$ and B belongs to $\mathfrak{K}$.

Proof. Evident by virtue of theorems 2.1, 2.2, 3.1 and 3.2.

After Alexander's theorem 3.3 has been established, it is ready to transfer the theorems 2.3 and 2.4 concerning convex mapping to the corresponding theorems concerning star-like mapping. The latter theorems can be derived, of course, also directly from theorem 3.1.

THEOREM 3.4. *Let $f(z)$ map $|z|<1$ univalently onto a domain star-like with respect to $f(0)=0$. Then, for $|z|<1$,*

$$\frac{1-|z|}{1+|z|} \leqq \Re z \frac{f'(z)}{f(z)} \leqq \frac{1+|z|}{1-|z|},$$

$$\frac{1-|z|}{1+|z|} \leqq \left| z \frac{f'(z)}{f(z)} \right| \leqq \frac{1+|z|}{1-|z|},$$

$$\frac{|z|}{(1+|z|)^2} \leqq \left| \frac{f(z)}{f'(0)} \right| \leqq \frac{|z|}{(1-|z|)^2}.$$

For any assigned z_0 with $0<|z_0|<1$, the equality sign in the left and right inequalties in every estimation can appear only for the functions of the form $cf_*(z; z_0)$ and $cf^*(z;z_0)$ with any non-vanishing constant c, respectively, where

$$f_*(z; z_0) = \frac{z}{(z+z_0/|z_0|)^2} \quad \text{and} \quad f^*(z; z_0) = f_*(z; -z_0) = \frac{z}{(z-z_0/|z_0|)^2}.$$

The extremal functions $f_*(z; z_0)$ and $f^*(z; z_0)$ map $|z|<1$ onto the whole plane which is cut along infinite half-rays centered at the origin and with the inclinations $-\arg z_0$ and $\pi - \arg z_0$, respectively.

THEOREM 3.5. *Let $f(z)$ map $|z|<1$ univalently onto a domain star-like with respect to $f(0)=0$. Then, for $|z|<1$,*

$$\arg \left| \frac{f(z)}{f'(0)z} \right| \leqq 2 \arcsin |z|.$$

For any assigned z_0 with $0<|z_0|<1$, the quantity $\arg(f(z)/(f'(0)z))$ (the branch vanishing at the origin and continued harmonically) becomes $-2\arcsin|z_0|$ and $+2\arcsin|z_0|$ if and only if $f(z)$ is of the form $cf_*(z; z_0)$ and $cf^*(z; z_0)$ with any non-vanishing constant c, respectively, where

$$f_*(z; z_0) = \frac{z}{(z-z_0(1-i\sqrt{1/|z_0|^2-1}))^2}$$

and $f^*(z; z_0) = \overline{f_*(\bar{z}; \bar{z}_0)}$, i.e.

$$f^*(z; z_0) = \frac{z}{(z-z_0(1+i\sqrt{1/|z_0|^2-1}))^2}.$$

The extremal functions $f_*(z; z_0)$ and $f^*(z; z_0)$ map $|z|<1$ onto the whole plane which is cut along infinite half-rays centered at the origin and with the inclinations $\pi/2 - \arcsin|z_0| - \arg z_0$ and $\pi/2 + \arcsin|z_0| + \arg z_0$, respectively.

4. Preliminary lemmas on elliptic functions.

In two subsequent sections we shall deal with analytic functions defined in an annulus

$$R_q: \qquad (0<)q<|z|<1.$$

While the Poisson kernel (complex form) involved in Herglotz representation for functions analytic in a circle is an elementary and really a linear function of its both arguments, it will be replaced in case of the annulus R_q as the basic domain by certain elliptic zeta-functions involved in the corresponding Villat-Stieltjes representation. Accordingly, some results concerning these functions will become necessary for establishing the distortion theorems which correspond to those stated above in the case of the unit circle. In order to avoid the interruption of our main discourse and for the sake of completeness, we summarize in the present section some lemmas necessary for later use. Throughout the present paper, we suppose that the notations on elliptic functions concern always the Weierstrassian theory constructed with primitive periods

$$2\omega_1 = 2\pi \quad \text{and} \quad 2\omega_3 = 2i \lg \frac{1}{q},$$

unless a contrary is explicitly stated. In particular, the quantities $e_\nu = \wp(\omega_\nu)$ $(\nu = 1, 2, 3)$ are then real and $\eta_1 = \zeta(\omega_1)$ is real positive while the quantity $\eta_3 = \zeta(\omega_3)$ is purely imaginary, and the e's satisfy the inequality

$$e_1 > e_2 > e_3.$$

LEMMA 4.1. *For any real φ, the function defined by*

$$P(z; \varphi) = \Im\left(\zeta(i \lg z + \varphi) + \frac{\eta_3}{\lg q} \lg z\right)$$

remains positive throughout the annulus R_q while the function defined by

$$Q(z; \varphi) = \Im\left(\zeta_3(i \lg z + \varphi) + \frac{\eta_3}{\lg q} \lg z\right)$$

remains negative throughout R_q.[1]

Proof. It is evident that $P(z; \varphi)$ is regular and harmonic in R_q. It is expressed in two alternative ways:

$$P(re^{i\theta}; \varphi) = \frac{1-r^2}{1-2r\cos(\theta-\varphi)+r^2} + 2\sum_{n=1}^{\infty} \frac{q^{2n}(r^n - 1/r^n)}{1-q^{2n}} \cos n(\theta-\varphi) - \frac{\lg r}{\lg q}$$

$$= 1 + 2\sum_{n=1}^{\infty} \frac{r^n - q^{2n}/r^n}{1-q^{2n}} \cos n(\theta-\varphi) - \frac{\lg r}{\lg q}.$$

The first expression shows that $P(z; \varphi)$ has the boundary value vanishing everywhere along $|z| = 1$ except at a single point $z = e^{i\varphi}$ where it behaves like as the Poisson kernel $(1-|z|^2)/|e^{i\varphi} - z|^2$ which is positive for $|z| < 1$. The second expression shows that $P(z; \varphi)$ has the boundary value vanishing everywhere along $|z| = q$ without any exception. Consequently, $P(z; \varphi)$ is

1) Since the periodicity moduli of $\zeta(i \lg z + \varphi)$ as well as $i \lg z$ are real, the functions $P(z;\varphi)$ and $Q(z;\varphi)$ are both single-valued with respect to their both arguments, a fact which will be seen also in the proof of this lemma.

positive throughout R_q. The negativeness of $Q(z; \varphi)$ is an immediate consequence of the fact just verified. In fact, we have an identity

$$Q(z; \varphi) = \Im\Big(\zeta\Big(-i \lg \frac{q}{z} + \varphi\Big) - \eta_3 + \frac{\eta_3}{\lg q} \lg z \Big)$$

$$= - \Im\Big(\zeta\Big(i \lg \frac{q}{z} - \varphi\Big) + \frac{\eta_3}{\lg q} \lg \frac{q}{z} \Big) = - P\Big(\frac{q}{z}; -\varphi\Big)$$

and the point q/z lies in R_q.

LEMMA 4.2. *For any r with $q < r < 1$, the quantities $\Im\zeta(i \lg r + \psi)$ and $\Im\zeta_3(i \lg r + \psi)$, qua functions of ψ, are even periodic functions with the period 2π which are strictly decreasing and increasing, respectively, for $0 \leqq \psi \leqq \pi$.*

Proof. (Cf. [5].) It is evident that $\Im\zeta(i \lg r + \psi)$ is an even periodic function of ψ with period 2π. In order to show its decreasing character in $0 \leqq \psi \leqq \pi$, we observe its series forms expressed in two alternative ways:

$$\Im\zeta(i \lg r + \psi) = \frac{\eta_1}{\pi} \lg r + \frac{1}{2} \frac{1 - r^2}{1 - 2r \cos\psi + r^2} + \sum_{n=1}^{\infty} \frac{q^{2n}(r^n - 1/r^n)}{1 - q^{2n}} \cos n\psi$$

$$= \frac{\eta_1}{\pi} \lg r + \frac{1}{2} + \sum_{n=1}^{\infty} \frac{r^n - q^{2n}/r^n}{1 - q^{2n}} \cos n\psi.$$

By differentiating with respect to ψ, we get

$$\frac{\partial}{\partial\psi} \Im\zeta(i \lg r + \psi) = - \frac{r(1 - r^2) \sin\psi}{(1 - 2r\cos\psi + r^2)^2} - \sum_{n=1}^{\infty} n \frac{q^{2n}(r^n - 1/r^n)}{1 - q^{2n}} \sin n\psi$$

$$= - \sum_{n=1}^{\infty} n \frac{r^n - q^{2n}/r^n}{1 - q^{2n}} \sin n\psi.$$

Now $(\partial/\partial\psi)\Im\zeta(i \lg r + \psi) = - \Im\wp(i \lg r + \psi)$ is regular and harmonic in the annulus $q < |z| < 1$ with respect to $z = re^{-i\psi}$ (or also to $re^{i\psi}$). The first expression shows that it vanishes everywhere along $|z| = 1$ except at a single point $z = 1$ and it remains negative in a neighborhood of $z = 1$ contained in the lower half of the annulus. The second expression shows that it vanishes everywhere along $|z| = q$. Further, it vanishes along two segments lying on the real axis and contained in the annulus. Consequently, the quantity $(\partial/\partial\psi)\Im\zeta(i \lg r + \psi)$ is negative throughout the lower half of the annulus in the $re^{-i\psi}$- plane so that, for any fixed r with $q < r < 1$, $\Im\zeta(i \lg r + \psi)$ is strictly decreasing for $0 \leqq \psi \leqq \pi$. The aimed result on $\Im\zeta_3(i \lg r + \psi)$ follows from the relation

$$\Im\zeta_3(i \lg r + \psi) = \Im(\zeta(i \lg r - i \lg q + \psi) - \eta_3) = - \Im\zeta\Big(i \lg \frac{q}{r} - \psi\Big) - \Im\eta_3.$$

LEMMA 4.3. *For any r with $q < r < 1$, the quantities $|\sigma(i \lg r + \psi)|$ and $|\sigma_3(i \lg r + \psi)|$, qua functions of ψ, are even functions which, in the interval $0 \leqq \psi \leqq \pi$, attain their minima both at $\psi = 0$ and their maxima both at $\psi = \pi$.*

Proof. The quantities under consideration are expressed by

$$|\sigma(i\lg r+\psi)| = e^{(\eta_1/2\pi)(\psi^2-\lg^2 r)}r^{-1/2}|1-re^{-i\psi}| \cdot \prod_{n=1}^{\infty}\frac{|1-q^{2n}re^{-i\psi}||1-q^{2n}r^{-1}e^{i\psi}|}{(1-q^{2n})^2}$$

and

$$|\sigma_3(i\lg r+\psi) = e^{(\eta_1/2\pi)(\psi^2-\lg^2 r)}\prod_{n=1}^{\infty}\frac{|1-q^{2n-1}re^{-i\psi}||1-q^{2n-1}r^{-1}e^{i\psi}|}{(1-q^{2n})^2},$$

whence readily follows the result.

LEMMA 4.4. *The quantity defined by*

$$T(\gamma,\theta_0)=\int_{-\pi}^{\pi}e^{-(\eta_1/\pi)(2\theta_0-\gamma-2\pi)\varphi}\frac{\sigma(\varphi)\sigma(\varphi-\gamma)}{\sigma_3(\varphi-\theta_0+\pi)^2}d\varphi$$

satisfies the following relations:

$$T(\gamma,\theta_0+2\pi)=e^{-4\eta_1\theta_0}T(\gamma,\theta_0),$$
$$T(\gamma+2\pi,\theta_0)=-e^{2\eta_1(\gamma+\pi)}T(\gamma,\theta_0),$$
$$T(-\gamma,2\pi-\theta_0)=T(\gamma,\theta_0)$$

and, for any real θ_0,

$$T(0,\theta_0)>0.$$

Proof. In view of well-known formulas

$$\sigma_3(\varphi-\theta_0-\pi)=e^{-2\eta_1(\varphi-\theta_0)}\sigma_3(\varphi-\theta_0+\pi)$$

and

$$\sigma(\varphi-\gamma-2\pi)=-e^{-2\eta_1(\varphi-\gamma-\pi)}\sigma(\varphi-\gamma),$$

we readily get the first two equations. The third equation follows from the definition by applying the change of integration variable $\varphi|-\varphi$ and remembering that $\sigma(\psi)$ is an odd function while $\sigma_3(\psi)$ is an even function. The last inequality follows directly; namely we have

$$T(0,\theta_0)=\int_{-\pi}^{\pi}e^{-(\eta_1/2\pi)(2\theta_0-2\pi)\varphi}\frac{\sigma(\varphi)^2}{\sigma_3(\varphi-\theta_0+\pi)^2}d\varphi>0.$$

This lemma can be proved also by expressing $T(\gamma,\theta_0)$ in terms of Jacobian theta-functions. In fact, we have

$$T(\gamma,\theta_0)=\frac{4\pi^2\vartheta_0^2}{\vartheta_1'^2}e^{(\eta_1/2\pi)(\gamma^2-2(\pi-\theta_0)^2)}\int_{-\pi}^{\pi}\frac{\vartheta_1\left(\frac{\varphi}{2\pi}\right)\vartheta_1\left(\frac{\varphi-\gamma}{2\pi}\right)}{\vartheta_0\left(\frac{\varphi-\theta_0+\pi}{2\pi}\right)^2}d\varphi,$$

the parameter associated to theta-functions being q, or expressed in the product form

$$T(\gamma, \theta_0) = \frac{4\pi^2\vartheta_0^2}{\vartheta_1'^2} e^{(\eta_1/2\pi)(\gamma^2 - 2(\pi-\theta_0)^2)}$$

$$\cdot 4q^2\int_{-\pi}^{\pi} \sin\frac{\varphi}{2}\sin\frac{\varphi-\gamma}{2}\prod_{n=1}^{\infty}\frac{(1-2q^{2n}\cos\varphi+q^{4n})(1-2q^{2n}\cos(\varphi-\gamma)+q^{4n})}{(1+2q^{2n-1}\cos(\varphi-\theta_0)+q^{4n-2})^2}\,d\varphi.$$

The functional equations as well as an inequality in the lemma are directly evident from the last expression.

LEMMA 4.5. *Let* $T(\gamma, \theta_0)$ *be defined as in lemma* 4.4. *The equation*

$$T(\theta_0 - \pi, \theta_0) = 0$$

has in the interval $0 \leqq \theta_0 < 2\pi$ *a single root* $\theta_0 = 0$.

Proof. The validity of $T(-\pi, 0) = 0$ follows readily from the equations in lemma 4.4. In fact, we get

$$T(-\pi, 0) = T(-\pi, 2\pi) = T(\pi, 0) = -T(-\pi, 0).$$

Now the equation $T(\theta_0 - \pi, \theta_0) = 0$ is equivalent to the equation

$$\int_{-\pi}^{\pi} K(\varphi, \theta_0)\,d\varphi = 0$$

where the integrand defined by

$$K(\varphi, \theta_0) = \sin\frac{\varphi}{2}\cos\frac{\varphi-\theta_0}{2}\prod_{n=1}^{\infty}\frac{(1-2q^{2n}\cos\varphi+q^{4n})(1+2q^{2n}\cos(\varphi-\theta_0)+q^{4n})}{(1+2q^{2n-1}\cos(\varphi-\theta_0)+q^{4n-2})^2}$$

is periodic in φ with period 2π. Suppose first $0 < \theta_0 \leqq \pi$. Then, in the equation

$$0 = \int_{-\pi}^{\pi} K(\varphi, \theta_0)d\varphi = \int_{-\pi}^{\pi} K(\varphi + \theta_0, \theta_0)\,d\varphi$$

$$= \int_{-\theta_0}^{\theta_0} K(\varphi+\theta_0, \theta_0)\,d\varphi + \int_{\theta_0}^{\pi}(K(\varphi+\theta_0, \theta_0) + K(-\varphi+\theta_0, \theta_0))\,d\varphi,$$

the integrand of the first integral of the last member is evidently positive throughout $-\theta_0 < \varphi < \theta_0$ while that of the second integral is equal to

$$\cos\frac{\varphi}{2}\prod_{n=1}^{\infty}\frac{1+2q^{2n}\cos\varphi+q^{4n}}{(1+2q^{2n-2}\cos\varphi+q^{4n-2})^2}\Big(\sin\frac{\varphi+\theta_0}{2}\prod_{n=1}^{\infty}(1-2q^{2n}\cos(\varphi+\theta_0)+q^{4n})$$

$$-\sin\frac{\varphi-\theta_0}{2}\prod_{n=1}^{\infty}(1-2q^{2n}\cos(\varphi-\theta_0)+q^{4n})\Big)$$

which is also positive throughout its interval of integration by virtue of the inequalities $\sin((\varphi+\theta_0)/2) > \sin((\varphi-\theta_0)/2)$ and $\cos(\varphi+\theta_0) > \cos(\varphi-\theta_0)$ valid there. Hence θ_0 with $0 < \theta_0 \leqq \pi$ cannot be a root of $T(\theta_0 - \pi, \theta_0) = 0$. Suppose next $\pi < \theta_0 < 2\pi$. By means of the relation in lemma 4.4, we get

$$T(\theta_0 - \pi, \theta_0) = T(2\pi - \theta_0 - \pi, 2\pi - \theta_0).$$

The right member of the last equation does not vanish for $0 < 2\pi - \theta_0 < \pi$, as shown just above. Hence again θ_0 with $\pi < \theta_0 < 2\pi$ cannot be a root of

$T(\theta_0 - \pi, \theta_0) = 0$. Thus it is concluded that a unique root of the equation $T(\theta_0 - \pi, \theta_0) = 0$ in $0 \leqq \theta_0 < 2\pi$ is $\theta_0 = 0$.

LEMMA 4.6. $\eta_1 + \pi e_3 < 0.$

Proof. From the series expansion of $\wp$-function

$$\wp(u) = -\frac{\eta_1}{\pi} + \frac{1}{4}\operatorname{cosec}^2\frac{u}{2} - 2\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}\cos nu,$$

we get, by putting $u = \omega_3 = -i \lg q$,

$$e_3 + \frac{\eta_1}{\pi} = -\frac{q}{(1-q)^2} - \sum_{n=1}^{\infty}\frac{nq^n(1+q^{2n})}{1-q^{2n}} = -2\sum_{n=1}^{\infty}\frac{nq^n}{1-q^{2n}} < 0.$$

Or alternatively, from a general formula

$$\eta_1 + \omega_1\wp(\omega_3) = \frac{\pi^2}{2\omega_1}\sum_{n=1}^{\infty}\operatorname{cosec}^2\frac{(2n-1)\pi\omega_3}{2\omega_1},$$

we get, by merely specifying as $\omega_1 = \pi$ and $\omega_3 = -i \lg q$,

$$\eta_1 + \pi e_3 = -2\pi\sum_{n=1}^{\infty}\frac{q^{2n-1}}{(1-q^{2n-1})^2} < 0.$$

The transference between two series forms for $\eta_1 + \pi e_3$ is also immediate.

5. Mapping of an annulus onto a domain bounded by convex curves.

We first derive an integral representation for a function mapping the aunulus

R_q : $(0 <) q < |z| < 1$

onto a ring domain bounded by two convex curves. It corresponds to the representation derived in theorem 2.1 in case of the unit circle. Here also we shall use a method of approximation since the representation holds for a function with simpler boundary nature.

THEOREM 5.1. *Let $f(z)$ be a single-valued analytic function mapping the annulus R_q univalently onto a ring domain in such a manner that the image of $|z| = 1$ bounds a convex domain containing in its interior the image of $|z| = q$ which itself bounds a convex domain or may possibly degenerate to a rectilinear segment. Then there exist increasing functions $\rho(\varphi)$ and $\tau(\varphi)$ defined for $-\pi \leqq \varphi < \pi$ and both with total variation equal to unity such that an integral representation of Villat-Stieltjes type*

$$1 + z\frac{f''(z)}{f'(z)} = \frac{2}{i}\int_{-\pi}^{\pi}(\zeta(i\lg z + \varphi)\,d\rho(\varphi) - \zeta_3(i\lg z + \varphi)\,d\tau(\varphi)) + ic^*$$

holds for $z \in R_q$ where c^* is a real constant defined by

$$c^* = \frac{2\eta_1}{\pi}\int_{-\pi}^{\pi}\varphi(d\rho(\varphi) - d\tau(\varphi)).$$

Proof. The proof of the present theorem can be performed through its whole process quite similarly as that of theorem 2.1. Namely, we first approximate the image-domain $f(R_q)$ by a sequence of ring domains $\{D_\nu\}$ of the same convexity nature as $f(R_q)$ in the sense of domain-kernel. Let the modulus of D_ν be denoted by $-\lg q_\nu$ and the mapping function $f_\nu(z)$ of R_{q_ν}: $q_\nu < |z| < 1$ onto D_ν be normalized, for instance, by $f_\nu(\sqrt{q}) = f(\sqrt{q})$. We may suppose that every annulus R_{q_ν} contains the point $\sqrt{q}$ and that every $f_\nu(z)$ admits of the representation of the form to be aimed. In fact, it suffices for this purpose to consider, for instance, a sequence of domains bounded by two convex rectilinear polygons (convexity concerning the curves themselves) and converging to $f(R_q)$ as the domain-kernel. We then have, for any $z \in R_q$,

$$1 + z\frac{f_\nu''(z)}{f_\nu'(z)} = \frac{2}{i}\int_{-\pi}^{\pi}(\zeta(i\lg z + \varphi; q_\nu)\,d\rho_\nu(\varphi)$$
$$- \zeta_3(i\lg z + \varphi; q_\nu)\,d\tau_\nu(\varphi)) + ic_\nu^*$$

with

$$\rho_\nu(\varphi) = \frac{1}{2\pi}\int_{-\pi}^{\varphi} d\arg df_\nu(e^{i\varphi}) \qquad \text{and} \qquad \tau_\nu(\varphi) = \frac{1}{2\pi}\int_{-\pi}^{\varphi} d\arg df_\nu(qe^{i\varphi}),$$

where c_ν^* is a real constant defined by

$$c_\nu^* = \frac{2\eta_1(q_\nu)}{\pi}\int_{-\pi}^{\pi}\varphi(d\rho_\nu(\varphi) - d\tau_\nu(\varphi))$$

and the parameter q_ν associated to the zeta-functions as well as to η_1 means that the primitive quasi-periods are 2π and $-2i\lg q_\nu$; cf. [7], [8] or [9]. Since we have

$$d\rho_\nu(\varphi) \geqq 0, \quad d\tau_\nu(\varphi) \geqq 0 \quad \text{and} \quad \int_{-\pi}^{\pi} d\rho_\nu(\varphi) = \int_{-\pi}^{\pi} d\tau_\nu(\varphi) = 1,$$

we can conclude similarly as in the proof of theorem 2.1 that the limit process over a suitable subsequence $\{\kappa_\nu\}$ of $\{\nu\}$ leads to the desired result.

COROLLARY 1. *For $f(z)$ satisfying the condition in theorem* 5.1, *there holds*

$$f(z) = A\int^{z} z^{ic^*-1}\exp\Big(2\int_{-\pi}^{\pi}(-\lg\sigma(i\lg z + \varphi)\,d\rho(\varphi)$$
$$+ \lg\sigma_3(i\lg z + \varphi)\,d\tau(\varphi))\Big)dz + B,$$

$A(\neq 0)$ and B being certain constants and c^* a real constant defined in theorem 5.1; *$\rho(\varphi)$ and $\tau(\varphi)$ are the same functions as in theorem* 5.1.

COROLLARY 2. *If $f(z)$ satisfies the condition in theorem* 5.1, *the image of every concentric circumference $|z| = r$ $(q < r < 1)$ by $f(z)$ is a convex curve.*

Proof. From the representation of theorem 5.1 there follows

$$\frac{d}{d\theta}\arg df(re^{i\theta}) = \Re\Big(1 + re^{i\theta}\frac{f''(re^{i\theta})}{f'(re^{i\theta})}\Big)$$

$$= 2\int_{-\pi}^{\pi} \Im(\zeta(i \lg (re^{i\theta}) + \varphi)\, d\rho(\varphi) - \zeta_3(i \lg (re^{i\theta}) + \varphi)\, d\tau(\varphi))$$

$$= 2\int_{-\pi}^{\pi} (P(re^{i\theta}; \varphi)\, d\rho(\varphi) - Q(re^{i\theta}; \varphi)\, d\tau(\varphi))$$

where P and Q are the functions defined in lemma 4.1. Since by this lemma $P(re^{i\theta}; \varphi) \geqq 0$ and $Q(re^{i\theta}; \varphi) \leqq 0$, we have $(d/d\theta) \arg df(re^{i\theta}) \geqq 0$ which proves the convexity of the image-curve of $|z| = r$. Further the equality sign in the last inequality may be, of course, excluded.

In theorem 5.1 as well as its corollary 1, it has been supposed that $f(z)$ is single-valued in R_q. The condition that $f(z)$ considered there remains invariant under the substitution $\lg z \,|\, \lg z + 2\pi i$ implies that for any r with $q < r < 1$ the relation

$$(\mathbf{M}) \qquad \int_{|z|=r} z^{ic^*-1} \exp\Big(2\int_{-\pi}^{\pi} (-\lg \sigma(i \lg z + \varphi)\, d\rho(\varphi) + \lg \sigma_3(i \lg z + \varphi)\, d\tau(\varphi))\Big) dz = 0$$

must be valid, in which the value of the left member is really independent of the choice of r. Taking this monodromy condition into account, it will be shown that the converse of theorem 5.1 (or rather its corollary 1) is also valid.

THEOREM 5.2. *Let $\rho(\varphi)$ and $\tau(\varphi)$ defined for $-\pi \leqq \varphi < \pi$ be increasing functions with total variation equal to unity. Then the function defined by*

$$f(z) = A\int^{z} z^{ic^*-1} \exp\Big(2\int_{-\pi}^{\pi} (-\lg \sigma(i \lg z + \varphi)\, d\rho(\varphi) + \lg \sigma_3(i \lg z + \varphi)\, d\tau(\varphi))\Big) dz + B$$

with a real constant

$$c^* = \frac{2\eta_1}{\pi} \int_{-\pi}^{\pi} \varphi(d\rho(\varphi) - d\tau(\varphi))$$

and with any complex constants $A (\neq 0)$ and B maps the annulus R_q univalently onto a domain bounded by two convex curves, of which the inside one originating from $|z| = q$ may possibly degenerate to a rectilinear segment, provided the monodromy condition (**M**) *holds for an r (and then necessarily for any r) with $q < r < 1$.*

Proof. It is evident that $f(z)$ is regular in R_q and its single-valuedness is assured by the condition (**M**). Further, for any r with $q < r < 1$, we have

$$\frac{d}{d\theta} \arg df(re^{i\theta}) = \Re\Big(1 + re^{i\theta} \frac{f''(re^{i\theta})}{f'(re^{i\theta})}\Big)$$

$$= 2\int_{-\pi}^{\pi} (P(re^{i\theta}; \varphi)\, d\rho(\varphi) - Q(re^{i\theta}; \varphi)\, d\tau(\varphi)) > 0;$$

cf. the proof of corollary 2 of theorem 5.1. This shows that every image-curve of $|z| = r$ and hence both boundary components of the image are convex. Finally, again for any r with $q < r < 1$, we have

$$\begin{aligned}
&\int_{-\pi}^{\pi} d \arg df(re^{i\theta}) \\
&= 2\int_{-\pi}^{\pi} d\theta\, \Im \int_{-\pi}^{\pi} (\zeta(i\lg r - \theta + \varphi)\, d\rho(\varphi) - \zeta_3(i \lg r - \theta + \varphi)\, d\tau(\varphi)) \\
&= 2\Im \int_{-\pi}^{\pi} \left(\lg \frac{-\sigma(i\lg r - \pi + \varphi)}{-\sigma(i\lg r + \pi + \varphi)}\, d\rho(\varphi) - \lg \frac{-\sigma_3(i\lg r - \pi + \varphi)}{-\sigma_3(i\lg r + \pi + \varphi)}\, d\tau(\varphi)\right) \\
&= 2\Im \int_{-\pi}^{\pi} ((\pi i - 2\eta_1(i\lg r + \varphi))\, d\rho(\varphi) + 2\eta_1(i\lg r + \varphi)\, d\tau(\varphi)) = 2\pi,
\end{aligned}$$

whenece follows the univalency of $f(z)$ in R_q.

Based on corollary 1 of theorem 5.1 and theorem 5.2, it is concluded that the representation given in the corollary accompanied by the monodromy condition (**M**) is characteristic to a function possessing the mapping property stated in theorem 5.1.

By making use of this representation, we shall first derive a distortion inequality for the quantity $1 + \Re(zf''(z)/f'(z))$. Though the result which will be obtained is not sharp, it has an analogue in case of star-like mapping for which the corresponding result is sharp.

THEOREM 5.3. *Let $f(z)$ possess the mapping property stated in theorem 5.1. Then, for any $z \in R_q$,*

$$\begin{aligned}
&2\Im(\zeta(i\lg|z| + \pi) - \zeta_3(i\lg|z| + \pi)) \\
&\qquad < 1 + \Re z\frac{f''(z)}{f'(z)} < 2\Im(\zeta(i\lg|z|) - \zeta_3(i\lg|z|)),
\end{aligned}$$

the equality sign in every inequality being excluded.

Proof. The inequalities with $\leqq$ instead of $<$ follow from the integral representation established in theorem 5.1, by taking into account the monotoneity of the zeta-functions involved which has been shown ln lemma 4.2. That the equality sign is excluded will be shown as follows. Now, if the equality sign would appear in the left inequality at a point $z_0 = r_0 e^{i\theta_0} \in R_q$ with $0 \leqq \theta_0 < 2\pi$, then $d\rho(\varphi)$ and $d\tau(\varphi)$ were both zero except at a single point $\theta_0 - \pi$ where $\rho(\varphi)$ and $\tau(\varphi)$ both show a jump with the height equal to unity. Hence we would have the relation

$$1 + z\frac{f''(z)}{f'(z)} = \frac{2}{i}(\zeta(i\lg z + \theta_0 - \pi) - \zeta_3(i\lg z + \theta_0 - \pi)) + ic^*$$

with

$$c^* = \frac{2\eta_1}{\pi}\int_{-\pi}^{\pi} \varphi(d\rho(\varphi) - d\tau(\varphi)) = 0,$$

which leads to

$$\begin{aligned}
f(z) &= A\int^z \frac{\sigma_3(i\lg z + \theta_0 - \pi)^2}{\sigma(i\lg z + \theta_0 - \pi)^2}\frac{dz}{z} + B \\
&= A\int^z (\wp(i\lg z + \theta_0 - \pi) - e_3)\, d\lg z + B
\end{aligned}$$

$$= Ai(\zeta(i \lg z + \theta_0 - \pi) + e_3 i \lg z) + B',$$

$A(\neq 0)$, B and B' being constants. The substitution $\lg z \mid \lg z + 2\pi i$ causes an additive constant to $f(z)$ which is equal to $-2Ai(\eta_1 + \pi e_3)$. But this periodicity modulus of $f(z)$ is not equal to zero, as shown in lemma 4.6. Hence $f(z)$ cannot be single-valued in R_q, contrary to the assumption. Similarly, if the equality sign would appear in the right inequality at a point $z_0 = r_0 e^{i\theta_0} \in R_q$ with $0 \leqq \theta_0 < 2\pi$, then we would have

$$f(z) = Ai(\zeta(i \lg z + \theta_0) + e_3 i \lg z) + B'$$

with constants $A(\neq 0)$ and B'. However, this function is not single-valued, again contrary to the assumption.

In case of the unit circle as the basic simply-connected domain, a function of the kind now considered is usually normalized by $f(0) = 0$. However, in case of an annulus, the image of each boundary component is a continuum so that several types of normalization will be possible. It would be desirable at any rate to consider the whole class of functions characterized by the condition stated in theorem 5.3. But, as already announced, the result obtained there is not sharp. In what follows, we shall deal with a subclass which is restricted by an additional condition that the image of the inside circumference is a rectilinear segment. In general, the functional $1 + zf''(z)/f'(z)$ remains invariant under the substitution $f(z) \mid Af(z) + B$ with any constants $A(\neq 0)$ and B. Further, its form remains invariant also under the substitution $z \mid e^{i\lambda} z$ with any real constant λ, while it changes its sign alone under the substitution $z \mid q/z$. In particular, it is a matter of indifference for this functional whether the position of the image segment under consideration as well as the antecedent of its one end point are preassigned, so far as the extremal problem is concerned.

We now give a distortion theorem of a general nature and show that the bounds are sharp. Moreover, by imposing, without loss of generality, a merely formal additional condition of normalization, we determine the whole family of extremal functions.

THEOREM 5.4. *Let $f(z)$ map the annulus R_q univalently onto a convex domain cut along a rectilinear segment originating from $|z| = q$. Suppose that the end points of this segment lie at $f(q)$ and $f(qe^{i\gamma})$, i.e. the zero points of $f'(z)$ on $|z| = q$ be q and $qe^{i\gamma}$ where $\gamma = \gamma[f]$ is a real parameter depending on respective $f(z)$. Then, for any $z \in R_q$,*

$$2\Im\zeta(i \lg |z| + \pi) \leqq 1 + \Re z \frac{f''(z)}{f'(z)}$$
$$+ \Im(\zeta_3(i \lg z) + \zeta_3(i \lg z + \gamma)) \leqq 2\Im\zeta(i \lg |z|).$$

Let $f(z)$ satisfy besides the condition imposed above a further condition of normalization that the image of $|z| = q$ is a horizontal slit lying on the real axis and $f(q)$ is its right end point. Let $z_0 = r_0 e^{i\theta_0}$ with $0 \leqq \theta_0 < 2\pi$ be any preassigned point in the annulus R_q.

(i) *The equality sign in the left estimation then holds at z_0 if and only if*

$f(z)$ is of the form $af_*(z;\theta_0)+b$ with any real constants $a>0$ and b where

$$f_*(z;\theta_0)=ie^{i(\theta_0-\gamma_*(\theta_0)/2)}\cdot\int_q^z z^{i(\eta_1/\pi)(2\theta_0-\gamma_*(\theta_0)-2\pi)-1}\frac{\sigma_3(i\lg z)\sigma_3(i\lg z+\gamma_*(\theta_0))}{\sigma(i\lg z+\theta_0-\pi)^2}dz$$

and $\gamma_*(\theta_0)$ denotes the value of γ associated to $f_*(z;\theta_0)$ which is determined as a root of the transcendental equation

$$T(\gamma,\theta_0)\equiv\int_{-\pi}^{\pi}e^{-(\eta_1/\pi)(2\theta_0-\gamma-2\pi)\varphi}\frac{\sigma(\varphi)\sigma(\varphi-\gamma)}{\sigma_3(\varphi-\theta_0+\pi)^2}d\varphi=0,\quad -\pi\leqq\gamma<\pi.$$

(ii) *The equality sign in the right estimation holds at z_0 if and only if $f(z)$ is of the form $af^*(z;\theta_0)+b$ with any real constants $a>0$ and b where $f^*(z;\theta_0)=f_*(z;\theta_0+\pi)$,[2] i.e.*

$$f^*(z;\theta_0)=-ie^{i(\theta_0-\gamma^*(\theta_0)/2)}\cdot\int_q^z z^{i(\eta_1/\pi)(2\theta_0-\gamma^*(\theta_0))-1}\frac{\sigma_3(i\lg z)\sigma_3(i\lg z+\gamma^*(\theta_0))}{\sigma(i\lg z+\theta_0)^2}dz$$

and $\gamma^*(\theta_0)=\gamma_*(\theta_0+\pi)$ denotes the value[3] of γ associated to $f^*(z;\theta_0)$ which is determined as a root of the equation

$$T(\gamma,\theta_0+\pi)\equiv\int_{-\pi}^{\pi}e^{-(\eta_1/\pi)(2\theta_0-\gamma)\varphi}\frac{\sigma(\varphi)\sigma(\varphi-\gamma)}{\sigma_3(\varphi-\theta_0)^2}d\varphi=0,\quad -\pi\leqq\gamma<\pi.$$

Proof. Based on theorem 5.1, we have

$$1+z\frac{f''(z)}{f'(z)}=\frac{2}{i}\int_{-\pi}^{\pi}\zeta(i\lg z+\varphi)d\rho(\varphi)-\frac{1}{i}(\zeta_3(i\lg z)+\zeta_3(i\lg z+\gamma))+ic^*,$$

where c^* is a real constant defined by

$$c^*=\frac{2\eta_1}{\pi}\left(\int_{-\pi}^{\pi}\varphi\, d\rho(\varphi)-\gamma\right)$$

and $\rho(\varphi)$ is an increasing function with total variation equal to unity. In fact, the differential $d\tau(\varphi)$ involved in the representation given in theorem 5.1 remains zero except at $\varphi=0$ and $\varphi=\gamma$ where $\tau(\varphi)$ jumps by 1/2. We thus get

$$1+\Re z\frac{f''(z)}{f'(z)}+\Im(\zeta_3(i\lg z)+\zeta_3(i\lg z+\gamma))=2\int_{-\pi}^{\pi}\Im\zeta(i\lg z+\varphi)d\rho(\varphi),$$

whence readily follows the distortion inequality by remembering lemma 4.2.

(i) If the equality sign in the left estimation holds at $z_0=r_0e^{i\theta_0}$, then the process given above shows that $d\rho(\varphi)$ must vanish except at a single value $\varphi=\theta_0-\pi$ whence follows

$$1+z\frac{f''(z)}{f'(z)}=\frac{2}{i}\zeta(i\lg z+\theta_0-\pi)-\frac{1}{i}(\zeta(i\lg z)+\zeta_3(i\lg z+\gamma))+ic^*$$

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2) Though θ_0 has been once restricted by $0\leqq\theta_0<2\pi$, the function $f_*(z;\theta_0)$ may be regarded as depending on θ_0 periodically with period 2π.

3) $\gamma^*(\theta_0)$ may be regarded as a periodic function of θ_0 with period 2π; cf. footnote 2).

with $\gamma = \gamma[f]$ where the real constant c^* is given by

$$c^* = \frac{2\eta_1}{\pi}\int_{-\pi}^{\pi} \varphi(d\rho(\varphi) - d\tau(\varphi)) = \frac{2\eta_1}{\pi}\Big(\theta_0 - \pi - \frac{\gamma}{2}\Big).$$

Integration with respect to $\lg z$ then implies

$$\lg(zf'(z)) = -2\lg\sigma(i\lg z + \theta_0 - \pi) + \lg\sigma_3(i\lg z) + \lg\sigma_3(i\lg z + \gamma) + ic^*\lg z + \lg c,$$

$\lg c$ being an integration constant.[4] Further integration of the exponential of the last expression with respect to z leads to

$$f(z) = c\int_q^z z^{ic^*-1}\frac{\sigma_3(i\lg z)\sigma_3(i\lg z + \gamma)}{\sigma(i\lg z + \theta_0 - \pi)^2}dz + b,$$

$b(= f(q))$ being an integration constant which is real by assumption. It is evident that the function of the last-mentioned form possesses the extremal property under consideration provided it is single-valued in R_q. It remains only to determine the value[5] of $\arg c$ and to obtain the relation connecting γ with θ_0, by observing the monodromy condition. Now, in view of its integral representation, $f(z)$ must be a function mapping R_q onto a polygonal ring domain which is bounded by a straight line corresponding to $|z| = 1$ and a rectilinear slit corresponding to $|z| = q$, the latter lying on the real axis. The inclination of the former boundary component is equal to

$$\begin{aligned}\arg df(e^{i\varphi}) &= \frac{\pi}{2} + \arg(e^{i\varphi}f'(e^{i\varphi}))\\ &= \frac{\pi}{2} + \arg c + \arg\frac{\sigma_3(-\varphi)\sigma_3(-\varphi+\gamma)}{\sigma(-\varphi+\theta_0-\pi)^2}\\ &= \frac{\pi}{2} + \arg c,\end{aligned}$$

since, for any real u, $\sigma(u)$ is real and $\sigma_3(u)$ is real positive. On the other hand, we have

$$\begin{aligned}\arg df(qe^{i\varphi}) &= \frac{\pi}{2} + \arg(qe^{i\varphi}f'(qe^{i\varphi}))\\ &= \frac{\pi}{2} + \arg c + \arg\frac{\sigma_3(i\lg q - \varphi)\sigma_3(i\lg q - \varphi + \gamma)}{\sigma(i\lg q - \varphi + \theta_0 - \pi)^2} + c^*\lg q\\ &= \frac{\pi}{2} + \arg c - i\eta_3(2\theta_0 - \gamma - 2\pi)\\ &\qquad + \arg\Big(\sin\frac{\varphi}{2}\sin\frac{\varphi-\gamma}{2}\Big) + c^*\lg q;\end{aligned}$$

here the use is made of the well-known formulas

$$\sigma(i\lg q - u) = e^{-\eta_3 u}\sigma_3(u)\sigma(i\lg q)$$

4) The ambiguity of an additive integral multiple of $2\pi i$ in $\lg c$ is a matter of indifference, since it causes a unique value of c.

5) Cf. footnote 4).

and

$$\sigma_3(i \lg q - u) = e^{\eta_3(u - i \lg q)} \sigma(u)/\sigma(i \lg q)$$

in which $\sigma(i \lg q)$ and η_3 are both purely imaginary and $\sigma_3(u)$ is real positive for any real u while $\sigma(u)$ is of the same sign as $\sin(u/2)$ for any real u. We further know the Legendre relation

$$\eta_1 i \lg \frac{1}{q} - \eta_3 \pi = \frac{\pi i}{2} \quad \text{i.e.} \quad i\eta_3 = \frac{1}{2} + \frac{\eta_1}{\pi} \lg q$$

so that, by substituting the value of c^* determined above, we get

$$\arg df(qe^{i\varphi}) = \frac{\pi}{2} + \arg c - \left(\theta_0 - \frac{\gamma}{2} - \pi\right) + \arg\left(\sin\frac{\varphi}{2} \sin\frac{\varphi - \gamma}{2}\right).$$

But, in view of the assumption, we have

$$\arg df(qe^{i\varphi}) = \arg\left(\sin\frac{\varphi}{2} \sin\frac{\varphi - \gamma}{2}\right) + \frac{\pi}{2}(1 - \operatorname{sgn}\gamma)$$

and hence

$$\arg c = \frac{\pi}{2} \operatorname{sgn}\gamma + \theta_0 - \frac{\gamma}{2}.$$

The last relation implies further the inclination of the boundary straight line corresponding to $|z| = 1$, i.e.

$$\arg df(e^{i\varphi}) = \theta_0 - \frac{\gamma}{2} + \frac{\pi}{2}(1 + \operatorname{sgn}\gamma).$$ [6]

Finally, in order to derive the connection between θ_0 and γ, we apply the monodromy condition to $f(z)$ which is equivalent to the invariance of $f(qe^{i\varphi})$ under the substitution $\varphi | \varphi + 2\pi$. The condition is then written in the form

$$\int_{-\pi}^{\pi} e^{-c^*\varphi} \frac{\sigma_3(i \lg q - \varphi)\sigma_3(i \lg q - \varphi + \gamma)}{\sigma(i \lg q - \varphi + \theta_0 - \pi)^2} d\varphi = 0$$

or

$$T(\gamma, \theta_0) = 0.$$ [7]

By the way, it is readily shown that, for any assigned θ_0, the last equation has surely a root $\gamma = \gamma(\theta_0)$. In fact, from the second relation in lemma 4.4 we get, in particular,

$$T(\pi, \theta_0) = -T(-\pi, \theta_0).$$

Since $T(\gamma, \theta_0)$ is evidently continuous with respect to γ in $-\pi \leqq \gamma \leqq \pi$, it

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6) An alternative and rather direct method for determining this value of inclination will be mentioned in a remark supplemented below.

7) This condition is nothing but the monodromy condition (**M**) applied to $f(z)$ now in consideration.

vanishes at least once in $-\pi \leqq \gamma < \pi$.[8] On the other hand, it is to be noted that γ can never be equal to zero for any θ_0, as seen from the last relation of lemma 4.4, a fact which is a matter of course by virtue of its own meaning. Thus the proof of the sharpness of the lower estimation has been completely performed out.

(ii) It is only necessary to note that any extremal function of the upper estimation is given by

$$af^*(z;\ \theta_0) + b = af_*(z;\ \theta_0 + \pi) + b$$

associated to $\gamma^*(\theta_0) = \gamma_*(\theta_0 + \pi)$.

It is to be noted that extremal functions in theorem 5.4 depend on $\theta_0 = \arg z_0$ but not on $|z_0|$. The extremal function $f_*(z;\ \theta_0)$ maps R_q univalently onto a half-plane bounded by a straight line with the inclination $\theta_0 - \gamma_*(\theta_0)/2 + (\pi/2)(1 + \operatorname{sgn} \gamma_*(\theta_0))$ which is cut along a slit lying on the real axis. The point at infinity as a boundary point of the image-domain corresponds to $-z_0/|z_0| = -e^{i\theta_0}$ and the end-points of the slit originate from q and $qe^{i\gamma_*(\theta_0)}$. The extremal function $f^*(z;\ \theta_0)$ maps R_q univalently onto a half-plane bounded by a straight line with the inclination $\theta_0 - \gamma^*(\theta_0)/2 - (\pi/2)(1 - \operatorname{sgn} \gamma^*(\theta_0))$ which is cut along a slit lying on the real axis. The point at infinity as a boundary point of the image-domain corresponds to $z_0/|z_0| = e^{i\theta_0}$ and the end points of the slit originate from q and $qe^{i\gamma^*(\theta_0)}$.

It will here be supplemented that the inclination of the boundary straight line originating from $|z| = 1$ by $f_*(z;\ \theta_0)$ can be obtained rather directly by means of an alternative method. For this purpose, we note that $f_*(z;\ \theta_0)$ is regular throughout the closed annulus $q \leqq |z| \leqq 1$ except at a single point $-e^{i\theta_0}$ where it has a pole of the first order. Hence its boundary behavior implies readily its univalency in the annulus provided it is single-valued, a fact which has been implicitly availed in the above proof. Consequently, for any r with $q < r < 1$, we must have

$$\int_{-\pi}^{\pi} d \arg df_*(re^{i\varphi};\ \theta_0) = 2\pi = \int_{-\pi}^{\pi} d\varphi,$$

so that the harmonic function

$$\arg f'_*(z;\ \theta_0) \equiv \arg df_*(z;\ \theta_0) - \arg z - \frac{\pi}{2}$$

and hence the analytic function $\lg f'_*(z;\ \theta_0)$ are also single-valued in R_q. Applying the monodromy criterion to the last function, we thus get

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \lg f'_*(e^{i\varphi};\ \theta_0)\, d\varphi = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lg f'_*(qe^{i\varphi};\ \theta_0)\, d\varphi,$$

both members of the last equation being interpreted as the limits for $r \to 1 - 0$ and $r \to q + 0$ of the integral along $|z| = r$. This equation leads, after

8) It seems very probable that the equation $T(\gamma, \theta_0) = 0$ has a unique root in $-\pi \leqq \gamma < \pi$ for any assigned θ_0.

separation of the imaginary part, to the relation

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\arg df_*(e^{i\varphi};\,\theta_0)\,d\varphi = \frac{1}{2\pi}\int_{-\pi}^{\pi}\arg df_*(qe^{i\varphi};\,\theta_0)\,d\varphi.$$

Consequently, denoting by v the inclination under consideration,[9] we get if $\gamma_* > 0$,

$$\frac{1}{2\pi}((v-2\pi)(\theta_0-\pi+\pi)+v(\pi-\theta_0+\pi))$$
$$=\frac{1}{2\pi}(0(0+\pi)+\pi(\gamma_*-0)+2\pi(\pi-\gamma_*)),$$

and if $\gamma_* < 0$,

$$\frac{1}{2\pi}((v-2\pi)(\theta_0-\pi+\pi)+v(\pi-\theta_0+\pi))$$
$$=\frac{1}{2\pi}(-\pi(\gamma_*+\pi)+0(0-\gamma_*)+\pi\pi),$$

i.e., written together,

$$v=\theta_0-\frac{\gamma_*}{2}+\frac{\pi}{2}(1+\operatorname{sgn}\gamma_*).$$

The distortion inequality of theorem 5.4 involves a quantity $\gamma=\gamma[f]$ depending on respective function $f(z)$. By eliminating this quantity, we obtain the following result.

THEOREM 5.5. *Under the same condition as in theorem* 5.4, *the inequality*

$$2\Im\zeta(i\lg|z|+\pi)-\Im\zeta_3(i\lg|z|+\pi)$$
$$\leqq 1+\Re z\frac{f''(z)}{f'(z)}+\Im\zeta_3(i\lg z)\leqq 2\Im\zeta(i\lg|z|)-\Im\zeta_3(i\lg|z|)$$

holds for any $z\in R_q$. *The estimation is sharp. More precisely, under a further normalization condition as in theorem* 5.4, *the left equality can appear only at* z_0 *with* $\arg z_0=0$ *and the extremal function is then given by* $af_*(z;0)+b$ *where*

$$f_*(z;0)=\int_q^z z^{-i\eta_1-1}\frac{\sigma_3(i\lg z)\,\sigma_3(i\lg z-\pi)}{\sigma(i\lg z-\pi)^2}\,dz,$$

while the right equality can appear only at z_0 *with* $\arg z_0=\pi$ *and the extremal function is then given by* $af^*(z;\pi)+b$ *where*

$$f^*(z;\pi)=\int_q^z z^{i\eta_1-1}\frac{\sigma_3(i\lg z)\,\sigma_3(i\lg z+\pi)}{\sigma(i\lg z+\pi)^2}\,dz;$$

$a>0$ *and* b *are in every case any real constants.*

Proof. In order to show the distortion inequality, it is only necessary to note that the quantity $\Im\zeta_3(i\lg|z|+\psi)$ is, qua function of ψ, periodic with period 2π, even in $|\psi|\leqq\pi$ and strictly increasing for $0\leqq\psi\leqq\pi$, a fact which has been proved in lemma 4.2. In fact, we then get

9) It suffices to consider v with respect to mod 2π.

$$\Im\zeta_3(i\lg|z|) \leqq \Im\zeta_3(i\lg|z|+\gamma) \leqq \Im\zeta_3(i\lg|z|+\pi)$$

whence immediately follows the result by virtue of theorem 5.4. Now the left equality can appear at $z_0 = r_0e^{i\theta_0}$ if and only if

$$\Im\zeta_3(i\lg z_0 + \gamma_*(\theta_0)) = \Im\zeta_3(i\lg|z_0|+\pi),$$

whence follows

$$\gamma_*(\theta_0) - \theta_0 = -\pi$$

since $-\pi \leqq \gamma_*(\theta_0) < \pi$ and $0 \leqq \theta_0 < 2\pi$. But, as shown in lemma 4.5, the equation $T(\theta_0 - \pi, \theta_0) = 0$ has in $0 \leqq \theta_0 < 2\pi$ a unique root $\theta_0 = 0$. Consequently, for the extremal function for the estimation from below we must have

$$\theta_0 = 0 \quad \text{and} \quad \gamma_*(\theta_0) = -\pi,$$

On the other hand, the right equality can appear at $z_0 = r_0e^{i\theta_0}$ if and only if

$$\Im\zeta_3(i\lg z_0 + \gamma^*(\theta_0)) = \Im\zeta_3(i\lg|z_0|),$$

whence follows

$$\gamma^*(\theta_0) - \theta_0 = -\pi(1 - \operatorname{sgn}\gamma^*(\theta_0))$$

since $-\pi \leqq \gamma^*(\theta_0) < \pi$ and $0 \leqq \theta_0 < 2\pi$. But, by virtue of lemma 4.4, either of the equations $T(\theta_0, \theta_0+\pi) = 0$ and $T(\theta_0 - 2\pi, \theta_0 + \pi) = 0$ is equivalent to $T(\theta_0 - 2\pi, \theta_0 - \pi) = 0$. Hence θ_0 must satisfy the last equation, whence, by means of lemma 4.5, we have

$$\theta_0 = \pi \quad \text{and} \quad \gamma^*(\theta_0) = \pi$$

Thus the proof is completed.

The image of $|z| = q$ for every extremal function $f_*(z; 0)$ or $f^*(z; \pi)$ is a vertical straight line which lies right- or left-side of the image-point of $z = q$ for $f_*(z; 0)$ or $f^*(z; \pi)$, respectively.

By taking lemma 4.2 into account, we can derive as a corollary of theorem 5.5 estimations from below and from above for $1 + \Re(zf''(z)/f'(z))$ of which the bounds depend only on $|z|$. However, the result so obtained has been already derived in theorem 5.3 for a wider class of functions. It has been proved, moreover, that the bounds so obtained are in any case never sharp within single-valued functions.

Finally another remark will be supplemented. Considering exclusively the functional $1 + \Re(zf''(z)/f'(z))$, we have derived some analogues of the first distortion inequality in theorem 2.3. Similar discussion may be performed on the remaining distortion inequalities in theorem 2.3 as well as the rotation inequality in theorem 2.4. Here we shall illustrate the circumstance by turning our attention to an analogue of the third inequality of theorem 2.3. For the functional $|f'(z)|$ the normalization has been made in case of the unit circle by assigning the value of $|f'(0)|$. Correspondingly, in case of the annulus, a suitable normalization condition must be imposed in order to determine the multiplicative constant factor A in the representation for $f(z)$

stated in corollary 1 of theorem 5.1. For instance, it will be plausible to impose such a condition that either $\frac{1}{2\pi}\int_{-\pi}^{\pi} f'(qe^{i\theta})\,d\theta$ or $f'(\sqrt{q})$ has a preassigned value. However, we suppose here for the sake of convenience another conditition that this constant A (or merely its absolute value) is itself preassigned. The result which will be derived is not best possible but it can be still regarded as a generalization of the corresponding result in the simply-connected case, as shown below.

THEOREM 5.6. *Let $f(z)$ satisfy the condition in theorem 5.1 and be expressed by the representation mentioned in its corollary 1. Then, for any $q < |z| < 1$,*

$$\frac{1}{|z|}\left|\frac{\sigma_3(i\lg|z|)}{\sigma(i\lg|z|+\pi)}\right|^2 < \left|\frac{f'(|z|)}{A}\right| < \frac{1}{|z|}\left|\frac{\sigma_3(i\lg|z|+\pi)}{\sigma(i\lg|z|)}\right|^2,$$

the equality sign in every inequality being excluded.

Proof. We put $|z| = r$. From the representation for $f(z)$, we get

$$\left|\frac{f'(r)}{A}\right| = \left|r^{ic^*-1}\exp\Big(2\int_{-\pi}^{\pi}(-\lg\sigma(i\lg r+\varphi)\,d\rho(\varphi)+\lg\sigma_3(i\lg r+\varphi)\,d\tau(\varphi))\Big)\right|$$

$$= \frac{1}{r}\exp\Big(2\int_{-\pi}^{\pi}(-\lg|\sigma(i\lg r+\varphi)|\,d\rho(\varphi)+\lg|\sigma_3(i\lg r+\varphi)|\,d\tau(\varphi))\Big)$$

since c^* is real. The inequalities with $\leqq$ instead of $<$ follow readily by virtue of lemma 4.3. If the equality sign would appear in the left inequality, then $d\rho(\varphi)$ and $d\tau(\varphi)$ were both zero except at $-\pi$ and 0, respectively, where $\rho(\varphi)$ and $\tau(\varphi)$ both show a jump with the height equal to unity. Hence we would have

$$f'(z) = Az^{-2\eta_1 i-1}\frac{\sigma_3(i\lg z)^2}{\sigma(i\lg z-\pi)^2}.$$

For this function, we have

$$d\arg df(e^{i\varphi}) = d\rho(\varphi) \qquad \text{and} \qquad d\arg df(qe^{i\varphi}) = d\tau(\varphi)$$

except at $-\pi$ and 0, respectively. But the behaviors of these functions explained above imply that the function $f(z)$ cannot be univalent and, moreover, it cannot be single-valued. Similarly, the equality sign cannot appear also in the right inequality within the class of functions here considered, since we would have

$$f'(z) = Az^{2\eta_1 i-1}\frac{\sigma_3(i\lg z-\pi)^2}{\sigma(i\lg z)^2}$$

if it had appeared.

As mentioned above, though the estimation of theorem 5.7 is not precise, it yields a precise estimation in the simply-connected case by the passage to limit: $q\to 0$. In fact, the lower and upper bounds in the estimation just proved then tend to $1/(1+|z|)^2$ and $1/(1-|z|)^2$, respectively. Consequently, if we suppose that the image-domain of R_q degenerates to a simply-connected convex domain pricked at a single point, we shall have as a limit an estimation of the form

$$\frac{1}{(1+|z|)^2} \leqq \left| \frac{f'(|z|)}{A_0} \right| \leqq \frac{1}{(1-|z|)^2} \qquad (|z|<1)$$

and hence, in particular, $|A_0| = |f'(0)|$. Since A_0 does not depend on the rotation about the origin in the z-plane, the last estimation further implies

$$\frac{1}{(1+|z|)^2} \leqq \left| \frac{f'(z)}{f'(0)} \right| \leqq \frac{1}{(1-|z|)^2} \qquad (|z|<1)$$

which coincides just with the third distortion inequality of theorem 2.3. Further, as $q \to 0$, we have

$$z^{-2\eta_1 i-1} \frac{\sigma_3(i \lg z)^2}{\sigma(i \lg z - \pi)^2} \to e^{-\pi^2/12} \frac{1}{(1+z)^2}$$

and

$$z^{2\eta_1 i-1} \frac{\sigma_3(i \lg z - \pi)^2}{\sigma(i \lg z)^2} \to - e^{-\pi^2/12} \frac{1}{(1-z)^2}$$

in which the limit functions, necessarily single-valued, are constant multiples of the derivatives of respective extremal functions for the point $z_0 = |z_0|$ in the third distortion inequality of theorem 2.3.

6. Mapping of an annulus onto a star-like ring domain.

We first introduce the notion of star-likeness in case of doubly-connected domains. A doubly-connected domain is said to be star-like with respect to a point if the part, contained in the domain, of any half-line starting at this point consists of a singie segment which may be possibly infinite. According to the definition, the reference point of a doubly-connected star-like domain does not belong to the domain, while it lies at an interior point in simply-connected case. However, any simply-connected star-like domain may be regarded as a degenerate form of a doubly-conneted star-like domain whose one boundary component consists of the reference point alone which behaves as a removable singularity.

In order to derive an analogue of Alexander's theorem, we follow a similar way as in the simply-connected case. Namely, we first establish an integral representation characteristic to a function which maps an annulus onto a star-like ring domain.

THEOREM 6.1. *Let $f(z)$ be a single-valued analytic function mapping the annulus R_q: $(0<)q<|z|<1$ univalently onto a ring domain star-like with respect to the origin in such a manner that the boundary component originating from $|z|=q$ separates the origin and the image-domain. Then there exist increasing functions $\rho(\varphi)$ and $\tau(\varphi)$ defined for $-\pi \leqq \varphi < \pi$ and both with total variation equal to unity such that an integral representation of Villat-Stieltjes type*

$$z\frac{f'(z)}{f(z)} = \frac{2}{i}\int_{-\pi}^{\pi} (\zeta(i \lg z + \varphi)\, d\rho(\varphi) - \zeta_3(i \lg z + \varphi)\, d\tau(\varphi)) + ic^*$$

holds for $z \in R_q$ where c^* is a real constant defined by

$$c^* = \frac{2\eta_1}{\pi}\int_{-\pi}^{\pi} \varphi(d\rho\ \varphi) - d\tau(\varphi)).$$

Proof. The present theorem relates to theorem 3.1 in quite a similar manner that theorem 5.1 relates to theorem 2.1. The procedure of the proof can be performed also so and hence its full statement may be omitted.

COROLLARY 1. *For $f(z)$ satisfying the condition in theorem* 6.1, *there holds*

$$f(z) = Cz^{ic^*}\exp\Big(2\int_{-\pi}^{\pi}(-\lg\sigma(i\lg z+\varphi)\,d\rho(\varphi)+\lg\sigma_3(i\lg z+\varphi)\,d\tau(\varphi))\Big),$$

$C(\neq 0)$ being a certain constant and c^* a real constant defined in theorem 6.1; *$\rho(\varphi)$ and $\tau(\varphi)$ are the same functions as in theorem* 6.1.

COROLLARY 2. *If $f(z)$ satisfies the condition in theorem* 6.1, *the image of every concentric circumference $|z| = r\ (q < r < 1)$ by $f(z)$ is a curve star-like with respect to the origin.*

Proof. Similar as the proof of corollary 2 of theorem 5.1. We have only to consider $\arg f(re^{i\theta})$ instead of $\arg df(re^{i\theta})$.

THEOREM 6.2. *Let $\rho(\varphi)$ and $\tau(\varphi)$ defined for $-\pi \leqq \varphi < \pi$ be increasing functions with total variation equal to unity. Then the function defined by*

$$f(z) = Cz^{ic^*}\exp\Big(2\int_{-\pi}^{\pi}(-\lg\sigma(i\lg z+\varphi)\,d\rho(\varphi)+\lg\sigma_3(i\lg z+\varphi)\,d\tau(\varphi))\Big)$$

with a real constant

$$c^* = \frac{2\eta_1}{\pi}\int_{-\pi}^{\pi}\varphi(d\rho(\varphi)-d\tau(\varphi))$$

and with any constant $C(\neq 0)$ maps the annulus R_q univalently onto a ring domain star-like with respect to the origin, of which the inside boundary component originating from $|z| = q$ separates the origin and the image-domain.

Proof. It is evident that $f(z)$ is regular in R_q. Moreover, it is single-valued. In fact, the substitution $\lg z \,|\, \lg z + 2\pi i$ causes a multiplicative factor to $f(z)$ which is equal to

$$\begin{aligned}
&\exp\Big(-2\pi c^* + 2\int_{-\pi}^{\pi}\Big(\lg\frac{\sigma(i\lg z+\varphi)}{\sigma(i\lg z+\varphi-2\pi)}\,d\rho(\varphi)\\
&\qquad\qquad + \lg\frac{\sigma_3(i\lg z+\varphi-2\pi)}{\sigma_3(i\lg z+\varphi)}\,d\tau(\varphi)\Big)\Big)\\
&= \exp\Big(-2\pi c^* + 2\int_{-\pi}^{\pi}((-\pi i + 2\eta_1(i\lg z+\varphi-\pi))\,d\rho(\varphi)\\
&\qquad\qquad - 2\eta_1(i\lg z+\varphi-\pi)\,d\tau(\varphi))\Big)\\
&= \exp\Big(-2\pi c^* - 2\pi i + 4\eta_1\int_{-\pi}^{\pi}\varphi(d\rho(\varphi)-d\tau(\varphi))\Big) = 1.
\end{aligned}$$

For any r with $q < r < 1$, we get

$$\begin{aligned}
\frac{d}{d\theta}\arg f(re^{i\theta}) &= \Re re^{i\theta}\frac{f'(re^{i\theta})}{f(re^{i\theta})}\\
&= 2\int_{-\pi}^{\pi}(P(re^{i\theta};\varphi)\,d\rho(\varphi) - Q(re^{i\theta};\varphi)\,d\tau(\varphi)),
\end{aligned}$$

where P and Q are the functions defined in lemma 4.1. By virtue of the same lemma the last expression is positive so that the star-likeness is verified. Finally, again for any r with $q < r < 1$, we have in a similar manner as in the proof of theorem 5.2 the relation

$$\int_{-\pi}^{\pi} d \arg f(re^{i\theta})$$
$$= 2\int_{-\pi}^{\pi} d\theta \Im \int_{-\pi}^{\pi} (\zeta(i \lg r - \theta + \varphi)\, d\rho(\varphi) - \zeta_3(i \lg r - \theta + \varphi)\, d\tau(\varphi)) = 2\pi,$$

whence follows the univalency of $f(z)$ in R_q.

Based on corollary 1 of theorem 6.1 and theorem 6.2, it is concluded that the representation given in the corollary is characteristic to a function possessing the mapping property stated in theorem 6.1. By comparing this set of theorems with the set of corollary 1 of theorem 5.1 and theorem 5.2, we can mention an analogue of Alexander's theorem 3.3 which shows an explicit connection between two classes of mapping functions under consideration.

THEOREM 6.3. *Let* $\mathfrak{K} = \{f_c(z)\}$ *be the class consisting of functions with the property mentioned in theorem* 5.1, *and* $\mathfrak{St} = \{f_s(z)\}$ *the class consisting of functions with the property mentioned in theorem* 6.1 *which is normalized by a further condition that the constant term of the Laurent expansion vanishes, i.e. the condition*

$$\int_{|z|=r} f_c(z) \frac{dz}{z} = 0$$

to be valid for an r *with* $q < r < 1$. *Then, they are connected by the relation*

$$1 + z\frac{f_c''(z)}{f_s'(z)} = z\frac{f_s'(z)}{f_s(z)}$$

More precisely, for any $f_c(z) \in \mathfrak{K}$, *the function defined by*

$$f_s(z) = Czf_c'(z)$$

with any constant $C(\neq 0)$ *belongs to* $\mathfrak{St}$ *and, conversely, for any* $f_s(z) \in \mathfrak{St}$, *the function defined by*

$$f_c(z) = A\int^z f_s(z)\frac{dz}{z} + B$$

with any constants $A(\neq 0)$ *and* B *belongs to* $\mathfrak{K}$.

Proof. Evident by virtue of theorems 5.1, 5.2, 6.1 and 6.2. The condition that the constant term of the Laurent expansion of $f_s(z)$ vanishes assures, of course, the single-valuedness of the corresponding $f_c(z)$.

We have considered in the preceding section a subclass of $\mathfrak{K}$ by imposing an additional condition that the image of $|z| = q$ is a rectilinear segment. It will be shown, by the way, that the functional equation of theorem 6.3 which establishes the connection between two classes $\mathfrak{K}$ and $\mathfrak{St}$ remains valid between this subclass of $\mathfrak{K}$ and the corresponding subclass of $\mathfrak{St}$.

THEOREM 6.4. *Let* $f_c(z) \in \mathfrak{K}$ *satisfy the condition that the image of* $|z| = q$ *is a rectilinear segment* l_c. *Then the function defined by*

$$f_s(z) = zf_c'(z)$$

belongs to $\mathfrak{St}$ *and the image of* $|z| = q$ *by* $f_s(z)$ *is also a rectilinear segment* l_s *which passes through the origin and is perpendicular to* l_c. *If the end points of* l_c *lie at* $f_c(q)$ *and* $f_c(qe^{i\gamma})$, *then* $f_s(q)$ *and* $f_s(qe^{i\gamma})$ *are the boundary elements lying at the origin on the opposite banks of* l_s. *Conversely, let* $f_s(z) \in \mathfrak{St}$ *satisfy the conditions that the image of* $|z| = q$ *is a rectilinear segment* l_s *and that the constant term of the Laurent expansion vanishes. Then the function defined by*

$$f_c(z) = \int^z f_s(z) \frac{dz}{z}$$

belongs to $\mathfrak{K}$ *and the image of* $|z| = q$ *by* $f_c(z)$ *is also a rectilinear segment* l_c *perpendicular to* l_s. *If the bonudary elements lying at the origin on the opposite banks of* l_s *are* $f_s(q)$ *and* $f_s(qe^{i\gamma})$, *then the end points of* l_c *lie at* $f_c(q)$ *and* $f_c(qe^{i\gamma})$.

Proof. By virtue of the general theorem 6.3, it is only necessary to consider the boundary behaviors of respective functions along $|z| = q$. In any case, we get, for any r with $q < r < 1$, the relation

$$\arg f_s(re^{i\theta}) = \arg df_c(re^{i\theta}) - \frac{\pi}{2}.$$

Letting r tend to $q + 0$, this implies

$$\arg f_s(qe^{i\theta}) = \arg df_c(qe^{i\theta}) - \frac{\pi}{2}$$

along $|z| = q$ except at $z = q$ and $z = qe^{i\gamma}$ where $\arg f_s(z)$ as well as $\arg df_c(z)$ have jumps with the height equal to π. On the other hand, the single-valuedness of $f_c(z)$ is equivalent to the condition that the constant term of the Laurent expansion of $f_s(z)$ vanishes. The last condition implies further that the segment l_s passes through the origin, a fact which follows, of course, merely from the star-likeness of the image by $f_s(z)$ with respect to the origin. The whole proposition of the theorem is now a simple consequence of these facts.

The boundary correspondence between images of $|z| = q$ by the corresponding functions may be described in a slightly precise manner. In the following, the inclination of every segment will be taken with respect to mod 2π. First, let the inclination of the part of l_c originating from the arc $0 \leqq \theta < \gamma$ be χ_c and consequently that of the part from the complementary arc $\gamma \leqq \theta < 2\pi$ be $\chi_c + \pi$. Then the images of these arcs by the corresponding $f_s(z)$ lie on the ray starting from the origin with the inclinations $\chi_c - \pi/2$ and $\chi_c + \pi/2$, respectively. Conversely, let the ray starting from the origin and bearing the image-segment of the arc $0 \leqq \theta < \gamma$ have the inclination χ_s and consequently let the ray bearing its remaining part have the inclination $\chi_s - \pi$. Then the images of these arcs by the corresponding $f_c(z)$ have the inclinations $\chi + \pi/2$ and $\chi_s - \pi/2$, respectively.

Let the end points of the image-segment of $|z| = q$ by $f_c(z)$ originate, as above, from q and $qe^{i\gamma}$. Then, for the corresponding $f_s(z)$, we have

$$z\frac{f_s'(z)}{f_s(z)}=\frac{2}{i}\int_{-\pi}^{\pi}\zeta(i\lg z+\varphi)\,d\rho(\varphi)-\frac{1}{i}(\zeta_3(i\lg z)+\zeta_3(i\lg z+\gamma))+ic^*$$

where c^* is a real constant given by

$$c^*=\frac{2\eta_1}{\pi}\Big(\int_{-\pi}^{\pi}\varphi\,d\rho(\varphi)-\gamma\Big).$$

It is evident that the quantity

$$z\frac{f_s'(z)}{f_s(z)}=\frac{d\lg f_s(z)}{d\lg z}$$

is purely imaginary everywhere along $|z|=q$. Hence the equation

$$f_s'(qe^{i\theta})=0$$

for determining the arguments of two points $qe^{i\delta_j}$ $(j=1,2)$ which correspond to the end points of l_s is equivalent to

$$\Im\frac{qe^{i\theta}f_s'(qe^{i\theta})}{f_s(qe^{i\theta})}=0$$

which may be written as

$$2\int_{-\pi}^{\pi}\Big(\Re\zeta(i\lg q-\theta+\varphi-)\ \frac{\eta_1}{\pi}\varphi\Big)d\rho(\varphi)$$
$$-\Big(\Re\zeta_3(i\lg q-\theta)+\Re\zeta_3(i\lg-\theta+\gamma)-\frac{2\eta_1}{\pi}\gamma\Big)=0,$$

i.e.

$$2\int_{-\pi}^{\pi}\Big(\Re\zeta_3(\theta-\varphi)+\frac{\eta_1}{\pi}\varphi\Big)d\rho(\varphi)=\Re(\zeta(\theta)+\zeta(\theta+\gamma))+\frac{2\eta_1}{\pi}\gamma.$$

The last equation expresses the connection between γ and each of $\theta=\delta_j$ $(j=1,2)$.

Now, based on theorem 6.3, we can state an analogue of theorem 5.3 for the class $\mathfrak{St}$. Namely, for this class, we have only to replace the functional $1+\Re(zf''(z)/f'(z))$ by $\Re(zf'(z)/f(z))$ in the estimation of theorem 5.3. In particular, the equality sign in every inequality is again excluded. But, in order to assure the correspondence between the classes $\mathfrak{K}$ and $\mathfrak{St}$, the latter is restricted by an additional codition that the constant term of the Laurent expansion vanishes. Consequently, there remains the possibility that the estimation thus obtained is sharp within a wider class consisting of functions which need not satisfy this additional restriction. This is really the case, as shown in the following theorem.

THEOREM 6.5. *Let $f(z)$ possess the mapping character stated in theorem 6.1. Then, for any $z\in R_q$,*

$$2\Im(\zeta(i\lg|z|+\pi)-\zeta_3(i\lg|z|+\pi))\leqq\Re z\frac{f'(z)}{f(z)}\leqq2\Im(\zeta(i\lg|z|)-\zeta_3(i\lg|z|)).$$

Every estimation is sharp. More precisely, let $z_0=r_0e^{i\theta_0}$ with $0\leqq\theta_0<2\pi$ be any preassigned point in R_q.

(i) *The left equality holds at z_0 if and only if $f(z)$ is of the form $cf_*(z;\theta_0)$ with a constant $c\neq0$ where*

$$f_*(z;\ \theta_0) = \frac{\sigma_3(i\lg z + \theta_0 - \pi)^2}{\sigma(i\lg z + \theta_0 - \pi)^2} \equiv \wp(i\lg z + \theta_0 - \pi) - e_3.$$

(ii) *The right equality holds at* z_0 *if and only if* $f(z)$ *is of the form* $cf^*(z;\ \theta_0)$ *with a constant* $c \neq 0$ *where* $f^*(z;\ \theta_0) = f_*(z;\ \theta_0 - \pi)$, *i.e.*

$$f^*(z;\ \theta_0) = \frac{\sigma_3(i\lg z + \theta_0)^2}{\sigma(i\lg z + \theta_0)^2} \equiv \wp(i\lg z + \theta_0) - e_3.$$

Proof. From the representation of theorem 6.1, the estimation follows readily by means of the same procedure as used in the proof of theorem 5.3.

(i) If the equality sign in the left inequality appears at z_0, then, by remembering again the procedure used in the proof of theorem 5.3, we have the relation

$$z\frac{f'(z)}{f(z)} = \frac{2}{i}(\zeta(i\lg z + \theta_0 - \pi) - \zeta_3(i\lg z + \theta_0 - \pi))$$

which leads to

$$f(z) = c\frac{\sigma_3(i\lg z + \theta_0 - \pi)^2}{\sigma(i\lg z + \theta_0 - \pi)^2} = c(\wp(i\lg z + \theta_0 - \pi) - e_3),$$

$c(\neq 0)$ being a constant. It is evident that the function of the last-mentioned form possesses the extremal property.

(ii) It is only necessary to note that any extremal function for estimation from above is evidently given by

$$cf^*(z;\ \theta_0) = cf_*(z;\ \theta_0 + \pi).$$

The extremal function $f_*(z;\ \theta_0)$ maps R_q univalently onto the whole plane cut along a finite slit and an infinite slit, both lying on the positive real axis. The end points of the finite slit lie at $0 = f_*(-qe^{i\theta_0};\ \theta_0)$ and $e_2 - e_3 = f_*(qe^{i\theta_0};\ \theta_0) > 0$ while those of the infinite slit lie at $e_1 - e_3 = f_*(e^{i\theta_0};\ \theta_0) > 0$ and $\infty = f_*(-e^{i\theta_0};\ \theta_0)$. On the other hand, R_q is mapped by $f^*(z;\ \theta_0)$ univalently onto the same domain as by $f_*(z;\ \theta_0)$ but the antecedent of the point $f_*(p;\ \theta_0)$ by the mapping function $f^*(z;\ \theta_0)$ is $-p$, i.e. the relation $f^*(z;\ \theta_0) = f_*(-z;\ \theta_0)$ holds identically.

The Laurent expansions of the extremal functions are

$$f_*(z;\ \theta_0) = -\frac{\eta_1}{\pi} - e_3 + \sum_{n=-\infty}^{\infty}{}' (-1)^n \frac{ne^{-in\theta_0}}{1 - q^{2n}} z^n$$

and

$$f^*(z;\ \theta_0) = -\frac{\eta_1}{\pi} - e_3 + \sum_{n=-\infty}^{\infty}{}' \frac{ne^{-in\theta_0}}{1 - q^{2n}} z^n$$

where every summation extends over all the integers except $n = 0$. As already remarked, the constant term in every expansion does not vanish, what is explicitly seen from these expansions. In fact,

$$-\frac{\eta_1}{\pi} - e_3 = 2\sum_{n=1}^{\infty} \frac{nq^n}{1 - q^{2n}} > 0;$$

cf. lemma 4.6.

We are now in position to derive the analogues of theorems 5.4 and 5.5, by replacing the convexity by the star-likeness. Since theorem 6.4 has been established, these analogues can be readily obtained for the subclass of $\mathfrak{St}$ considered in theorem 6.4. However, the distortion inequalities thus obtained will remain valid without an additional condition that the constant term of the Laurent expansion vanishes. Accordingly, we shall derive them directly by means of theorem 6.1. The proofs will be performed, however, in quite a similar way.

THEOREM 6.6. *Let $f(z)$ map the annulus R_q univalently onto a ring domain star-like with respect to the origin in such a manner that $f(q)$ and $f(qe^{i\gamma})$ are the boundary elements with the same affix zero, where $\gamma = \gamma[f]$ is a real parameter depending on respective $f(z)$. Then, for $q < |z| < 1$,*

$$2\Im\zeta(i\lg|z|+\pi) \leqq \Re z\frac{f''(z)}{f'(z)} + \Im(\zeta_3(i\lg z)+\zeta_3(i\lg z+\gamma)) \leqq 2\Im\zeta(i\lg|z|).$$

Let $f(z)$ satisfy besides the condition imposed above a further normalization condition that the image of $|z|=q$ is a vertical slit lying on the imaginary axis and $f(q)=0$ lies on its right bank or may possibly be the lower end point of the slit. Let $z_0 = r_0e^{i\theta_0}$ with $0 \leqq \theta_0 < 2\pi$ be any preassigned point in the annulus R_q.

(i) *The equality sign in the left estimation holds then at z_0 if and only if $f(z)$ is of the form $af_*(z;\theta_0,\gamma)$ with any real positive constant a where*

$$f_*(z;\theta_0,\gamma) = e^{i(\theta_0-\gamma/2+(\pi/2)\mathrm{sgn}\gamma)}z^{i(\eta_1/\pi)(2\theta_0-\gamma-2\pi)}\frac{\sigma_3(i\lg z)\sigma_3(i\lg z+\gamma)}{\sigma(i\lg z+\theta_0-\pi)^2}$$

for $\gamma \neq 0$ and $f_*(z;\theta_0,0)$ is understood to be $f_*(z;\theta_0,-0)$. Here γ with $-\pi \leqq \gamma < \pi$ denotes the value of the parameter associated to respective function.

(ii) *The equality sign in the right estimation holds at z_0 if and only if $f(z)$ is of the form $af^*(z;\theta_0,\gamma)$ with any real positive constant a where $f^*(z;\theta_0,\gamma) = f_*(z;\theta_0+\pi,\gamma)$, i.e.*

$$f^*(z;\theta_0,\gamma) = -e^{i(\theta_0-\gamma/2+(\pi/2)\mathrm{sgn}\gamma)}z^{i(\eta_1/\pi)(2\theta_0-\gamma)}\frac{\sigma_3(i\lg z)\sigma_3(i\lg z+\gamma)}{\sigma(i\lg z+\theta_0)^2}$$

for $\gamma \neq 0$ and $f^*(z;\theta_0,0) \equiv f^*(z;\theta_0,-0)$. Here γ with $-\pi \leqq \gamma < \pi$ denotes the value of the parameter associated to respective function.

Proof. We can proceed in quite similar way as in the proof of theorem 5.4. In fact, based on theorem 6.1, we have only to modify it formally, i.e. to replace the functional $1+zf''(z)/f'(z)$ by $zf'(z)/f(z)$ and accordingly $zf'(z)$ by $f(z)$. The circumstance is much simpler, since the monodromy condition for the present case is always satisfied; cf. the proof of theorem 6.2. In the following lines, we shall describe only a sketch concerning the part on extremal functions.

(i) We first have

$$z\frac{f'(z)}{f(z)} = \frac{2}{i}\zeta(i\lg z+\theta_0-\pi) - \frac{1}{i}(\zeta_3(i\lg z)+\zeta_3(i\lg z+\gamma)) + ic^*$$

with

$$c^* = \frac{2\eta_1}{\pi}\left(\theta_0 - \pi - \frac{\gamma}{2}\right),$$

whence follows, by integration with respect to $\lg z$,

$$f(z) = cz^{ic^*} \frac{\sigma_3(i \lg z)\sigma_3(i \lg z + \gamma)}{\sigma(i \lg z + \theta_0 - \pi)^2},$$

$c(\neq 0)$ being an integration constant. In order to determine the value of $\arg c$, we take into account the supposition that the point $f(q) = 0$ lies on the right bank or at the lower end point of the image-slit. We thus get

$$\begin{aligned} \frac{\pi}{2} &= \arg f(qe^{+i0}) \\ &= \arg c + c^* \lg q + \arg \frac{\sigma_3(i \lg q)\sigma_3(i \lg q + \gamma)}{\sigma(i \lg q + \theta_0 - \pi)^2} \\ &= \arg c - \left(\theta_0 - \frac{\gamma}{2} - \pi\right) + \arg \sin \frac{-\gamma}{2}, \end{aligned}$$

i. e.

$$\arg c = \frac{\pi}{2} \operatorname{sgn} \gamma + \theta_0 - \frac{\gamma}{2}.$$

By the way, the inclination of the half-line originating from $|z| = 1$ is given by

$$\arg f(e^{i\varphi}) = \arg c + \arg \frac{\sigma_3(-\varphi)\sigma_3(-\varphi + \gamma)}{\sigma(-\varphi + \theta_0 - \pi)^2} = \arg c.$$

(ii) It is only necessary to note that any extremal function for the upper estimation is given by $af^*(z;\, \theta_0, \gamma) = af_*(z;\, \theta_0 + \pi, \gamma)$.

The extremal function $f_*(z;\, \theta_0, \gamma)$ maps R_q univalently onto the whole plane cut along a finite slit lying on the imaginary axis and an infinite slit lying on the half-line with the inclination equal to $\theta_0 - \gamma/2 + (\pi/2) \operatorname{sgn} \gamma$. The point at infinity as a boundary point corresponds to $-z_0/|z_0| = -e^{i\theta_0}$ and the origin as two boundary elements originates from q and $qe^{i\gamma}$. The extremal function $f^*(z;\, \theta_0, \gamma)$ maps R_q univalently onto the whole plane cut along a finite slit lying on the imaginary axis and an infinite slit lying on the half-line with the inclination equal to $\theta_0 - \gamma/2 + \pi + (\pi/2) \operatorname{sgn} \gamma$. The point at infinity as a boundary point corresponds to $z_0/|z_0| = e^{i\theta_0}$ and the origin as two boundary elements originates from q and $qe^{i\gamma}$.

In case of the class considered in theorem 5.5, the value of the parameter γ has been so determined for extremal function that its single-valuedness should be assured. However, in the present case the determination of the parameter is unnecessary so that, for any assigned value θ_0, there exists a family of extremal functions depending on this parameter as an indeterminate constant which may be arbitrarily chosen. Among these extremal functions, there exist particular ones with the vanishing constant term of the Laurent expansion which are characterized by the equations $T(\gamma,\, \theta_0) = 0$ and $T(\gamma,\, \theta_0 + \pi) = 0$, respectively.

THEOREM 6.7. *Under the same condition as in theorem* 6.6, *the inequality*

$$2\Im\zeta(i\lg|z|+\pi)-\Im\zeta_3(i\lg|z|+\pi)$$

$$\leqq \Re z\frac{f'(z)}{f(z)}+\Im\zeta_3(i\lg z)\leqq 2\Im\zeta(i\lg|z|)-\Im\zeta_3(i\lg|z|)$$

holds for any $z\in R_q$. The estimation is sharp. More precisely, under the same additional normalization as in theorem 6.6, *the left equality can appear at z_0 if $f(z)$ is of the form $af_*(z;\theta_0)$ where*

$$f_*(z;\theta_0)=ie^{i(\theta_0+\pi\,\mathrm{sgn}(\theta_0-\pi))/2}z^{i(\eta_1/\pi)(\theta_0-\pi)}\frac{\sigma_3(i\lg z)\sigma_3(i\lg z+\theta_0-\pi)}{\sigma(i\lg z+\theta_0-\pi)^2}$$

for $\theta_0\neq\pi$ and $f_*(z;\pi)\equiv f_*(z;\pi-0)$, while the right equality can appear at z_0 if $f(z)$ is of the form $af^*(z;\theta_0)$ where

$$f^*(z;\theta_0)=-aie^{i\theta_0/2}z^{i(\eta_1/\pi)(\theta_0+\pi+\pi\,\mathrm{sgn}(\theta_0-\pi))}\frac{\sigma_3(i\lg z)\sigma_3(i\lg z+\theta_0)}{\sigma(i\lg z+\theta_0)^2}$$

for $\theta_0\neq\pi$ and $f^*(z;\pi)\equiv f^*(z;\pi-0)$, a being in each case a positive real constant.

Proof. From theorem 6.6 together with lemma 4.2, the estimation follows readily. The determination of extremal functions can be performed in a similar way as in theorem 5.6. Since the single-valuedness of every function $f(z)$ under consideration is assured indifferent of the value of parameter γ, it is only necessary to remark that, for extremal functions from below and from above, we must have

$$\gamma-\theta_0=-\pi \qquad \text{and} \qquad \gamma-\theta_0=-\pi(1+\mathrm{sgn}\,(\theta_0-\pi)),$$

respectively

The image of $|z|=1$ by $f_*(z;\theta_0)$ or $f^*(z;\theta_0)$ lies on the half-line whose inclination is $(\theta_0+\pi+\pi\,\mathrm{sgn}\,(\theta_0-\pi))/2$ or $(\theta_0-\pi)/2$, respectively.

Similarly as in the remark for the preceding theorem, the extremal function $f_*(z;\theta_0)$ or $f^*(z;\theta_0)$ has the vanishing constant term of the Laurent expansion if and only if $\theta_0=0$ or $\theta_0=\pi$, respectively; cf. the proof of theorem 5.5.

Finally, we shall give a result corresponding to theorem 5.6; cf. the remark antecedent to this theorem. It is to be noted that, contrary to the convex case, the distortion inequality of the star-like case is accompanied actually by extremal functions.

THEOREM 6.8. *Let $f(z)$ satisfy the condition in theorem* 6.1 *and be expressed by the representation mentioned in its corollary* 1. *Then, for $q<|z|<1$,*

$$\left|\frac{\sigma_3(i\lg|z|)}{\sigma(i\lg|z|+\pi)}\right|^2\leqq\left|\frac{f(|z|)}{C}\right|\leqq\left|\frac{\sigma_3(i\lg|z|+\pi)}{\sigma(i\lg|z|)}\right|^2.$$

The left and right equality signs hold if and only if $f(z)$ is given by

$$f_*(z)=Cz^{-2\eta_1 i}\frac{\sigma_3(i\lg z)^2}{\sigma i\lg z-\pi)^2}$$

and $f^*(z) = f_*(-z)$, *i.e.*

$$f^*(z) = Cz^{2\eta_1 i}\frac{\sigma_3(i \lg z - \pi)^2}{\sigma(i \lg z)^2},$$

respectively.

Proof. By modifying the proof process of theorem 5.6, we can readily verify the present proposition. In fact, it is only necessary to replace $zf'(z)/A$ in theorem 5.6 by $f(z)/C$. Since no more integration is required, there exist really the extremal functions within the class under consideration.

For the sake of brevity, we may suppose $C = 1$. Then the extremal functions $f_*(z)$ and $f^*(z)$ map R_q univalently onto the whole plane cut along a finite slit and an infinite slit, both lying on the real axis. The end points of these slits lie at $f_*(-q) = f^*(q) < 0$, $f_*(q) = f^*(-q) = 0$, $f_*(1) = f^*(-1) > 0$ and $f_*(-1) = f^*(1) = \infty$.

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