EXISTENCE OF GREATEST DECOMPOSITION OF A SEMIGROUP

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§ 1. Let us consider letters x_1, x_2, ..., x_n placed in a row permitting repetition, for example, $x_2 x_1^2 x_3^3 x_n x_1 \cdots x_{n-1}^4$. Such a form is called a monomial of x_1, ..., x_n and is denoted by $f(x_1, \cdots, x_n)$ etc. If x_1, ..., x_n are adopted as elements of a semigroup T, then $f(x_1, \cdots, x_n)$ represents a product of elements in T. Suppose that a semigroup T fulfils a suitable system of equalities:

$$f_\lambda(x_1, \cdots, x_n) = g_\lambda(x_1, \cdots, x_n), \quad \lambda \in \Lambda,$$

$$\text{for all } x_1, \cdots, x_n \in T,$$

where x_1, ..., x_n vary independently and each side of the equalities needs not contain all of letters x_1, ..., x_n, but letters appearing in both sides are x_1, ..., x_n; for example, $f_1(x, y) = xy$, $g_1(x, y) = x$. Then T is called a semigroup with monomial conditions $f_\lambda = g_\lambda$, $\lambda \in \Lambda$. Of course a one-element semigroup $\{x\}$ is one of this kind.
In this short note, we shall prove the existence of greatest decomposition of a semigroup S to a semigroup T with $f_\lambda = g_\lambda$, $\lambda \in \Lambda$, which turns out to be an expansion of the theorem in the previous paper [1].

§ 2. Now let D be all decompositions of S to a semigroup T with $f_\lambda = g_\lambda$, $\lambda \in \Lambda$, and $\overset{d}{\sim}$ be a congruence relation arising $d \in D$. The following lemma is clear.

Lemma 1. $\overset{d}{\sim}$ is a congruence relation arising a decomposition d of S to a semigroup T with $f_\lambda = g_\lambda$, $\lambda \in \Lambda$, if and only if

(1) $x \overset{d}{\sim} x$,
(2) $x \overset{d}{\sim} y$ implies $y \overset{d}{\sim} x$,
(3) $x \overset{d}{\sim} y$ implies $xz \overset{d}{\sim} yz$ and $zx \overset{d}{\sim} zy$,
(4) $f_\lambda(x_1, x_2, \cdots, x_n) \overset{d}{\sim} g_\lambda(x_1, x_2, \cdots, x_n)$, $\lambda \in \Lambda$.

Theorem. D is a complete lattice.

Proof. We define $d_\alpha \gtrsim d_\beta$ to mean that $x \overset{d_\alpha}{\sim} y$ implies $x \overset{d_\beta}{\sim} y$. Then D is a partly ordered set and D contains a least element, i.e. a mapping of all elements of S into one class. In order to verify that D is a complete lattice, it is sufficient to show that any subset D' of D has a least upper bound in D [2]. Now we define $x \overset{d_1}{\sim} y$ to mean $x \overset{d}{\sim} y$ for all $d \in D'$. Since every $\overset{d}{\sim}$ is a congruence relation, it is proved easily that $\overset{d_1}{\sim}$ is also so, that is, (1) $x \overset{d_1}{\sim} x$, (2) $x \overset{d_1}{\sim} y$ implies $y \overset{d_1}{\sim} x$, (3) $x \overset{d_1}{\sim} y$ implies $xz \overset{d_1}{\sim} yz$ and $zx \overset{d_1}{\sim} zy$. Moreover $f_\lambda(x_1, x_2, \cdots, x_n) \overset{d_1}{\sim} g_\lambda(x_1, x_2, \cdots, x_n)$ because $f_\lambda(x_1, x_2, \cdots, x_n) \overset{d}{\sim} g_\lambda(x_1, x_2, \cdots, x_n)$ for all $d \in D'$. Obviously $x \overset{d_1}{\sim} y$ implies $x \overset{d}{\sim} y$ for all $d \in D'$; hence a decomposition d_1 is an upper bound of D'. Let d_1' be any upper bound of D'. Then $x \overset{d_1'}{\sim} y$ implies $x \overset{d}{\sim} y$ for all $d \in D'$ so that $x \overset{d_1}{\sim} y$, that is to say, $d_1' \gtrsim d_1$; d_1 is a least upper bound of D'. Thus the proof of the theorem has been completed. Accordingly we have

Corollary. There is a greatest decomposition of a semigroup S to a semigroup T with $f_\lambda = g_\lambda$, $\lambda \in \Lambda$.

§ 3. We shall give several important examples of T.

1. Left singular semigroup, i.e. a semigroup satisfying $xy = x$,

$$f_1(x, y) = xy, \quad g_1(x, y) = x.$$

Right singular semigroup, i.e. a semigroup satisfying ,

$$f_1(x, y) = xy, \quad g_1(x, y) = y.$$

2. Commutative semigroup,

$$f_1(x, y) = xy, \quad g_1(x, y) = yx.$$

3. Idempotent semigroup,

$$f_1(x)=x^2 \quad , \quad g_1(x)=x \ .$$

4. Semilattice

$$f_1(x,y)= xy \quad , \quad g_1(x,y)=yx \ ,$$

$$f_2(x) = x^2 \quad , \quad g_2(x,y)=x \ .$$

5. A semigroup satisfying a condition $xyx=x$ for all x , y ,

$$f_1(x,y)=xyx \quad , \quad g_1(x,y)=x \ .$$

6. A semigroup satisfying a condition $(xy)^n=x^n y^n$,

Addenda

We should like to correct the incompleteness of Lemma 2 involved in our previous paper [1], p. 109. In the proof of Lemma 2, there is an omission to prove commutativity. But the general theory in this paper will show that the Lemma is true. We express many thanks to Professor A. H. Clifford for his kind guidance and his pointing out our omission.

References

[1] T. Tamura & N. Kimura, On decompositions of a commutative semigroup, Kōdai Math. Sem. Rep., 1954, pp. 109-112.

[2] G. Birkhoff, Lattice theory, p. 49.

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