ON A PROPOSITION OF VON NEUMANN

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1. J. von Neumann wrote, without proof, the following proposition in his monumental paper [3; footnote 10, p. 123]: "We could show that the operation $x \to x^{|p|q|\cdots}$ depends only on x and A (and not on the sequence $p, q, \cdots$)", where A is an abelian W^*-subalgebra of a semi-finite factor M acting on a separable Hilbert space H, and where $p, q, \cdots$ is a sequence of projections in A generating the subalgebra A.

In this note, we shall give a proof of the above mentioned proposition as a consequence of a result due to one of the authors [5; § 4]. Furthermore, we wish to give a characterization of a maximal abelian subalgebra in a semi-finite W^*-algebra by means of conditional expectation.

2. Let M be a semi-finite W^*-algebra with regular gage μ. For any W^*-subalgebra A of M there exists a *non-negative linear operation* $x \to x^e$ (with $\|x^e\| \leq \|x\|$) from M into A satisfying that

(i) $x^e = I^e x$ *for* $x \in A$, (ii) $x^{*e}x^e \leqq (x^*x)^e$ *and* $x^e = x^{ee}$,

(iii) $x_\alpha^e \uparrow x^e$ *if* $x_\alpha \uparrow x$, (iv) $(x^e y)^e = (xy^e)^e = x^e y^e$;

cf. [1; Théorème 8]. When it satisfies $I^e = I$, we have called it to be normal expectation; cf. [2]. Further, the above operation $x \to x^e$ (not always $I^e = I$) satisfies that

(v) $\mu(x^e y) = \mu(xy^e)$ *for every* $x \in M$ *and* μ*-integrable* $y \in M$;

cf. [1]. Conversely if a non-negative linear operation $x \to x^\varepsilon$ (with $I^\varepsilon \leq I$) from M into A satisfies (i), (ii) and (v), then it also satisfies (iii), (iv) and $x^\varepsilon = x^e$ for every $x \in M$, cf. [5; Theorem 1], i.e. the $x \to x^e$ is uniquely determined by A and μ. We shall call the operation $x \to x^e$ to be *conditional expectation* (*of* x) *conditioned by* A.

Let $p, q, \cdots, r$ be arbitary finite set of projections in M. According to von Neumann [3], we define the operations $x \to x^{|p}$, $x^{|p|q}$ from M into itself such as $x^{|p} = pxp + (1-p)x(1-p)$, $x^{|p|q} = (x^{|p})^{|q}$, and by successive application $x^{|p|q|\cdots|r}$ is defined which are normal expectations. Furthemore, if $p, q, \cdots, r, \cdots$ is a sequence of mutually commutative projections in M, then it is possible to introduce an operation $x \to x^{|p|q|\cdots}$ for μ-integrable operators $x \in M$, with respect to the metric convergence (i.e. L^2-mean convergence)

(1) $$x^{|p|q|\cdots} = \lim_r x^{|p|q|\cdots|r};$$

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cf. [2; Lemma 2.1.6] and [5; Theorem 5]. Let C_p denote the associate subalgebra of the operation $x \to x^{|p}$, in the sense of Nakamuru-Turumaru [2], which is the set of all elements $x \in M$ with $x^{|p} = x$, then C_p is a W^*-subalgebra of M consisting of all elements which commute with p. It is known [5; Theorem 4] that the operation $x \to x^{|p|q|\cdots|r}$ coincides with the conditional expectation conditioned by $C_p \cap C_p \cap \cdots \cap C_r$, and that each element of $C_p \cap C_q \cap \cdots \cap C_r$ commutes with every $p, q, \cdots, r$ (it is to be noted that the process corresponds to the process of taking the diagonal block of finite matrix; cf. [3; Introduction]). Furthermore, we have

THEOREM 1. *If $p, q, \cdots$ is a sequence of mutually commutative projections in M, then the operation $x \to x^{|p|q|\cdots}$ defined on the algebra M_μ consisting of μ-integrable $x \in M$ is uniquely extended to the conditional expectation $x \to x^e$ on M conditioned by*

$$(2) \qquad C = C_p \cap C_q \cap \cdots$$

and $x \to x^e$ is independent of the choice of the regular gage μ, i.e. (v) *holds for any regular gage of A, and C is the set of all elements in M commuting with $p, q, \cdots$.*

It is sufficient to prove the conditional expection $x \to x^e$ being independent of the choice of the gage μ, because the other parts follow from the previous part of this theorem and [5; Corollary 5.1]. Let A be the W^*-subalgebra of M generated by $\{p, q, \cdots\}$, then we have

$$(3) \qquad C = A' \cap M$$

where A' is the commutor of A. Since the centre $M' \cap M$ of M is contained in C, by Dixmier [1; Proposition 8] (v) holds for any regular gage of M.

3. Let A be an abelian W^*-subalgebra of a semi-finite W^*-algebra M acting on a separable Hilbert space. Then, by the well known theorem of J. von Neumann [4], there exists a sequence $p, q, \cdots$ of projections in A which generates the algebra A, i.e. $\{p, q \cdots\}'' = A$. Clearly by the definition of the $C (= C_p \cap C_q \cap \cdots)$, we have (3) and C is uniquely determined only by A in the equation (3). If M is a semi-finite factor, then the regular gage μ is essentially unique and the algebre M_μ coincides with the algebra (denoted by M_0) consisting of all the elements of finite rank in the sense of von Neumann [3] which is independent of the regular gage μ. Thus the von Neumann's proposition follows immediately from Theorem 1:

THEOREM 2 (von Neumann). *If M is a semi-finite factor acting on a separable Hilbert space H, and if A is an abelian W^*-algebra generated by its sequence of projections $p, q, \cdots$, then the operation $x \to x^{|p|q|\cdots}$ defined on the algebra M_0 (consisting of $x \in M$ with finite rank) depends only on x and A, and is independent of the choice or the order of $p, q, \cdots$.*

In this theorem, if M is a semi-finite W^*-algebra with regular gage μ and acting on H, then the same fact holds for M_μ without that the operation $x \to x^{|p|q|\cdots}$ depends also on the gage μ, because the algebra M_μ depends on μ. In general, we shall call the operation $x \to x^{|p|q\cdots}$ to be *von Neumann's operation defined by A.*[1)]

4. An abelian W^*-subalgebra A of a W^*-algebra M is called to be *maximally abelian* in M if

$$A = A' \cap M, \tag{4}$$

or each element of C of (3) is contained in A. If $A, M, p, q, \cdots$ are as the beginning in § 3, and if A and μ are maximally abelian in M and regular gage of M, respectvely, then by Theorem 1 the conditional expectation of μ-integrable $x \in M$ conditioned by A coincides with the von Neumann's operation $x \to x^{|p|q|\cdots}$ defined by A. Conversely, if the conditional expectation $x \to x^e$ of the x conditioned by A coincides with the operation $x \to x^{|p|q|\cdots}$, then Theorem 1 implies that A satisfies (4), i.e. A is maximally abelian in M. Indeed, putting $C = A' \cap M$ and denoting $x \to x^{\mathcal{E}}$ the conditional expectation conditioned by C, then by Theorem 1 we have $x^{\mathcal{E}} = x^e$ for every $x \in M$. Since A is the direct sum of $M^e = \{x^e;\ x \in M\}$ and $\{\lambda(I - I^e)$; λ being complex numbers$\}$, cf. [5; Theorem 1], $A = C$. Thus we have the following:

THEOREM 3. *Let M be a semi-finite W^*-algebra with a regular gage μ and acting on a separable Hilbert space. Then an abelian W^*-subalgebra A of M is maximally abelian in M if and only if the conditional expectation of $x \in M_\mu$ conditioned by A coincides with the von Neumann's operation $x \to x^{|p|q|\cdots}$ defined by A.*

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1) If M is a finite W^*-algebra (not always a factor) acting on a separable Hilbert sprce H, and if A is an abelian W^*-subalgebra of M, then the von Neumann's operation defined by A, which coincides with the conditional expectation conditioned by $A' \cap M$, depends only on A and defined on every $x \in M$.