ON GRÖTZSCH'S EXTREMAL AFFINE MAPPING

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We shall consider a class of continuously differentiable mappings. If a mapping T attains the minimum of maximal dilatation in a given class of mappings, then T is called extremal. Let R and R' be two rectangles. H. Grötzsch proved that the unique extremal quasiconformal mapping of R onto R' with vertices corresponding to each other is the affine mapping. This affine mapping is called Grötzsch's affine mapping and will always be denoted by the capital letter T in the sequel.

In the present note we shall establish an extremal property of T, from which Grötzsch's original extremality can be easily deduced. Our method of proof is the one usually used in the theory of minimal surfaces or in the calculus of variations. Hence there is no new idea in our proof. Though our proof will be performed under very special situations, the method is far general. Hence we may believe that its analogues or applications will give some essential improvements in the global theory of quasiconformal mappings in a future.

Let $\mathfrak{H}$ be a class of mappings S such that S is continuously differentiable mapping from R onto R', homotopic to T and carries each side of R onto a side of R'. Here one-to-one correspondence of S is not assumed. We define the Dirichlet functional by

$$I_R(S) = \iint_R (|p_S|^2 + |q_S|^2)\, dx\, dy,$$

where p_S and q_S denote the complex derivatives of S.

Theorem. *If $S \in \mathfrak{H}$ and $S \neq T$, then $I_R(S) > I_R(T)$.*

Proof. Let $S \in \mathfrak{H}$ and $I_R(S) \leqq I_R(T)$ and let $\{S_t\}$ be a one parameter family of mappings defined by

$$S_t = (1-t)T + tS, \qquad 0 \leqq t \leqq 1.$$

Evidently $S_t \in \mathfrak{H}$ holds. $I_R(S_t)$ is computed as follows:

$$\frac{I_R(S_t) - I_R(T)}{t} = (-2+t) I_R(T) + t\, I_R(S) + 2(1-t) \iint_R \Re (p_T \overline{p_S} + q_T \overline{q_S})\, dx\, dy.$$

Thus we have

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$$\left.\frac{dI_R(S_t)}{dt}\right|_{t=0} = -2\,I_R(T) + 2\iint_R \Re\,(p_T\,\overline{p_S} + q_T\,\overline{q_S})\,dx\,dy.$$

By the Schwarz inequality we have

$$\left.\frac{dI_R(S_t)}{dt}\right|_{t=0} \leqq -2\,I_R(T) + 2\sqrt{I_R(T)\,I_R(S)} \leqq 0.$$

We shall now prove that $dI_R(S_t)/dt|_{t=0} = 0$ implies $S = T$, which contradicts our hypothesis $S \neq T$. If the equality sign occurs, then there must hold the following facts:

$p_T\,\overline{p_S} + q_T\,\overline{q_S}$ is real non-negative, $|p_S| = k\,|p_T|$ and $|q_S| = k\,|q_T|$ hold for a suitable positive constant k for any point of R and $I_R(T) = I_R(S)$. These imply that $p_T = p_S$ and $q_T = q_S$, whence follows $S = T$. Thus we have

$$\left.\frac{dI_R(S_t)}{dt}\right|_{t=0} < 0,$$

if $S \neq T$.

Next we shall compute the first variation of $I_R(S)$ for any S. Let D_ε be any infinitesimal deformation of R onto itself defined by

$$z \to z - \varepsilon\,h(z).$$

Here $h(z)$ is assumed to be continuously differentiable and ε is a complex number with sufficiently small modulus. We shall denote a mapping $z \to z - \varepsilon\,h(z) \to S(z - \varepsilon\,h(z))$ by $D_\varepsilon\,S$. Then we have

$$I_R(D_\varepsilon\,S) - I_R(S) = \Re\left(\varepsilon\iint_R p_S\,\overline{q_S}\,h_{\bar z}\,dx\,dy\right) + O(\varepsilon^2);$$

cf. Ahlfors' paper and also a paper due to Gerstenhaber–Rauch.

For any sufficiently small t, S_t can be obtained from T by an infinitesimal deformation D_t. Continuous differentiability of $h(z)$ is really assured by that of S_t. And moreover $\Im h(z) = 0$ on two horizontal sides and two vertical sides of R, respectively, since S_t belongs to the class $\mathfrak{H}$.

Let the sides of the rectangles be parallel to the coordinate axes and of length a, b and a', b', respectively. Then the Grötzsch's extremal affine mapping T is given by

$$T: \qquad w = \frac{1}{2}\left(\frac{a'}{a} + \frac{b'}{b}\right)z + \frac{1}{2}\left(\frac{a'}{a} - \frac{b'}{b}\right)\bar z,$$

whence $p_T\,\overline{q_T}$ is a constant equal to $K = (1/4)\cdot((a'/a)^2 - (b'/b)^2)$ on R.

Therefore we have by integration by parts

$$\frac{I_R(S_t) - I_R(T)}{t} = K\,\Re\iint_R h_{\bar z}\,dx\,dy + O(t)$$

$$= K\frac{1}{2}\left(\Im\int_0^a (h(x,\,0) - h(x,\,b))\,dx + \Re\int_0^b (h(a,\,y) - h(0,y))\,dy\right) + O(t).$$

The first member of the last expression is equal to zero by the boundary behavior of $h(z)$. Therefore, letting t tend to zero, there follows an absurdity relation

$$\left.\frac{dI_R(S_t)}{dt}\right|_{t=0} = 0.$$

This completes the proof of our theorem.

COROLLARY. *The affine mapping T has Grötzsch's extremality.*

Proof. Let $\mathfrak{H}'$ be a subclass of $\mathfrak{H}$ such that any element of $\mathfrak{H}'$ satisfies the homeomorphism. Then, if $S \in \mathfrak{H}'$, then

$$\iint_R (|p_S|^2 - |q_S|^2)\, dx\, dy = \iint_R (|p_T|^2 - |q_T|^2)\, dx\, dy = a' b'.$$

Suppose, on the contrary, that $|q_S|/|p_S| \leqq |q_T|/|p_T| \equiv k$ (constant). Then

$$\iint_R |p_S|^2\, dx\, dy \leqq \iint_R |p_T|^2\, dx\, dy.$$

Thus we have

$$I_R(S) \leqq \iint_R |p_S|^2\, dx\, dy \cdot (1 + k^2) \leqq I_R(T).$$

This leads to a contradiction by our theorem, unless S coincides with T. Thus there must hold the Grötzsch's extremality of T, that is,

$$\sup \frac{|q_S|}{|p_S|} > k = \frac{|q_T|}{|p_T|}.$$

Similarly we can discuss the cases such that both R and R' are doubly connected domains or the tori with one distinguished point.

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