MEAN CONVERGENCE OF A FOURIER SERIES AND A FOURIER TRANSFORM

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1. Let $f(x)$ be a function of $L_1(-\pi,\pi)$, and suppose that $f(x)$ belongs to L_r on a subinterval (a,b) of $(-\pi,\pi)$, $r>1$. We consider about the convergence in mean (L_r) over (a,b) of the Fourier series of $f(x)$. It is well known that if (a,b) coincides with $(-\pi,\pi)$, then the Fourier series of $f(x)$ converges to $f(x)$ in mean L_r over $(-\pi,\pi)$.

Thus if we define $\lambda(x)$ as unity in (a,b) and 0 outside of (a,b), then the Fourier series of $f(x)\lambda(x)$ which is of $L_r(-\pi,\pi)$ converges in mean (L_r) to $f(x)\lambda(x)$ over $(-\pi,\pi)$, and consequently converges in mean (L_r) to $f(x)$ on (a,b). Since $f(x)\lambda(x)$ and $f(x)$ is identical in (a,b), the Fourier series of $f(x)\lambda(x)$ and of $f(x)$ are uniformly equiconvergent in $(a+\varepsilon, b-\varepsilon)$, ε being a positive number arbitrarily small but fixed. Hence the Fourier series of $f(x)$ converges in mean (L_r) to $f(x)$ in $(a+\varepsilon, b-\varepsilon)$, $\varepsilon>0$.

But the Fourier series of $f(x)$ does not necessarily converge in mean (L_r) on (a,b), which is implied in the fact that will be stated later. (3. (3.5)) And thus we shall consider additional conditions on the behaviors of $f(x)$ at vicinities of $x=a$ and $x=b$ for the mean convergence (L_r) in (a,b).

Also we consider the similar problem in the theory of Fourier transforms. For the sake of convenience we first treat the Fourier transform case. Though we can treat the Fourier series case by similar arguments, we shall deduce it from theorems for Fourier transforms.

2. <u>Theorem 1. Let</u> $1\leq p\leq 2$ <u>and</u> $p\leq r$, <u>and</u> $f(x)\in L_p(-\infty,\infty)$, $f(x)\in L_r(a,b)$. <u>If there exist constants</u> s_1 <u>and</u> s_2 <u>such that</u>

$$(2.1)\quad \int_0^t |f(b+x)-s_2|\,dx = O(t^{\alpha}),\quad t>0,$$

$$(2.2)\quad \int_0^t |f(a-x)-s_1|\,dx = O(t^{\alpha}),\quad t>0,$$

<u>for small</u> t, <u>where</u> $\alpha>1-\frac{1}{r}$, <u>then it holds that</u>

$$(2.3)\quad \lim_{N\to\infty}\int_a^b |R_N(x)|^r\,dx = 0.$$

<u>Here we denote</u>

$$(2.4)\qquad R_N(x) = \int_{-N}^{N}\varphi(t)e^{-ixt}\,dt - \int_{-N}^{N}\psi(t)e^{-ixt}\,dt,$$

$\varphi(t)$ <u>and</u> $\psi(t)$ <u>being Fourier transforms of</u> $f(x)$ <u>and</u> $\lambda(x)f(x)$ <u>respectively,</u>

$$\varphi(t)\sim\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ixt}f(x)\,dx$$

$$(2.5)\qquad \psi(t)\sim\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ixt}f(x)\lambda(x)\,dx$$

<u>and</u> $\lambda(x)=1$ <u>in</u> $a<x<b$; 0 <u>outside of</u> (a,b).

From Theorem 2 the following theorem is obtained immediately.

<u>Theorem 2. Let</u> $1\leq p\leq r\leq 2$. <u>If</u> $f(x)\in L_p(-\infty,\infty)$ <u>and</u> $f(x)\in L_r(a,b)$ <u>and further</u> (2.1) <u>and</u> (2.2) <u>holds for</u> $\alpha>1-\frac{1}{r}$, <u>then</u>

$$\lim_{N\to\infty}\int_a^b\left|\frac{1}{\sqrt{2\pi}}\int_{-N}^{N}\varphi(t)e^{-ixt}\,dt - f(x)\right|^r dx = 0$$

<u>as</u> $N\to\infty$, <u>where</u> $\varphi(t)$ <u>is the Fourier transform of</u> $f(x)$.

Since $\lambda(x)f(x)\in L_r(-\infty,\infty)\ (r\leq 2)$

$\frac{1}{\sqrt{2\pi}}\int_{-N}^{N}\psi(t)e^{-ixt}dt$ converges in mean (L_r) to $\lambda(x)f(x)$ and particularly $\frac{1}{\sqrt{2\pi}}\int_{-N}^{N}\psi(t)e^{-ixt}dt$ converges in mean to $f(x)$ in (a, b), $\psi(t)$ being the Fourier transform of $\lambda(x)f(x)$, and Theorem 1 shows Theorem 2.

Proof of Theorem 1. We may assume without loss of generality that $a=-h$, $b=h$ $(h>0)$. Then the Fourier transform of $\lambda(x)$ is

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\lambda(x)e^{itx}dx=\sqrt{\frac{2}{\pi}}\,\frac{\sin ht}{t},$$

and the Fourier transform $\psi(t)$ of $\lambda(x)f(x)$ $(\in L_p(-\infty,\infty))$ is

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\varphi(u)\frac{\sin h(u-t)}{u-t}\,du.$$

Thus we have

$$(2.6)\quad R_N(x)$$
$$=\int_{-N}^{N}\varphi(t)e^{-itx}dt$$
$$-\int_{-N}^{N}e^{-itx}\frac{1}{\pi}\int_{-\infty}^{\infty}\varphi(u)\frac{\sin h(u-t)}{u-t}du$$

Since

$$\lambda(x)=\frac{1}{\pi}\lim_{T\to\infty}\int_{-T}^{T}\frac{\sin ht}{t}e^{ixt}dt,$$

which is 1 in $(-h, h)$, we have for $-h<x<h$,

$$\pi R_N(x)$$
$$=\lim_{T\to\infty}\int_{-T}^{T}\frac{\sin ht}{t}e^{ixt}dt\int_{-N}^{N}\varphi(u)e^{-ixu}du$$
$$-\int_{-\infty}^{\infty}\frac{\sin ht}{t}dt\int_{-N}^{N}\varphi(t+u)e^{-ixu}du$$
$$=\lim_{T\to\infty}\left[\int_{-T}^{T}\frac{\sin ht}{t}e^{ixt}dt\int_{-N}^{N}\varphi(u)e^{-ixu}du\right.$$
$$\left.-\int_{-\infty}^{\infty}\frac{\sin ht}{t}e^{ixt}dt\int_{-N+t}^{N+t}\varphi(u)e^{-ixu}du\right]$$
$$=-\lim_{T\to\infty}\int_{-T}^{T}\frac{\sin ht}{t}e^{ixt}dt\int_{N}^{N+t}\varphi(u)e^{-ixu}du$$
$$+\lim_{T\to\infty}\int_{-T}^{T}\frac{\sin ht}{t}e^{ixt}dt\int_{-N}^{-N+t}\varphi(u)e^{-ixu}du$$

$$(2.7)\quad =-I_1+I_2,$$

say. Let $\mu(u)=\mu(u;N,t)=1$ for $N<u<N+t$, $=0$ otherwise.

Then the Fourier transform of $\mu(u)$ is

$$\frac{1}{\sqrt{2\pi}}\;\frac{e^{ixN}(1-e^{itx})}{-ix}$$

Thus we can write

$$\int_{N}^{N+t}\varphi(u)e^{-ixu}du$$
$$=\int_{-\infty}^{\infty}\varphi(u)\mu(u)e^{-ixu}$$
$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x+y)\frac{e^{-Ny}(e^{ity}-1)}{iy}dy,$$

and hence

$$I_1=\frac{1}{\sqrt{2\pi}}\lim_{T\to\infty}\int_{-T}^{T}\frac{\sin ht}{t}e^{ixt}dt\int_{-\infty}^{\infty}f(x+y)\frac{e^{iNy}(e^{ity}-1)}{iy}dy$$

$$(2.8)\quad =\frac{1}{\sqrt{2\pi}}\lim_{T\to\infty}\int_{-\infty}^{\infty}f(x+y)e^{iNy}dy\int_{-T}^{T}\frac{(e^{ity}-1)}{iy}\frac{\sin ht}{t}e^{ixt}dt.$$

We shall now prove that the interchange of the order of limit and integration is legitimate. Fix the value of x for a moment such that $h>x>-h$.

If $A>|x+h|, |x-h|$ and $A>1$, then the integral

$$\int_{-T}^{T}(e^{ity}-1)\frac{\sin ht}{t}e^{ixt}dt$$

converges boundedly (with respect to y). Thus if $|x|\neq h$, then

$$(2.9)\quad \lim_{T,T'\to\infty}\int_{|y|>A}f(x+y)\frac{e^{iNy}}{iy}dy\cdot\int_{T'>|t|>T}(e^{ity}-1)\frac{\sin ht}{t}e^{ixt}dt=0.$$

We take a positive number η such that

$h-x>\eta$, $x+h>\eta$

and

(2.9) $\int_{-\eta}^{\eta} |f(x+y)|\, dy < \varepsilon,$

where ε is any assigned positive number. Then

$$\limsup_{T,T'\to\infty} \Big| \int_{-A}^{A} f(x+y) e^{iNy} dy \int_{T'>|x_i|>T} \frac{e^{ity}-1}{iy} \frac{\sin ht}{t} e^{ixt} dt \Big|$$

$$\leqq \limsup_{T,T'\to\infty} \Big| \int_{-\eta}^{\eta} \Big| + \limsup_{T,T'\to\infty} \Big| \int_{A>|y|>\eta} \Big|$$

(2.10) $= J_1 + J_2,$

say. In J_1, $|y| < \eta$. And

$$\Big(\int_{-T'}^{-T} + \int_{T}^{T'}\Big)(e^{ity}-1)\frac{\sin ht}{t} e^{ixt} dt$$

$$= \frac{1}{i}\Big(\int_{-T'}^{-T} + \int_{T}^{T'}\Big) \frac{(\cos(x+y) - \cos xt)\sin ht}{t} dt$$

$$+ \Big(\int_{-T'}^{-T} + \int_{T}^{T'}\Big) \frac{(\sin(x+y)t - \sin xt)\sin ht}{t} dt$$

(2.11) $= \frac{1}{i} J_3 + J_4,$

say. J_4 being zero, since the integr and is an odd function. We have

$$J_3 = \frac{1}{2}\Big(\int_{-T'}^{-T} + \int_{T}^{T'}\Big) \frac{\sin(x+y+h)t - \sin(x+h)t}{t} dt$$

(2.12) $+ \frac{1}{2}\Big(\int_{-T'}^{-T} + \int_{T}^{T'}\Big) \frac{\sin(x+y-h)t - \sin(x-h)t}{t} dt.$

The former integral of (2.12) is further

$$= \Big(\int_{-(x+y+h)T'}^{-(x+y+h)T} + \int_{(x+y+h)T}^{(x+y+h)T'}\Big) \frac{\sin u}{u} du$$

(2.13) $- \Big(\int_{-(x+h)T'}^{-(x+h)T} + \int_{(x+h)T}^{(x+h)T'}\Big) \frac{\sin u}{u} du$

say, remembering $x+h > 0$, $x+y+h > 0$ which is

$$\int_{-(x+y+h)T'}^{-(x+h)T} - \int_{-(x+y+h)T}^{-(x+h)T} - \int_{(x+h)T}^{(x+y+h)T} + \int_{(x+h)T'}^{(x+y+h)T'}$$

$$= O\Big(\int_{(x+h)T'}^{(x+y+h)T'} \frac{du}{u}\Big) + O\Big(\int_{(x+h)T}^{(x+y+h)T} \frac{du}{u}\Big)$$

$$= O\Big(\log\Big(1 + \frac{y}{x+h}\Big)\Big)$$

$$= O(y),$$

for small y. The same estimation will be obtained for the last integral of (2.12). Hence

(2.14) $J_3 = O(y)$

for small y for fixed x.
Putting (2.13) into J_1, we get

$$|J_1|$$

$$\leqq \limsup_{T,T'\to\infty} \Big| \int_{-\eta}^{\eta} f(x+y) \frac{e^{iNy}}{y} dy \int_{T'>|x|>T} (e^{ity}-1) \frac{\sin ht}{t} e^{ixt} dt \Big|$$

$$\leqq \int_{-\eta}^{\eta} |f(x+y)| \frac{1}{y} O(y)\, dy$$

$$= O\Big(\int_{-\eta}^{\eta} |f(x+y)|\, dy\Big).$$

Hence by (2.9), there exists a constant C (which may depend on x) such that

(2.14) $|J_1| \leqq C\varepsilon$

Next we estimate J_2. We devide the integral containing J_3 as follows:

$$\limsup_{T,T'\to\infty} \Big| \int_{A>|y|>\eta} J_3 f(x+y) e^{iNy} \frac{dy}{y} \Big|$$

$$= \limsup_{T,T'\to\infty} \Big| \int_{|x+h+y|>\eta',\, A>|y|>\eta} + \int_{|x+h+y|<\eta',\, A>|y|>\eta} \Big|$$

$$= \limsup_{T,T'\to\infty} |J_5 + J_6|,$$

say, where η' is chosen so that

$$\int_{|x+h+y|<\eta'} |f(x+y)|\, dy < \varepsilon,$$

ε being any given positive number. The similar arguments as in the estimation of J_1 show that

$$(2.15)\qquad \limsup_{T,T'\to\infty} |J_6| \leqq C\varepsilon,$$

C being a constant.(1) In J_5, every integral in (2.13) is clearly convergent boundedly to zero as $T, T'\to\infty$, since $|x+y+h|>\eta'$ and $x+h>0$. Noticing $|y|>\eta$, we get

$$(2.16)\qquad \lim_{T,T'\to\infty} J_5 = 0.$$

(2.15) and (2.16) show that

$$(2.17)\qquad |J_2| \leqq C\varepsilon.$$

Thus putting (2.16), (2.17) and combining with (2.9), we get, since ε is arbitrary,

$$(2.18)\qquad \lim_{T,T'\to\infty} \int_{-\infty}^{\infty} f(x+y)\frac{e^{iNy}}{y}\, dy \cdot \int_{T'>|t|>T} (e^{ity}-1)\frac{\sin ht}{t} e^{ixt}\, dt = 0$$

for $|x|\neq h$. This means that the change of order of limit and integration in (2.8) can be permissible. Thus we get

$$I_1 = \sqrt{\frac{\pi}{2}} \int_{-\infty}^{\infty} f(x+y)\frac{e^{iNy}}{iy}\, dy \cdot \frac{1}{\pi}\lim_{T\to\infty}\int_{-T}^{T} \frac{\sin ht}{t}(e^{ity}-1)e^{ixt}\, dt.$$

Now since

$$\frac{1}{\pi}\lim_{T\to\infty}\int_{-T}^{T}\frac{\sin ht}{h} e^{i(x+y)t}\, dt = \begin{cases} 0, & |x+y|>h \\ 1, & |x+y|<h \end{cases}$$

$$\frac{1}{\pi}\lim_{T\to\infty}\int_{-T}^{T}\frac{\sin ht}{t} e^{ixt}\, dt = \begin{cases} 0, & |x|>h, \\ 1, & |x|<h, \end{cases}$$

we have, for $-h<x<h$,

$$I_1 = I_1(x)$$

$$= i\sqrt{\frac{\pi}{2}}\left(\int_{-x+h}^{\infty} f(x+y)\frac{e^{iNy}}{y}\, dy + \int_{-\infty}^{-x-h} f(x+y)\frac{e^{iNy}}{y}\, dy\right)$$

$$= i\sqrt{\frac{\pi}{2}}\left(\int_{h}^{\infty} f(y)\frac{e^{iN(y-x)}}{y-x}\, dy + \int_{-\infty}^{-h} f(y)\frac{e^{iN(y-x)}}{y-x}\, dy\right)$$

$$= i\sqrt{\frac{\pi}{2}}\, e^{-iNx}\left(\int_{h}^{\infty} \frac{f(y)}{y-x}e^{iNy}\, dy + \int_{-\infty}^{-h}\frac{f(y)}{y-x}e^{iNy}\, dy\right)$$

$$(2.19)\qquad = i\sqrt{\frac{\pi}{2}}\, e^{-iNx}(J_7 + J_8),$$

say.

Let δ be any positive number such that $\delta<2h$ and put

$$J_7 = \left(\int_{h}^{h+\delta} + \int_{h+\delta}^{\infty}\right)\frac{f(y)}{y-x}e^{iNy}\, dy$$

$$(2.20)\qquad = J_9 + J_{10},$$

say. The integrand of J_{10} is absolutely integrable and

$$\int_{h+\delta}^{\infty}\left|\frac{f(y)}{y-x}\right| dy \leqq \frac{h+\delta}{\delta}\int_{h+\delta}^{\infty}\frac{|f(y)|}{y}\, dy.$$

Thus for $-h<x<h$, by Riemann-Lebesque theorem

$$\lim_{N\to\infty} J_{10} = 0$$

holds boundedly and consequently for every $r>1$

$$(2.21)\qquad \lim_{N\to\infty}\int_{-h}^{h} |J_{10}|^r\, dx = 0.$$

Now we consider the integral

$$K = \int_{h}^{h+\delta}\frac{e^{iNy}}{y-x}\, dy.$$

$$\left|\int_{h}^{h+\delta}\frac{e^{iNy}}{y-x}\, dy\right| \leqq \int_{h}^{h+\delta}\frac{dy}{y-x} = \log\left(1+\frac{\delta}{h-x}\right),$$

and so that

$$\int_{-h}^{h} K^r\, dx = \int_{-h}^{h}\log^r\left(1+\frac{\delta}{h-x}\right) dx$$

$$= \int_0^{2h} \log^r\left(1+\frac{\delta}{u}\right) du = \delta \int_{\delta/2h}^{\infty} \log^r(1+v) \frac{dv}{v^2}$$

$$= \delta \left(\int_{\delta/2h}^{\xi} + \int_{\xi}^{V} + \int_{V}^{\infty} \right),$$

where ξ and V be chosen so that

$$\log^r(1+v) < v^{\mu}, \quad (0<\mu<1), \text{ for } v > V,$$

$$0 < \xi < 1.$$

Then

$$\delta \int_V^{\infty} \log^r(1+v) \frac{dv}{v^2} \leqq \delta \int_V^{\infty} \frac{dv}{v^{2-\alpha}} \leqq C\delta,$$

$$\delta \int_{\xi}^{V} \log^r(1+v) \frac{dv}{v^2} = C\delta,$$

and

$$\delta \int_{\delta/2h}^{\xi} \frac{\log^r(1+v)}{v^2} dv \leqq \delta \int_{\delta/2h}^{\xi} \frac{v^r}{v^2} dv$$

$$\leqq \delta \int_0^{\xi} \frac{dv}{v^{2-r}} = C\delta,$$

noticing $r>1$. Hence we get

$$(2.22) \quad \int_{-h}^{h} |K|^r dy \leqq C\delta.$$

Hence we have

$$\int_{-h}^{h} |J_q|^r dx = \int_{-h}^{h} dx \left| \int_h^{h+\delta} \frac{f(y)}{y-x} e^{iNy} dy \right|^r$$

$$= \int_{-h}^{h} dx \left| \int_h^{h+\delta} \frac{f(y)-S_2}{y-x} e^{iNy} dy + S_2 \int_h^{h+\delta} \frac{e^{iNy}}{y-x} dy \right|^r$$

$$\leqq C \int_{-h}^{h} dx \left| \int_h^{h+\delta} \frac{f(y)-S_2}{y-x} e^{iNy} dy \right|^r$$

$$+ C \int_{-h}^{h} |K|^r dx$$

$$(2.23) \quad \leqq C \int_{-h}^{h} dx \left(\int_h^{h+\delta} \frac{|f(y)-S_2|}{y-x} dy \right)^r + C\delta$$

by (2.22). We shall estimate the first term. Now let $G(x) \geqq 0$ be any function. By Holder's inequality, we have

$$\int_{-h}^{h} G(x) dx \int_h^{h+\delta} \frac{|f(y)-S_2|}{y-x} dy$$

$$= \int_h^{h+\delta} |f(y)-S_2| dy \int_{-h}^{h} \frac{G(x)}{y-x} dx$$

$$\leqq \int_h^{h+\delta} |f(y)-S_2| dy \left(\int_{-h}^{h} G^{r'}(x) dx \right)^{\frac{1}{r'}} \left(\int_{-h}^{h} \frac{dx}{(y-x)^r} \right)^{\frac{1}{r}}$$

$$(1/r + 1/r' = 1)$$

$$\leqq \frac{1}{r-1} \int_h^{h+\delta} |f(y)-S_2| dy \left(\int_{-h}^{h} G^{r'}(x) dx \right)^{\frac{1}{r'}} \frac{1}{(y-h)^{1-1/r}}.$$

Taking

$$G(x) = \left(\int_h^{h+\delta} \frac{|f(y)-S_2|}{y-x} dy \right)^{r-1},$$

we have

$$\int_{-h}^{h} dx \left(\int_h^{h+\delta} \frac{|f(y)-S_2|}{y-x} dy \right)^r$$

$$\leqq \frac{1}{r-1} \left\{ \int_{-h}^{h} dx \left(\int_h^{h+\delta} \frac{|f(y)-S_2|}{y-x} dy \right)^r \right\}^{\frac{r-1}{r}} \cdot$$

$$\cdot \int_h^{h+\delta} \frac{|f(y)-S_2|}{(y-h)^{1-1/r}} dy$$

from which it results

$$(2.24) \quad \int_{-h}^{h} dx \left(\int_h^{h+\delta} \frac{|f(y)-S_2|}{y-x} dy \right)^r$$

$$\leqq \frac{1}{r-1} \int_h^{h+\delta} \frac{|f(y)-S_2|}{(y-h)^{1-1/r}} dy^{(2)}.$$

If we put $\Phi(y) = \int_h^{y} |f(y)-S_2| dy$ $= \int_0^{y-h} |f(h+u)-S_2| du$, then by (2.1)

$$\Phi(y) = O((y-h)^{\alpha}), \quad \alpha > 1 - \frac{1}{r}.$$

And thus the right side of (2.24) becomes, by integration by parts,

$$\frac{1}{r-1} \left[\frac{\Phi(y)}{(y-x)^{1-1/r}} \right]_h^{h+\delta} + \frac{1}{r} \int_h^{h+\delta} \frac{\Phi(y)}{(y-h)^{2-1/r}} dy$$

$$\leqq C\delta^{\alpha-1+\frac{1}{r}} + C \int_h^{h+\delta} (y-h)^{\alpha-2+1/r} dy$$

$$\leqq C\delta^{\alpha-1+\frac{1}{r}}.$$

Hence putting this result into (2.23), we have

$$\int_{-h}^{h}|J_9|^r dx \leqq C\delta.$$

Combining with (2.21) and putting into (2.20) we get

$$\limsup_{N\to\infty}\int_{-h}^{h}|J_7|^r dx \leqq C\delta.$$

Since δ is arbitrary, we finally get

$$\lim_{N\to\infty}\int_{-h}^{h}|J_7|^r dx = 0$$

Under the assumption (2.2), we also have, by similar reasoning,

$$\lim_{N\to\infty}\int_{-h}^{h}|J_8|^r dx = 0.$$

Or we have

$$\lim_{N\to\infty}\int_{-h}^{h}|I_1|^r dx = 0.$$

Similarly we have

$$\lim_{N\to\infty}\int_{-h}^{h}|I_2|^r dx = 0$$

These results prove Theorem 2.

3. In this section we consider the Fourier series case. As we have stated in §1, we deduce corresponding theorems for Theorem 1 and 2.

Theorem 3. Let $f(x)\in L_p(-\pi,\pi)$, $1\leqq p<\infty$ and periodic with period 2π, and $f(x)\in L_r$ in a subinterval (a,b) of $(-\pi,\pi)$, r being not less than p. If $\alpha>1-\frac{1}{r}$ and there exist constants s_1 and s_2 such that

(3.1) $$\int_0^t |f(b+x)-s_2|dx = O(t^\alpha),\ t>0$$

(3.2) $$\int_0^t |f(a-x)-s_1|dx = O(t^\alpha),\ t>0$$

for small t, and we put

$$R_N(x) = s_N(x) - S_N(x),$$

where $S_N(x)$ is the partial sums of the Fourier series of $\lambda(x)f(x)$ ($\lambda(x) = 1$, in (a,b); $=0$, otherwise), then

(3.3) $$\lim_{N\to\infty}\int_a^b |R_N(x)|^r dx = 0.$$

Theorem 4. Let $f(x)\in L_p(-\pi,\pi)$, $1\leqq p<\infty$ and periodic with period 2π and $f(x)\in L_r$ in a subinterval (a,b) of $(-\pi,\pi)$, $p\leqq r$. If $\alpha>1-\frac{1}{r}$ and (3.1) and (3.2) hold, then

(3.4) $$\lim_{n\to\infty}\int_a^b |s_n(x)-f(x)|^r dx = 0.$$

Since by similar arguments, we can deduce these theorems from Theorem 1 and 2, we shall prove Theorem 4 only.

Proof of Theorem 4. We may suppose without loss of generality that $\alpha<1$ and $a=-h$, $b=h$. Let

$$l(t) = \begin{cases} 1-2|t|, & 1/2\geqq t\geqq -\frac{1}{2} \\ 0, & \text{otherwise}, \end{cases}$$

Then the Fourier transform of $l(t)$ is $\sqrt{\frac{2}{\pi}}\frac{\sin^2 x/4}{x^2/4}$. Thus we have

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\sin^2 x/4}{x^2/4}e^{i(n+t)x}dx = \sqrt{\frac{\pi}{2}}\,l(n+t)$$

and we put

$$\varphi_N(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ixt}\Big(\sum_{n=-N}^{N}c_n e^{inx}\Big)\frac{\sin^2 x/4}{x^2/4}dx,$$

where c_n is the Fourier coefficients of $f(x)$,

$$f(x)\sim\sum_{-\infty}^{\infty}c_n e^{inx}.$$

Then we have

$$\varphi_N(t) = c_n\sqrt{\frac{\pi}{2}}\,l(n+t),\ \text{for}\ n-\tfrac{1}{2}\leqq t\leqq -n+\tfrac{1}{2}$$

$$(n=0,\ \pm1,\ \cdots,\ \pm N)$$

$$= 0,\ \text{for}\ |t|>N+\tfrac{1}{2}.$$

Now $f(x)\frac{\sin^2 x/4}{x^2/4}\in L_q(-\infty,\infty)$ for every q such that $1\leqq q\leqq p$. For $f(x)\in L_q$ in every finite interval and

$$\int_{-\infty}^{\infty}\left|f(x)\frac{\sin^2 x/4}{x^2/4}\right|^q dx=\sum_{-\infty}^{\infty}\int_{2n\pi}^{2(n+1)\pi}\left|f(x)\frac{\sin^2 x/4}{x^2/4}\right|^q dx$$
$$\leqq C\int_{-\pi}^{\pi}|f(x)|^q dx+C\sum_{n=2}^{\infty}\frac{1}{n^{2q}}\int_{2n\pi}^{2(n+1)\pi}|f(x)|^q dx$$
$$\leqq C\int_{-\pi}^{\pi}|f(x)|^q dx+C\int_{-\pi}^{\pi}|f(x)|^q dx\sum_{1}^{\infty}\frac{1}{n^{2q}}$$
$$<\infty .$$

Hence we may assume, without loss of generality that $1\leqq p\leqq 2$. Hence $f(x)\frac{\sin^2 x/4}{x^2/4}$ has the Fourier transform $\varphi_\infty(t)$ and we can easily see that

$$\varphi_\infty(t)=\lim_{N\to\infty}\varphi_N(t)=c_n\sqrt{\frac{\pi}{2}}\,l(n+t)$$

for $n-\frac{1}{2}\leqq t\leqq n+\frac{1}{2}$, for $n=0,\pm1,\pm2,\dots$.
For if the Fourier transform of $f(x)\frac{\sin^2 x/4}{x^2/4}$ be $F(t)$, then $\{\sum_{-N}^{N}c_n e^{inx}-f(x)\}\frac{\sin^2 x/4}{x^2/4}$ has the Fourier transform $\varphi_N(t)-F(t)$, and by Hausdorff-Young theorem shows that

$$\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|\varphi_N(t)-F(t)|^{p'}dt\right\}^{\frac{1}{p'}}$$
$$\leqq\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left|\sum_{-N}^{N}c_n e^{inx}-f(x)\right|^p\cdot\left\{\frac{\sin^2 x/4}{x^2/4}\right\}^p dx\right\}^{\frac{1}{p}},$$

where $1/p+1/p'=1$ and if $p=1$, then $p'=\infty$ and the left side means $\max_{-\infty<t<\infty}|\varphi_N(t)-F(t)|$. Letting $N\to\infty$, the right hand side becomes (C being a constant independent of N)

$$\lim_{N\to\infty}\int_{-\infty}^{\infty}\left|\sum_{-N}^{N}c_n e^{inx}-f(x)\right|^p\left\{\frac{\sin^2 x/4}{x^2/4}\right\}^p dx$$
$$\leqq \limsup_{N\to\infty} C\int_{-\pi}^{\pi}\left|\sum_{-N}^{N}c_n e^{inx}-f(x)\right|^p dx$$
$$+\limsup_{N\to\infty}\left[\left(\sum_{k=0}^{\infty}+\sum_{-\infty}^{-2}\right)\int_{(2k+1)\pi}^{(2k+3)\pi}\left|\sum_{-N}^{N}c_n e^{inx}-f(x)\right|^p\left\{\frac{\sin^2 x/4}{x^2/4}\right\}^p dx\right]$$
$$\leqq C\limsup_{N\to\infty}\int_{-\pi}^{\pi}\left|\sum_{-N}^{N}c_n e^{inx}-f(x)\right|^p dx$$
$$+C\sum_{k=0}^{\infty}\frac{1}{\{(2k+1)\pi\}^p}\limsup_{N\to\infty}\int_{-\pi}^{\pi}\left|\sum_{-N}^{N}c_n e^{inx}-f(x)\right|^p dx=0.$$

Hence $\varphi_N(t)$ converges in mean $L_{p'}$ to $F(t)$, but $\lim_{N\to\infty}\varphi_N(t)=\varphi_\infty(t)$, and hence $F(t)$ is identical almost everywhere with $\varphi_\infty(t)$.

Now it is easily verified that if $f(x)$ satisfies the conditions (3.1) and (3.2), then $f(x)\frac{\sin^2 x/4}{x^2/4}$ also satisfies (3.1) and (3.2) with $S_2\frac{\sin^2 b/4}{b^2/4}$, $S_1\frac{\sin^2 a/4}{a^2/4}$ in place of S_2, S_1 respectively. Thus putting $f(x)\frac{\sin^2 x/4}{x^2/4}$ instead of $f(x)$ in Theorem 2, ($a=-h$, $b=h$), we have

$$(3.3)\qquad \int_{-h}^{h}|R_N(x)|^r dx\to 0.$$

But since if $\psi(t)$ be the Fourier transform of $f(x)\lambda(x)\frac{\sin^2 x/4}{x^2/4}$, then since $\frac{1}{\sqrt{2\pi}}\int_{-N}^{N}\psi(t)e^{-ixt}dt$ converges in mean L_r to $f(x)\lambda(x)\frac{\sin^2 x/4}{x^2/4}$, (3.3) shows that

$$(3.4)\qquad \lim_{N\to\infty}\int_{-h}^{h}\left|\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\varphi_\infty(t)e^{-ixt}dt-f(x)\frac{\sin^2 x/4}{x^2/4}\right|^r dx=0.$$

But since we have

$$\frac{1}{\sqrt{2\pi}}\int_{-N}^{N}\varphi_\infty(t)e^{-ixt}dt=\sqrt{\frac{\pi}{2}}\sum_{-N}^{N}c_n e^{inx}\frac{\sin^2 x/4}{x^2/4},$$

it results

$$\int_{-h}^{h}\left|\sum_{-N}^{N}c_n e^{inx}-f(x)\right|^r dx$$
$$\leqq C\int_{-h}^{h}\left|\sum_{-N}^{N}c_n e^{inx}-f(x)\right|^r\left\{\frac{\sin^2 x/4}{x^2/4}\right\}^r dx$$
$$=C\int_{-h}^{h}\left|\frac{1}{\sqrt{2\pi}}\int_{-N}^{N}\varphi_\infty(t)e^{-ixt}dt-f(x)\frac{\sin^2 x/4}{x^2/4}\right|^r dx$$

which tends to zero by (3.4) and this proves our theorem.

We shall mention here that Theorems 5 and 6 are cease to be true if .

This is shown by considering the following example: let

$$(3.5) \qquad f(x) = 0, \quad for \quad -\pi < x \leq 0$$
$$= x^{-1/r}, \quad for \quad 0 < x < \pi.$$

Then we have

$$S_n(x) = \frac{1}{\pi}\int_0^\pi \frac{1}{t^r} \frac{\sin(n+\frac{1}{2})(t-x)}{\sin(t-x)/2}\,dt$$

and we can prove that

$$\int_{-\pi}^{0} |S_n(x)|^r dx$$

does not tend to zero. We do not concern details here.

(1) C may be different on each occurrence.

(2) If the right hand side is finite, then so does the left. Strictly, we should take the integral concerning x over $(-h, h-\varepsilon)$ and let ε tend to $+0$.

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