ON RIEMANN SURFACES ADMITTING AN INFINITE CYCLIC CONFORMAL TRANSFORMATION GROUP

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In his lecture[1] M. Tsuji has introduced a new classification principle in the theory of Riemann surfaces as follows: If there is no single-valued analytic function with finite spherical area on a given Riemann surface, then the surface is said to belong to a null class O_{MD}. In the same lecture he also has offered several conjectural problems one of which is the same as our problem. Our discussions in the present paper chiefly depend upon the several important results in Heins' paper [1]. So we shall use his results with the abbreviations H. Th. 5.1 etc.

Let W be an open Riemann surface admitting an infinite cyclic group $\mathfrak{G}$ which is generated by a conformal transformation T which maps W onto itself. And further we shall confine ourselves to the case where W has only two ideal boundary components γ_1, γ_2 which are defined as the sets of limit points $\lim_{n\to\infty} T^n p$ and $\lim_{n\to\infty} T^{-n} p$, respectively. By H. Th. 13.1 there holds $W \in O_G$, and two ends W_1, W_2, corresponding to γ_1, γ_2, respectively, are both the Heins' ends of harmonic dimension one in Heins' sense. Our problem may be stated as follows:

Under what conditions can W be mapped onto a finite-sheeted covering surface of z-plane?

Suppose that W can be mapped upon a finite covering surface W' of the whole z-sphere with some defect points. Let $f(p)$ denote this mapping function. By H. Th. 4.2, there exist the limits

$$\lim_{n\to\infty} f(T^n p) \qquad \text{and} \qquad \lim_{n\to\infty} f(T^{-n} p).$$

Without loss of generality we may put $\lim_{n\to\infty} f(T^n p) = 0$. By H. Th. 4.2, on a suitable subend of W_1 the number of sheets is equal to the local degree $d(f, W_1)$ of f. When f runs through the class of the admitted mapping functions, there exists a minimum value of $d(f, W_1)$. Let this minimum value d_0 be attained by an extremal function f_0.

PROPOSITION 1. *With a suitable complex constant t of modulus less than one,*

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1) Prof. M. Tsuji has introduced this classification principle and the related problems at Kansûron Danwakai (Meeting of the researchers of function-theory) in Tokyo held at November 10, 1956.

we have a functional relation

$$\psi \circ f_0(Tp) = t\psi \circ f_0(p),$$

where ψ^{-1} is a suitable rational function.

Proof. Let W^1 be a subdomain of W_1 on which $|f_0(p)| < 1$ and Γ^1 the set on which $|f_0(p)| = 1$. We may evidently suppose that W^1 has Γ^1 as the whole relative boundary. Let $f_1(p)$ be a function defined by $f_1(Tp) = f_0(p)$. Then $f_1(p)$ maps $T(W^1)$ onto d_0-sheeted unit disc such that $\lim_{n\to\infty} f_1(T^n p) = 0$. $f_0(p)$ maps $T(W^1)$ onto a covering surface of a subdomain of $|z| < 1$ which satisfies a limit relation $\lim_{n\to\infty} f_0(T^n p) = 0$ and whose local degree $d(f_0, W_1)$ is equal to d_0. Therefore, by H. Th. 5.1, we have a functional relation $f_0(p) = \varphi \circ f_1(p)$ on $T(W^1)$. Here φ stands for an analytic function of z which is univalent and bounded, $|\varphi| < 1$, on $|z| < 1$ and $\varphi(0) = 0$. This functional relation can be continued to the whole surface W. Now we shall prove this fact. Let $p_1, \cdots, p_\nu$ be a set of points such that $Tp_1, \cdots, Tp_\nu$ lie on W^1 and $f_0(p_1) = \cdots = f_0(p_\nu) = z$. Then $f_1(T^2 p_\mu) = z$ by definition and hence $f_1(T^2 p_\mu) = w$, $\mu = 1, \cdots, \nu$. Let $\varphi(z)$ be $f_1 T^2 f_0^{-1}(z) = w$, then $\varphi(z)$ is analytic and independent of the choice of $f_0^{-1}(z) = p_\mu$, $\mu = 1, \cdots, \nu$. It can be written $\varphi(z) = f_0 T f_0^{-1}(z)$, and hence $f_0(p_\mu) = \varphi \circ f_1(p_\mu)$ holds for any $\mu = 1, \cdots, \nu$. Thus the functional relation $f_0(p) = \varphi \circ f_1(p)$ (or equivalently $f_0(Tp) = \varphi \circ f_0(p)$) holds on $T^{-1}(T(W^1)) = W^1$ and φ is an analytic function single-valued on a somewhat extended domain which includes $|z| < 1$. The same process is now available to extend the validity domain of the functional relation to the whole surface W. However, although the single-valuedness of $\varphi(z)$ for a global parameter z of z-plane can no more be guaranteed, the single-valuedness of φ can be established on a suitable Riemann surface Z which is a finite covering surface of the whole z-sphere and on which W' lies as a finite covering surface. Now we shall construct Z and prove that Z is a simply-connected surface without boundary. Let F be a fundamental domain for the covering transformation group $\mathfrak{G}$. Let Z be a Riemann surface obtained from W' by the successive identification process and prolongation defined as follows. Let $f_0(W^1)$ be a subsurface of W' lying on $|z| < 1$ and each point of which is connectible to an ideal boundary point $\lim_{n\to\infty} f(T^n p)$ by a continuous curve lying on the subsurface. If $f_0(p_1) = f_0(p_2) = a$, $|a| < 1$, $p_1, p_2 \in W^1$, then we identify these $f_0(p_\mu)$, $\mu = 1, 2$ and we denote the resulting punctured disc $0 < |z| < 1$ by $\{f_0(W^1)\}$. If $f_0(p_\mu) = a$, $f_0(Tp_\mu) = b$ and $Tp_\mu \in T^{-n} W^1$, where such points are finite in number, then we identify these $f_0(p_\mu)$ and we denote the resulting domain by $\{f_0(T^{-n-1}(W^1)\}$. Evidently $\{f_0(T^{-n-1}(W^1))\} - \overline{\{f_0(T^{-n}(W^1))\}}$ is a proper doubly connected domain. Thus that $\{f_0(T^{-n}(W^1))\}$ is topologically equivalent to a disc punctured at the center implies that the same fact remains true for $\{f_0(T^{-n-1}(W^1))\}$. Repeating this process ad infinitum, $f_0(T^{-n}F)$ tends to a point lying on $\lim_{n\to\infty} f_0(T^{-n}p)$. Thus the prolongation of the final domain obtained by the repetition of the process is a simply-connected Riemann surface Z without boundary. Evidently $\varphi(q)$ is single-valued on Z.

There is a single-valued analytic function ψ which maps Z univalently onto the whole w-sphere such that two points of Z, on which two ideal boundary points of W' lie, correspond to two points $w=0$ and $w=\infty$, respectively. Then we can consider that φ is single-valued on the whole w-sphere and $f_0(p)$ is transformed to the function $\psi\circ f_0(p)$. Thus we have a functional relation

$$\psi\circ f_0(Tp)=\varphi\circ\psi\circ f_0(p),$$

where φ is a single-valued analytic function on the whole w-sphere. Next we shall prove the univalency of $\varphi(w)$. Let $\varphi(a)=0$ hold for a suitable $a\ (\neq 0)$, then there is a set of points p_μ such that $\psi\circ f_0(p_\mu)=a$, and hence $\psi\circ f_0(Tp_\mu)=0$ whence follows $\psi\circ f_0(T^n p_\mu)=0$ for all n. This is absurd, since the number of sheets is finite. Thus φ must be univalent on the whole w-sphere, whence follows that φ is a linear function of w reducing to a form $\varphi=tw$ by the normalization. Thus our proposition 1 has been proved completely.

From this proposition 1 we have a fact that $\psi\circ f_0(p)$ does not take values zero and infinity on W and the number of sheets on every point $w=a(a\neq 0,\infty)$ is the same and coincides with the local degree d_0 of $f_0(p)$ on W_1.

For simplicity's sake we shall again use in the sequel the notations $f_0(p)$ and z-sphere instead of $\psi\circ f_0(p)$ and w-sphere.

Let $u(p)$ be a single-valued positive harmonic function on W^1 vanishing identically on Γ^1 on which $|f_0(p)|=1$, then by H. Th. 12.1 and the one-dimensionality of W^1 any such harmonic function has the form $ku(p)$ with a suitable positive constant k. Thus we have

$$\log\frac{1}{|f_0(p)|}=ku(p),\qquad k>0.$$

Now supposing a normalizing condition

$$\int_{\Gamma^1}\frac{\partial}{\partial\nu}u(p)ds=2\pi,\qquad \frac{\partial}{\partial\nu}:\ \text{inner normal derivative,}$$

then by the definition of local degree we have

$$d_0=\frac{1}{2\pi}\int_{T^n(\Gamma^1)}\frac{\partial}{\partial\nu}\log\frac{1}{|f_0(p)|}ds=\frac{k}{2\pi}\int_{T^n(\Gamma^1)}\frac{\partial}{\partial\nu}u(p)ds=k.$$

Therefore $d_0u(p)-\log|t|=d_0u(Tp)$ holds on W^1. Since $u(p)$ can be continued harmonically on the whole W by $u(p)=(1/d_0)\log(1/|f_0(p)|)$, $d_0u(p)-\log|t|=d_0\,(u(T_p))$ also remains true on W. Let $g(p)$ be an analytic function whose real part is equal to $u(p)$. Evidently $(f_0'(p)/f_0(p))d\rho\ (=-d_0g'(p)d\rho)$ being invariant for $\mathfrak{G}$, this is an Abelian differential of the first kind on a closed Riemann surface R constructed from W by an identification process W mod $\mathfrak{G}$, where ρ is a local uniformizing parameter at p. Let C be a cycle on R obtained from any curves joining p and Tp on W by the identification of p and Tp and forming a member of a canonical homology base. With exception of two cycles C and Γ^1, the periodicity modulus of $-d_0g'(p)d\rho$

along any cycle must have the form $2\pi i L$ with an integer L such that $|L| \leqq d_0$. We have

$$\int_{\Gamma^1} - d_0 g'(p) d\rho = - d_0 2\pi i \qquad \text{along } \Gamma^1,$$

$$\int_C - d_0 g'(p) d\rho = \log t, \quad |t| \neq 1, \quad \text{along } C.$$

These can be deduced by the single-valuedness of $f_0(p)$ and $u(p)$.

Let R be of genus p and $(\alpha_1, \alpha_2, \cdots, \alpha_p, \beta_1, \beta_2, \cdots, \beta_p)$ be a canonical homology base of one-dimensional homology group of R such that (α_j, β_j) forms the j-th conjugate pair of cycles. Let dw_j be an Abelian differential of the first kind on R such that

$$\int_{\alpha_k} dw_j = 2\pi i \delta_{j,k} \qquad \text{and} \qquad \int_{\beta_k} dw_j = 2\pi i \tau_{j,k}.$$

Evidently we have

$$- d_0 g'(p) d\rho = \sum_{j=1}^{p} A_j dw_j$$

with some complex constants A_j.

The first case with which we concern is the case that $\Gamma^1 = \alpha_p$ and hence $C = \beta_p$. Then we have

$$\begin{cases} - d_0 2\pi i = \int_{\alpha_p} - d_0 g'(p) d\rho = \int_{\alpha_p} \sum_{j=1}^{p} A_j dw_j = A_p 2\pi i, & \\ 2\pi i L_k = \int_{\alpha_k} - d_0 g'(p) d\rho = A_k 2\pi i, & k \neq p, \\ 2\pi i L_k' = \int_{\beta_k} - d_0 g'(p) d\rho = \sum_{j=1}^{p} A_j 2\pi i \tau_{j,k}, & k \neq p, \\ \log t = \int_{\beta_p} - d_0 g'(p) d\rho = 2\pi i \sum_{j=1}^{p} A_j \tau_{j,p}, & |t| \neq 1. \end{cases}$$

Thus we have the system of simultaneous algebraic equations

$$\text{(A)} \qquad \begin{cases} A_p = - d_0, & \\ A_k = L_k, & k \neq p, \\ \sum_{j=1}^{p} A_j \tau_{j,k} = L_k', & k \neq p, \end{cases}$$

where L_k and L_k' are integers satisfying $|L_k| \leqq d_0$ and $|L_k'| \leqq d_0$. Since no non-zero Abelian differential of the first kind has a real periodicity modulus along every cycle, we have that $\Im \sum_{j=1}^{p} A_j \tau_{j,k} = 0$ $(k = 1, \cdots, p-1)$ implies that $\Im \sum_{j=1}^{p} A_j \tau_{j,p} \neq 0$, and hence $|t| \neq 1$ is implied by the above epuations (**A**).

However in the general case we may suppose that all the L_k', $k = 1, \cdots, p-1$, reduce to zero, since the origin of z-sphere is not circumscribed by the image curve of $\beta_k (k \neq p)$ by the mapping function $f_0(p)$. (**A**) reduces then to the following simpler form

$$(\mathbf{B})\qquad \begin{cases} A_p = -d_0, \\ A_k = L_k, \quad k \neq p, \quad L_k\text{: integers such that } |L_k| \leq d_0, \\ \sum_{j=1}^{p} A_j \tau_{j,k} = 0, \qquad k \neq p. \end{cases}$$

Now we shall proceed to the converse problem. In the first place we shall treat the algebraic problem. If (**B**) has a system of solution $(A_1, \cdots, A_p)$ for a suitable system $(L_1, \cdots, L_{p-1}, d_0)$ whose members are all integers and $|L_k| \leq d_0$, then $(L_1, \cdots, L_{p-1}, d_0)$ is said to be an admissible integral system for (**B**). With this terminology we see that $(nL_1, \cdots, nL_{p-1}, nd_0)$ is also an admissible integral system for (**B**), if $(L_1, \cdots, L_{p-1}, d_0)$ is so. Among all the admissible integral systems for (**B**), there exists only one system for which minimum of d_0 is realized. To prove this we remember that the imaginary part of the Riemann matrix $(\tau_{j,k})_{j,k=1,\cdots,p}$ is symmetric and positive definite. Hence the imaginary part of $(\tau_{j,k})_{j,k=1,\cdots,p-1}$ is also positive definite and its determinant does not vanish. On the other hand, the last $p-1$ equations of (**B**) can be written in the following form

$$\sum_{j=1}^{p-1} L_j \tau_{j,k} = d_0 \tau_{p,k}, \qquad k = 1, \cdots, p-1.$$

This system of simultaneous equations has only at most one solution, which implies the uniqueness of the minimnm admissible integral system. Thus we have verified that any admissible integral system has the form $(nL_1, \cdots, nL_{p-1}, nd_0)$ with a suitable system $(L_1, \cdots, L_{p-1}, d_0)$, where n is a positive integer.

If (**B**) has an admissible integral system with minimum d_0, then there is only one Abelian differential of the first kind df whose periodicity moduli satisfy (**B**). The real part of the periodicity modulus of df along β_p does not reduce to zero, and hence we see that an Abelian integral of the first kind f satisfies a period relation

$$f(Tp) = f(p) + \log t, \qquad |t| \neq 1,$$

where $\log t$ is the periodicity modulus of df along β_p. Let $F(p)$ be $\exp(f(p))$, then $F(p)$ has no period along any α_k, $k = 1, \cdots, p$ and any β_k, $k = 1, \cdots, p-1$, by (**B**). However along β_p a multiplicative functional relation

$$F(Tp) = tF(p), \qquad |t| \neq 1,$$

holds. Thus $F(p)$ is a single-valued analytic function regular on W and has the local degree d_0 on each ideal boundary component of W. Therefore $F(p)$ is the desired mapping function which maps W onto a finite-sheeted (d_0-sheeted) covering surface of z-sphere.

By this mapping $F(p)$, the image Riemann surface W' has the following special structure: Two ideal boundaries lie on the points $z = 0$ and $z = \infty$, respectively. The origin $z = 0$ and infinity $z = \infty$ are two accumulation points of branch points. Moreover, near the origin and infinity, d_0 sheets whirl as in the origin and infinity for the function z^{d_0}. This is easy to deduce.

Evidently (**B**) can be solved always, when R is of genus one. When p is greater than one, (**B**) has no solution for the general closed Riemann surface. However, under some restrictions on the Riemann matrix, it is shown that (**B**) is solvable.

THEOREM 1. *A necessary and sufficient condition in order that there exists at least one non-constant single-valued analytic function on W by which W is mapped onto a finite covering surface of z-sphers is that the corresponding closed Riemann surface R has the Riemann matrix $(\tau_{j,k})_{j,k=1,\cdots,p}$ for which* (**B**) *has at least one admissible integral system.*

The second case with which we concern is the case that Γ^1 consists of α_{p-1} and α_p, and hence C is β_{p-1} or β_p. In this case we have the following simultaneous equations as (**B**)

$$\begin{cases} A_{p-1} + A_p = -d_0, & \\ A_k = L_k, \quad k = 1, \cdots, p, & |L_{p-1}| \leqq d_0 - 1, \quad |L_p| \leqq d_0 - 1, \quad |L_k| \leqq d_0 \\ & \text{for} \quad k \neq p-1, p \quad \text{and all the } L_k \text{ are integers,} \\ \sum_{j=1}^{p} A_j \tau_{j,k} = 0, & k \neq p-1, p, \\ \sum_{j=1}^{p} A_j \tau_{j,p-1} = \sum_{j=1}^{p} A_j \tau_{j,p}. & \end{cases}$$

Since the situation is the same as in the first case, we shall omit here the detailed discussions. This case is, however, useful to establish the conditions in the case where R is a bordered Riemann surface of finite topoiogical characters.

REFERENCE

[1] M. HEINS, Riemann surfaces of infinite genus. Ann. of Math. **55** (1952), 296—317.

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