SOME ESTIMATIONS ON THE SZEGÖ KERNEL FUNCTION

BY MITSURU OZAWA

Let B be an n-ply connected bounded planar domain bounded by n smooth curves Γ_i, $i=1, \cdots, n$. We shall denote the total of these curves by Γ. Let $L^2(\Gamma)$ be a class of the function $f(z)$ satisfying the following conditions:

(1) $f(z)$ is regular single-valued in B,

(2) $\int_\Gamma |f(z)|^2 ds_z$ is finite.

$L^2(\Gamma)$ is a Hilbert space with the following inner product

$$(f, g) = \int_\Gamma f(z)\overline{g(z)} ds_z, \qquad f, g \in L^2(\Gamma).$$

In $L^2(\Gamma)$, there is a reproducing kernel function $K(z, \overline{z_0})$— the *Szegö kernel function*—, where the reproducing property means the identity

$$f(z_0) = (f(z), K(z, \overline{z_0}))$$

valid for any $f(z)$ in $L^2(\Gamma)$. It is well known that the Szegö kernel satisfies the following relations:

$K(z_0, \overline{z_0}) \geqq 0$ and Hermitian property, that is, $K(z, \overline{z_0}) = \overline{K(z_0, \bar{z})}$.

In our present case, the equality sign does not occur in the first relation. These relations, together with the uniqueness of the Szegö kernel, are obtained under the effect of the reproducing property.

Let $L(z, z_0)$ be the so-called adjoint L-kernel associated with the Szegö kernel introduced by Garabedian, which satisfies the following conditions:

It is a single-valued analytic regular function except a point $z = z_0$ in B, having the expansion

$$L(z, z_0) = \frac{1}{2\pi(z - z_0)} + \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

around $z = z_0$, and on the boundary Γ of B it satisfies an important boundary relation

$$L(z, z_0) dz = i\overline{K(z, \overline{z_0})} ds.$$

Let $l(z, z_0)$ be $-\sum_{n=0}^{\infty} a_n (z - z_0)^n$ in the expansion of $L(z, z_0)$. This is a single-valued regular function belonging to the class $L^2(\Gamma)$ and satisfies the symmetric relation, namely $l(z, z_0) = l(z_0, z)$.

Received April 23, 1956.

Let $\mathfrak{B}$ be a class of single-valued regular functions satisfying the boundedness condition $|f(z)|\leqq 1$ in B and $f(z_0)=0$. Then $\mathrm{Max}_{\mathfrak{B}}|f'(z_0)|$ exists and is equal to $2\pi K(z_0, \overline{z_0})$. The extremal function is unique and maps B onto the n-times covered unit disc. From this extremal property we have that $K(z_0, \overline{z_0})$ is the monotone decreasing functional of the increasing basic domain.

For detailed explanations one can refer to the papers due to the following authors, namely Garabedian [1], Garabedian-Schiffer [2], Schifler [6] and Nehari [3], [4].

The first estimation. Let S be the length of Γ, then there holds an inequality

$$K(z_0, \overline{z_0})\cdot S\geqq 1.$$

It has been once proved by Ono (in Japanese) that B must be a simply connected domain if equality sign occurs.

For completeness we shall prove this by a quite different way. By the boundary relation for L and K, we have

$$\int_\Gamma |L(z, z_0)|^2 ds_z = -i\int_\Gamma L(z, z_0)K(z, \overline{z_0})dz$$
$$= K(z_0, \overline{z_0}).$$

Thus, by the Schwarz inequality and the residue theorem, there holds

$$1=\left|\int_\Gamma L(z, z_0)dz\right|^2\leqq\int_\Gamma |L(z, z_0)|^2 ds_z\int_\Gamma ds$$
$$\leqq K(z_0, \overline{z_0})\cdot S.$$

Let S' be the length of the outer boundary Γ_1 and B' the finite simply connected domain bounded by Γ_1. Then, by the monotoneity of $K(z_0, \overline{z_0})$ as a domain functional, we have

$$K_{B'}(z_0, \overline{z_0})\leqq K_B(z_0, \overline{z_0}),$$

the subscripts indicating the referred domains. Therefore we have a series of inequalities

$$\frac{1}{S}=K_B(z_0, \overline{z_0})\geqq K_{B'}(z_0, \overline{z_0})\geqq \frac{1}{S'}.$$

This is absurd, unless B coincides with B'.

Even if B were simply connected, the equality sign does not occur unless the configuration $[B, z_0]$ is suitably restricted. This fact is deduced by a well known fact, that is, $\lim_{z_0\to\Gamma} K(z_0, \overline{z_0})=+\infty$. Now we shall establish a precise result for occurrance of the equality sign. We may restrict B as a simply connected domain. Let $f(w)$ be a mapping function such that $|w|<1$ is mapped conformally upon B with $f(0)=z_0$. Then $(f'(w))^{1/2}$ is single-valued when we fix a branch of it. Let $S(r)$ be the length of the image curve of $|w| = r$. Then

$$S(r) = \int_{|w|=r} |f'(w)| \, |dw| .$$

Writing $(f'(w))^{1/2}$ as $\sum_{n=1}^{\infty} a_n w^n$, the usual calculation leads to a relation

$$S(r) = 2\pi \sum_{n=0}^{\infty} |a_n|^2 \cdot r^{2n+1}.$$

By the Jordan smoothness of Γ, we have

$$S(1) = 2\pi \sum_{n=0}^{\infty} |a_n|^2,$$

whence follows that

$$1 \leqq K(z_0, \overline{z_0}) \cdot S(1) = K(z_0, \overline{z_0}) \cdot 2\pi \sum_{n=0}^{\infty} |a_n|^2.$$

Referring to a result due to Garabedian, namely

$$2\pi K(z_0, \overline{z_0}) = \frac{1}{|a_0|^2}$$

in our terms, we have

$$1 \leqq 1 + 2\pi K(z_0, \overline{z_0}) \sum_{n=1}^{\infty} |a_n|^2.$$

If the equality sign occurs, then all the a_n with exception of a_0 are equal to zero, and conversely. Thus, if so, $f(w) = f(0) + a_0^2 w$, and hence the inverse function $F(z, z_0)$ of $f(w)$ can be described as

$$F(z, z_0) = \frac{1}{a_0^2}(z - z_0).$$

This shows that the domain B and the referred point z_0 must be a disc and its center, respectively, when the equality sign occurs in an inequality

$$K_B(z_0, \overline{z_0}) \geqq \frac{1}{S}.$$

The inverse statement of the above fact is evidently true.

The second estimation. We shall prove an inequality

$$K(z_0, \overline{z_0}) \leqq \frac{1}{4\pi^2} \int_\Gamma \frac{ds_z}{|z - z_0|^2}.$$

Let $[f, L]$ be a generalized inner product defined by

$$\int_\Gamma f(z)\overline{L(z, z_0)} ds_z, \qquad f(z) \in L^2(\Gamma).$$

By the boundary relation for K and L, we see

$$[f,\ L] = -i\int_{\Gamma} f(z)K(z,\ \overline{z_0})dz = 0.$$

This leads to a relation

$$(f(z),\ l(z,\ z_0)) = \frac{1}{2\pi}\left[f(z),\ \frac{1}{z-z_0}\right].$$

Putting $f(z) = l(z,\ w)$ and noting the boundary relation, we have

$$\begin{aligned}(l(z,\ w),\ l(z,\ z_0)) &= \frac{1}{4\pi^2}\int_{\Gamma}\frac{ds_z}{(z-w)\overline{(z-z_0)}} - \overline{\left(\frac{1}{2\pi i}\int_{\Gamma}\frac{K(z,\ \bar{w})}{z-z_0}dz\right)} \\ &= \frac{1}{4\pi^2}\int_{\Gamma}\frac{ds_z}{(z-w)\overline{(z-z_0)}} - K(w,\ \overline{z_0}).\end{aligned}$$

Especially setting $w = z_0$, we see

$$\int_{\Gamma}|l(z,\ z_0)|^2 ds_z + K(z_0,\ \overline{z_0}) = \frac{1}{4\pi^2}\int_{\Gamma}\frac{ds_z}{|z-z_0|^2},$$

which leads to our result.

Next we should examine the cases where the equality sign occurs. If so, $l(z,\ z_0)\equiv 0$ or $L(z,\ z_0) = 1/2\pi(z-z_0)$. Since $L(z,\ z_0)^2 dz$ is an invariant differential, we have

$$L(f(z),\ f(z_0))^2 df(z) = L(z,\ z_0)^2 dz$$

with a certain conformal mapping function $f(z)$. If $l(f(z),\ f(z_0))$ vanishes identically, then $l(z,\ z_0) = 0$ leads to a differential equation

$$\frac{f'(z)}{(f(z)-f(z_0))^2} = \frac{1}{(z-z_0)^2},$$

whose solution is given by

$$f(z) - f(z_0) = \frac{z-z_0}{c(z-z_0)+1},$$

c being a constant. This shows that the mapping funtion reduces to a linear function. On the other hand, the equality sign occurs for any disc and any inner point of this disc. This can be deduced by an easy calculation and is well known. Now putting $f(z)$ the Riemann mapping function from B onto $|w| < 1$, then we can conclude that B must be a disc when $l(z,\ z_0)\equiv 0$ happens. Here we must assume that the connectivity of B is one in number.

As is well known, $K(z,\ z_0)^2$ has the multivalued indefinite integral unless the connectivity number of B is equal to one, that is, there does not happen a fact that all the periods

$$\overline{\alpha_\nu(z_0)} = 4\pi\int_{\Gamma_\nu} K(z,\ \overline{z_0})^2 dz, \qquad \nu = 1,\ \cdots,\ n;\ n \gneqq 1,$$

reduce to zero simultaneously. By the boundary relation of K and L, we have

$$-4\pi\int_{\Gamma_\nu} L(z, z_0)^2 dz = \alpha_\nu(z_0).$$

However, if we assume that an identity

$$L(z, z_0) \equiv \frac{1}{2\pi(z-z_0)}$$

holds, then we have the following fact that all the $\alpha_\nu(z_0)$ reduce to zero, since there holds

$$\int_{\Gamma_\nu} \frac{1}{(z-z_0)^2} = 0 \qquad \text{for any } \nu \ (\nu = 1, \cdots, n).$$

This contradicts our hypothesis that the connectivity number of B is greater than one. Hence we have a result: If the equality sign occurs in our second estimation, then the connectivity number of B is equal to one and moreover B is a disc.

We should mention an interpretation obtained by the second estimation. Let B be an n-ply connected domain bounded by n circles $\Gamma_1 \cdots, \Gamma_n$. We shall put Γ_1 the outer boundary and B_1 the finite simply connected domain bounded by Γ_1 alone. Let B_j be an infinite domain bounded by a circle Γ_j. Then we have a subadditivity relation for the Szegö kernel functions of B_j, $j = 1, \cdots, n$, namely

$$K_B(z_0, \overline{z_0}) \leqq \sum_{j=1}^{n} K_{B_j}(z_0, \overline{z_0}),$$

because

$$K_{B_j}(z_0, \overline{z_0}) = \frac{1}{4\pi^2}\int_{\Gamma_j} \frac{ds_z}{|z-z_0|^2}$$

holds for any B_j.

In the general domain B,

$$\int_{\Gamma} |l(z, z_0)|^2 ds_z + K_B(z_0, \overline{z_0}) = \sum_{j=1}^{n}\int_{\Gamma_j} |l_{B_j}(z, z_0)|^2 ds_z + \sum_{j=1}^{n} K_{B_j}(z_0, \overline{z_0})$$

holds.

To investigate the location of the critical points of $K_B(z_0, \overline{z_0})$ seems very interesting. Topological or Morse theoretic tools are very useful, however the precise metrical result can not be obtained when we follow only these tools. We now mention a remarkable fact: Let B be a convex domain, then there is only one minimum point of the subharmonic function $K_B(z_0, \overline{z_0})$ continuous in B. There are no other critical points of $K_B(z_0, \overline{z_0})$. For the proof we need somewhat long but elementary considerations. For an annulus $r < |z| < 1$, there is a critical line $|z| = \sqrt{r}$. This is not so difficult to show. Details are omitted here.

The above two estimations may be considered as isoperimetric inequalities.

Let $L_n^2(\Gamma)$ be a subclass of $L^2(\Gamma)$ such that $f^{(n)}(t) = 1$ and $M_n = \inf \|f\|^2$ for $f \in L_n^2(L)$. The existence of an extremal function $F_n(z, t)$ and positivity of M_n is almost evident. Our problem is to seek the inner relations among these extremal functions.

Let $f(z)$ be an arbitrary function of $L_n^2(\Gamma)$, then $F_n(z, t) + \lambda\varphi(z)$ with $\varphi(z) \equiv f^{(n)}(t)F_n(z, t) - f(z)$ belongs to $L_n^2(\Gamma)$ for any λ. Thus we have

$$\|F_n\|^2 \leqq \|F_n + \lambda\varphi\|^2 = \|F_n\|^2 + 2\operatorname{Re}\lambda(F_n, \varphi) + |\lambda|^2\|\varphi\|^2.$$

The arbitrariness of λ leads to a fact that

$$(F_n, \varphi) = 0 \qquad \text{or} \qquad f^{(n)}(t)\|F_n\|^2 = (f, F_n).$$

Putting

$$H_n(z, \bar{t}) = \frac{F_n(z, t)}{n!M_n},$$

we have a sort of reproducing property

$$\frac{1}{n!} f^{(n)}(t) = (f(z), H_n(z, \bar{t})).$$

F_n is uniquely determined by its extremality. This is also almost obvious.

Let $H_n(z, \bar{t}) = \sum_{\mu=0}^{\infty} h_{n\mu}(z-t)^{\mu}$ in local, then $h_{nm} = \overline{h_{mn}}$, that is, the matrix (h_{mn}) is hermitian. $\{H_n\}_{n=0}^{\infty}$ is a complete system in the Hilbert space $L^2(\Gamma)$, since $(g(z), H_n(z, \bar{t})) = 0$ for any n implies that $g^{(n)}(t) = 0$ for any n and hence $g(z) \equiv 0$ on B.

Some slightly modified arguments from the ones carried out in a Schiffer's paper [6] lead to some important facts, namely

$H_n(z, \bar{t})$ is analytic on $\bar{B}$ and there exists an analytic function $L_n(z, t)$ such that $L_n(z, t)$ is regular except a point $z = t$ in B where it has a local expansion

$$L_n(z, t) = \frac{(-1)^n}{(z-t)^{n+1}} + \sum_{m=0}^{\infty} A_m(z-t)^m$$

and satisfies a boundary relation

$$L_n(z, t) = \overline{2\pi i H_n(t, \bar{z}) \frac{dt}{ds}}, \qquad t \in \Gamma.$$

We should remark that L_0 and L used in the earlier part are essentially the same but somewhat different in their normalizations.

From the last boundary relation we can proceed to a mapping problem as Schiffer has done. But we do not go further in this tendency.

$\{H_n(z, \bar{t})\}_{n=0}^{\infty}$ is linearly independent. If it is not so, then there are x_n, not all zero, such that

$$\sum_{n=0}^{N} x_n H_n(z, \bar{t}) = 0.$$

This leads to a relation

$$\sum_{n=0}^{N} x_n \overline{L_n(z,\ t)}\frac{dz}{ds} \equiv 0 \qquad \text{on } \Gamma, \qquad \text{or } \sum_{n=0}^{N} x_n L_n(z,\ t) \equiv 0 \qquad \text{on } \Gamma.$$

This remains true on the whole B. Thus we have

$$\sum_{n=0}^{N} \frac{x_n}{(z-t)^{n+1}} + \sum_{m=0}^{\infty}\sum_{n=0}^{N} x_n A_{mn}(z-t)^m \equiv 0$$

near t, whence follows that

$$\sum_{n=0}^{N} \frac{x_n}{(z-t)^{n+1}} \equiv 0$$

which leads to a fact the all x_n reduce to zero, which is absurd.

Without loss of generality we may assume that the origin is contained in B. Let $\Phi_n(z)$ be $H_n(z,\ \bar{0})$. We may expand $H_0(z,\ \bar{t})$ in terms of these $\Phi_n(z)$ as follows:

$$H_0(z,\ \bar{t}) = \sum_{\mu,\nu=0}^{\infty} q_{\mu\nu}\Phi_\mu(z)\overline{\Phi_\nu(t)} = \sum_{\mu,\nu=0}^{\infty} k_{\mu\nu} z^\mu \bar{t}^\nu.$$

The last expansion must be considered as a local one about the origin. Then we have

$$q_{\mu\nu} = \overline{q_{\nu\mu}}.$$

A short calculation leads to the following :

$$\sum_{n=0}^{\infty} q_{nm} h_{n\mu} = \delta_{m\mu},$$

$$k_{\mu\nu} = \sum_{m,n=0}^{\infty} q_{mn} h_{m\mu}\overline{h_{n\nu}} = \overline{h_{\mu\nu}} = h_{\nu\mu}.$$

All the necessary and sufficient tools have been prepared now and we can state our result as follows:

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ satisfy $\|f(z)\| \leqq 1$, then there holds

$$\left|\sum_{\nu=0}^{N} x_\nu a_\nu\right|^2 \leqq \sum_{\nu,\mu=0}^{N} x_\nu \overline{x_\mu} k_{\nu\mu}$$

for any complex numbers x_ν and for any integer N. Conversely if this condition is satisfied for a function $f(z)$ with local expansion $\sum_{n=0}^{\infty} a_n z^n$, then $f(z)$ can be continued analytically on the whole domain B and has the bounded norm $\|f\| \leqq 1$.

This result is an analogue of our earlier result in [5]. Since the situation is the same, we omit the proof here.

References

[1] P. R. Garabedian, Schwarz's lemma and the Szegö kernel function. Trans. Amer. Math. Soc. **67** (1949), 1—35.

[2] P. R. Garabedian, and M. Schiffer, On existence theorems of potential theory and conformal mapping. Ann. of Math. **52** (1950), 164—187.

[3] Z. Nehari, On bounded analytic functions. Proc. Amer. Math. Soc. **1** (1950), 268—275.

[4] Z. Nehari, Bounded analytic functions. Bull. Amer. Math. Soc. **57** (1951), 354—366.

[5] M. Ozawa, On functions of bounded Dirichlet integral. Kōdai Math. Sem. Rep. (1952), 95—98.

[6] M. Schiffer, On various types of orthogonalization. Duke Math. Journ. **17** (1950), 329—366.

Department of Mathematics,
Tokyo Institute of Technology.