ON TRANSLATIONS OF A SEMIGROUP

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Let S be a semigroup. We mean by a right translation of S a mapping φ of S into S such that

$$\varphi(xy) = x\varphi(y)$$

for every $x, y \in S$.

Analogously we define a left translation ψ as

$$\psi(xy) = \psi(x)y$$

for every $x, y \in S$.

The set of all right translations of forms a semigroup by usual product defined as $(\varphi_\beta \varphi_\alpha)(x) = \varphi_\beta(\varphi_\alpha(x))$; and the set is denoted by Φ which is called a right translation semigroup of S. Similarly a left translation semigroup Ψ of S is defined. If S is a commutative semigroup, the distinction between "right" and "left" is not required.

In this paper we shall arrange fundamental properties of a translation semigroup and shall mention that some special types of semigroups are characterized by their translation semigroups. We may omit dual proof for simplicity, namely, prove propositions only in the "right" case and do not in the "left"case.

Now we denote by M the set of all mappings of S into S. Naturally M is a semigroup and $\Phi \subset M$, $\Psi \subset M$.

Lemma 1. Φ and Ψ are semigroups with two-sided unit.

Proof. If $\varphi_1, \varphi_2 \in \Phi$, then it holds that

$$(\varphi_2\varphi_1)(xy) = \varphi_2(\varphi_1(xy))$$

$$= \varphi_2(x\varphi_1(y)) = x\varphi_2\varphi_1(y)$$

for every $x, y \in S$. Hence $\varphi_1\varphi_2 \in \Phi$. Whatever S is, an identical mapping φ_0 of S to S is included in Φ , and φ_0 is obviously a two-sided unit of Φ since

$$\varphi_0\varphi = \varphi\varphi_0 = \varphi$$

for every $\varphi \in \Phi$.

Let f_a be a mapping of S into S which is given as

$$f_a(x) = xa \qquad a \in S ,$$

and let g_a be a mapping defined as $g_a(x) = ax$. This f_a is called an inner right translation, and g_a is called an inner left translation. Denote by R the set of all f_a where a flows throughout S , and by L the set of all g_a, $a \in S$. Obviously

$$R \subset \Phi, \quad L \subset \Psi.$$

As well-known, S is homomorphic to L, and dually homomorphic to R.

Lemma 2. R is a right ideal of S and L is a left ideal of S.

Proof. For every $\varphi \in \Phi$ and $f_a \in R$,

$$(\varphi f_a)(x) = \varphi(f_a(x))$$

$$= \varphi(xa) = x\varphi(a) = f_{\varphi(a)}(x)$$

so that we get $\Phi R \subset R$.

Corollary. R and L are subsemigroups of Φ and Ψ respectively.

We have easily

Theorem 1. A mapping φ is a right translation if and only if $\varphi g_a = g_a \varphi$ for all left inner translation g_a ; and ψ is a left translation if and only if $\psi f_a = f_a \psi$ for all f_a .

Proof. If φ is a right translation of S , then, by the definition,

$$\varphi(g_a(x)) = \varphi(ax)$$

$$= a\varphi(x) = g_a(\varphi(x)),$$

and the converse is clear.

Theorem 2. A right translation φ of S maps a right ideal[1] to a right ideal. If $\varphi(S)$ is an image of S by φ and K is a right ideal of $\varphi(S)$, then the inverse image I of K by is a right ideal of S .

Proof. Let I be a right ideal of S . Take any $u \in J = \varphi(I)$, then $u = \varphi(x)$ for some $x \in I$. For every $z \in S$, since $zx \in I$,

$$zu = z\varphi(x) = \varphi(zx) \in J.$$

Hence J is a right ideal of S . Next, we shall prove the latter half of the theorem. Let I be the inverse image of a right ideal K of $\varphi(S)$. For $x \in I$ and $z \in S$,

$$\varphi(zx) = z\varphi(x) \in zK \subset K$$

proving that $zx \in I$ and I is a right ideal. Thus the proof of the theorem has been completed.

Theorem 2'. A left translation ψ of S maps a left ideal to a left ideal. If $\psi(S)$ is an image of S by ψ and K is a left ideal of $\psi(S)$, then the inverse image I of K by ψ is a left ideal of S .

Remark. A subsemigroup of S is not necessarily mapped to a subsemigroup of S . Really a following simple example shows this fact.

A semigroup S is composed of three elements and the product is defined as following table.

	a	b	c
a	a	a	a
b	a	b	c
c	a	c	b

Let φ be a mapping :

$$\varphi(a) = a,\ \varphi(b) = c,\ \varphi(c) = b.$$

Though φ is clearly a translation, the image $\{a, c\}$ of a subsemigroup $\{a, b\}$ is not a subsemigroup.

There are two mutually isomorphic semigroups : S with a right translation semigroup Φ , and S' with a right translation semigroup Φ' .

Theorem 3. Let f be an isomorphism of S to S' .

(1). If $\varphi \in \Phi$, then $f\varphi f^{-1} \in \Phi'$;
if $\psi \in \Psi$, then $f\psi f^{-1} \in \Psi'$,

(2) if $\varphi' \in \Phi'$, then $\varphi' = f\varphi f^{-1}$ for some $\varphi \in \Phi$,
if $\psi' \in \Psi'$, then $\psi' = f\psi f^{-1}$ for some $\psi \in \Psi$,

(3) Φ is isomorphic to Φ' , and Ψ is isomorphic to Ψ' .

Proof. (1) Since f and f^{-1} are isomorphisms,

$$f\varphi f^{-1}(xy) = f\varphi(f^{-1}(x)f^{-1}(y))$$
$$= f(f^{-1}(x)\varphi f^{-1}(y)) = ff^{-1}(x)f\varphi f^{-1}(y)$$
$$= x f\varphi f^{-1}(y).$$

(2) For $\varphi' \in \Phi$, letting $\varphi = f^{-1}\varphi' f$,

$$f\varphi f^{-1} = f(f^{-1}\varphi' f)f^{-1} = (ff^{-1})\varphi'(ff^{-1}) = \varphi'.$$

(3) Since $f\varphi_1 f^{-1} = f\varphi_2 f^{-1}$ implies $\varphi_1 = \varphi_2$, the correspondence $\varphi \to f\varphi f^{-1}$ is one to one. Now if $\varphi_1 \to f\varphi_1 f^{-1}$ and $\varphi_2 \to f\varphi_2 f^{-1}$,
then

$$(f\varphi_1 f^{-1})(f\varphi_2 f^{-1}) = f\varphi_1(f^{-1}f)\varphi_2 f^{-1} = f(\varphi_1\varphi_2)f^{-1}.$$

It has been proved that the correspondence $\varphi \to f\varphi f^{-1}$ is an isomorphism.

Theorem 4. The three conditions are all equivalent.

(1) Φ is homomorphic to R ,
(2) S contains a right unit,
(3) $\Phi = R$.

Proof. (1) $\to$ (2) Since Φ contains a two-sided unit, an identical mapping, R also contains it, i.e., there is $e \in S$ such that

$$f_e(x) = xe.$$

This e is a right unit of S .

(2) $\to$ (3) Let e be a right unit of S .
Then

$$\varphi(x) = \varphi(xe) = x\varphi(e) = f_{\varphi(e)}(x),$$

whence $\Phi \subset R$. With $R \subset \Phi$, we get $\Phi = R$.

(3) $\to$ (1) : trivial.

Theorem 4'. The three conditions are all equivalent.

(1) Ψ is homomorphic to L,
(2) S contains a left unit,
(3) $\Psi = L$.

Theorem 5. Φ (or Ψ) is isomorphic to S if and only if S has a two-sided unit.

Proof. Suppose that S has a two-sided unit. By Theorem 4, we have $\Phi = R$; on the other hand, we can see that $a \leftrightarrow f_a$ is one-to-one and S is isomorphic to R. Thus Φ and S are proved to be mutually isomorphic each other. Conversely if Φ is isomorphic to S, S has a two-sided unit, because Φ has a two-sided unit.

Lemma 3. A right (left) translation maps a right (left) zero, if exists, to a right (left) zero.

Proof. From $x0 = 0$ for all x, we get $x\varphi(0) = \varphi(0)$ whence $\varphi(0)$ is a right zero.

Lemma 4. A right (left) translation maps a left (right) zero, if exists, to itself.

Proof. Let 0 be a left zero. From $0x = 0$, we have

$$\varphi(0) = \varphi(0x) = 0\varphi(x) = 0.$$

Lemma 5. An element mapped to itself by all right (left) translations is, if exists, a left (right) zero.

Proof. Let a be a fixed point by all $\varphi \in \Phi$. Let us, especially, take $\varphi = f_x \in R$, then $a = f_x(a) = ax$ for all $x \in S$. Hence a is a left zero of S.

Thus we have

Theorem 6. A left (right) zero, if exists, of S is only one element which is invariant by all right (left) translations of S.

Corollary. If S has a two-sided zero 0, 0 is a fixed element under every translation φ and ψ.

Definition. A semigroup S is called a right singular semigroup, if the product xy of x and y is defined as $xy = y$. A left singular semigroup is defined as $xy = x$.

We shall discuss how a right (left) singular semigroup is characterized by the right (left) translation semigroup Φ (Ψ).

Theorem 7. $\Phi = M$ if and only if S is a right singular semigroup.

$\Psi = M$ if and only if S is a left singular semigroup.

Proof. At first, we shall prove that S is right singular if $\Phi = M$. Suppose that there are x_0 and $y_0 \in S$ such that

$$x_0 y_0 \neq y_0.$$

We can choose a mapping $\varphi (\in M)$ of S into S such that

$$\varphi(y_0) = y_0$$

and

$$\varphi(x_0 y_0) = y_0 \ ;$$

on the other hand, since φ is at the same time a right translation because $\Phi = M$, we get

$$y_0 = \varphi(x_0 y_0) = x_0 \varphi(y_0) = x_0 y_0 ,$$

contradicting with the assumption that $y_0 \neq x_0 y_0$. Hence $xy = y$ for all $x, y \in S$.

Conversely, if S is right singular, from $\varphi(xy) = \varphi(y)$ and $x\varphi(y) = \varphi(y)$, it follows that $\varphi(xy) = x\varphi(y)$ for every $\varphi \in M$. This shows $M \subset \Phi$; hence $\Phi = M$.

Theorem 8. S is a right singular semigroup if and only if

$$\Psi = L = \{\varphi_0\}$$

where φ_0 is an identical mapping.

Proof. If S is right singular, all elements of S are left units. We can prove the theorem easily by Theorem 4'.

Corollary. Φ consists of one element if and only if S is a left singular semigroup.

Lastly we shall relate to characterization of another special semigroup by its translation semigroup.

Theorem 9. If S is a semigroup with multiplication

$$xy = 0 \quad \text{for every } x \text{ and } y \in S,$$

then $\Phi = \Psi$ and Φ is composed of mappings φ of S into S satisfying $\varphi(0) = 0$.
Furthermore the converse is true.

Proof. In the semigroup with multiplication $xy = 0$, the mapping φ, which fulfils $\varphi(0) = 0$, is proved to be a right translation. In fact,

$$\varphi(xy) = \varphi(0) = 0, \quad x\varphi(y) = 0,$$

hence

$$\varphi(xy) = x\varphi(y).$$

Next, we shall prove the converse. Suppose that there are x_0 and $y_0 \in S$ such that $x_0 y_0 = z_0 \neq 0$. Take a mapping φ with the conditions:

$$\varphi(0) = 0, \ \varphi(y_0) = y_0, \text{ and } \varphi(z_0) = 0,$$

then, since φ is also a right translation,

$$0 = \varphi(z_0) = \varphi(x_0 y_0) = x_0 \varphi(y_0) = x_0 y_0$$

arriving at contradiction. The proof of the theorem has been completed.

Finally, by the way, we shall define a <u>substituted semigroup</u> of a semigroup. Given a semigroup S and a right translation φ of it, a new multiplicative system S_φ with multiplication $x \cdot y$,

$$x \cdot y = \varphi(x) y.$$

Theorem 10. S_φ is a semigroup.

Proof. Since φ is a right translation,

$$(x \cdot y) \cdot z = (\varphi(x) y) \cdot z = \varphi(\varphi(x) y) z = \varphi(x)\varphi(y) z$$

On the other hand,

$$x \cdot (y \cdot z) = x \cdot (\varphi(y) z) = \varphi(x)\varphi(y) z.$$

Hence we have $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.
Analogously if we define S_ψ as the system with $x * y = x\psi(y)$, we have

Theorem 10'. S_ψ is a semigroup.

S_φ and S_ψ are called <u>substituted semigroups</u> of S by a right translation φ and a left translation ψ respectively.

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1) If $SA \subset A$, we call A a right ideal of S.

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