A NOTE ON MIDDLE UNITARY SEMIGROUPS

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In this paper the author touches upon two problems, one of which is to determine what is the structure of $(\mathfrak{M})$-inversible semigroups and the other is of the general theory of special middle unitary semigroups. The proof of every theorem or lemma, however, is to be stated in another paper. In this paper, we use symbols $\dotplus$ or $\cdot\sum$ for the direct sum, i.e. the disjoint sum of sets. And moreover let A, B be two of any subsets of a semigroup, and use AB for the set $\{xy \mid x \in A, y \in B\}$.

1. In this paragraph, we define the relative inversibility of semigroups and completely determine the structure of semigroups having such a property in regard to their own middle units. Let S be a semigroup and N be a subset of S, then we shall say that S is relatively inversible, relatively left inversible or relatively right inversible in regard to N if S satisfies each of the following conditions.

(1) For any $x \in S$, there exists $x^* \in S$ with $xx^* = x^*x \in N$
(2) For any $x \in S$, there exists $x' \in S$ with $x'x \in N$
(3) For any $x \in S$, there exists $x'' \in S$ with $xx'' \in N$

We use the initial signature '(N)-inversible semigroup' for any semigroup relatively inversible in regard to N, and especially use '$(\mathfrak{M})$-inversible semigroup' for any semigroup relatively inversible in regard to the set consisting of all middle units of its own. It is clear that any semigroup is relatively left (right) inversible in regard to itself and that, generally speaking, even if a semigroup is both relatively right inversible and relatively left inversible in regard to its subset N, it is not necessarily relatively inversible in regard to N. However, if S is a middle unitary semigroup [1] and N is a subset consisting of all middle units of its own, the following lemma is satisfied.

[Lemma 1.1] If S is both relatively right inversible and relatively left inversible in regard to N, S is relatively inversible in regard to N.

Now any group is a semigroup relatively inversible in regard to its unit and it is well known to the general public that the semigroups relatively inversible in regard to all their own left (right) units are nothing but the right (left) groups [2].

Therefore, we may consider the $(\mathfrak{M})$-inversible semigroups to be semigroups more general than both the groups and the right (left) groups. Hereafter we use G for a $(\mathfrak{M})$-inversible semigroup, and M for the set consisting of all middle units of G.

[Lemma 1.2] $xx_1^* = x_1^*x \in M$, $xx_2^* = x_2^*x \in M$ for some $x, x_1^*, x_2^* \in G$ implies $xx_1^* = x_1^*x = x_2^*x = xx_2^*$

We give the following equivalent relation $\sim$ to the elements of G.

$a \sim b$ when and only when there exists such an element $x \in G$ as in $ax \in M$ and $bx \in M$.

In this case,
(1) $a \sim a$ for any $a \in G$.
(2) $a \sim b$ implies $b \sim a$.
(3) $a \sim b$, $b \sim c$ implies $a \sim c$.
(4) $a \sim b$, $c \sim d$ implies $ac \sim bd$.

are satisfied. Let Ω be the factor algebraic system and $\bar{a}$ be the residue class of G which contains the element a for any $a \in G$, then the following lemma holds good.

[Lemma 1.3] Ω is a group and its unit coincides with M.

Now if Γ stands for the set $\{x^2 \mid x \in M\}$, it coincides with the set consisting of all idempotent middle units of G, and consequently it is isomorphic to some outer product semigroup $R \times L$, where R is a right singular semigroup and L is a left singular semigroup [3]. Accordingly there exists an isomorphism ξ of Γ onto $R \times L$.

$$\Gamma \overset{\xi}{\cong} R \times L \qquad (1)$$

Secondly the mapping φ, which is the correspondence $a \longrightarrow (\bar{\bar{a}}, aa^*)$ for any $a \in G$, is a homomorphism G onto $\Omega \times \Gamma$ (where an element a^* is such as $aa^* = a^*a \in \Gamma$).

$$G \overset{\varphi}{\sim} \Omega \times \Gamma \qquad (2)$$

Consequently, by (1) and (2)

$$G \overset{\xi\varphi}{\sim} \Omega \times R \times L \qquad (3)$$

(Remark)

In general, by a quasi-$\mathfrak{T}$-group is meant an outer product semigroup of a group, a right singular semigroup and a left singular semigroup. Accordingly $\Omega \times R \times L$ is a quasi-$\mathfrak{T}$-group.

If we use $G(g, r, \ell)$ for the inverse image of $(g, r, \ell) \in \Omega \times R \times L$ by $\xi\varphi$, $G(g_1, r_1, \ell_1)\, G(g_2, r_2, \ell_2)$ is the set consisting of only one element for any $(g_1, r_1, \ell_1), (g_2, r_2, \ell_2) \in \Omega \times R \times L$. From the above observations the following lemma is concluded.

[Lemma 1.4] If G is a $(\mathfrak{M})$-inversible semigroup, there are sets $\{G_\alpha\}_{\alpha \in Q}$ having as each coefficient an element of a quasi-$\mathfrak{T}$-group Q, such as

$$G = \dot{\sum_{\alpha \in Q}} G_\alpha \; ;$$

$$G_\beta G_\gamma = \text{single element} \in G_{\beta\gamma} \quad \text{for any } \beta, \gamma \in Q.$$

Moreover, if $p_{(g, r, \ell)}$ stands for $G(g, r, \ell)\, G(g^{-1}, r, \ell)\, G(g, r, \ell)$

$$p_{(g_1, r_1, \ell_1)}\, p_{(g_2, r_2, \ell_2)} = p_{(g_1 g_2, r_2, \ell_1)}$$
for any $(g_1, r_1, \ell_1), (g_2, r_2, \ell_2) \in \Omega \times R \times L$.

Therefore, the following theorem is satisfied by the above observations and the [Lemma 1.4]

[Theorem 1.1] If G is a $(\mathfrak{M})$-inversible semigroup, there are sets $\{G_\alpha\}_{\alpha \in Q}$ having as each coefficient an element of a quasi-$\mathfrak{T}$-group Q and elements $\{p_\alpha\}_{\alpha \in Q}$, $p_\alpha \in G_\alpha$, such as

$$G = \dot{\sum_{\alpha \in Q}} G_\alpha \; ;$$

$$G_\beta G_\gamma = p_{\beta\gamma} \text{ for any } \beta, \gamma \in Q.$$

Since $\{p_\alpha\}_{\alpha \in Q}$ is clearly a quasi-$\mathfrak{T}$-group, G is nothing but a quasi-$\mathfrak{T}$-group in essence.

2. It seems to the author that fairly many semigroups treated by us contain no idempotents except their own middle units. For instance, the completely non-commutative semigroups, the quasi-$\mathfrak{T}$-groups, the right (left) simple semigroups and the $(\mathfrak{M})$-inversible semigroups are all such semigroups. Although both the regular semigroups and the right (left) regular semigroups, for example, the semigroup consisting of all positive integers, contain no idempotents except their own middle units, they are not necessarily $(\mathfrak{M})$-inversible. [4]

In this sense, the semigroups containing no idempotents except their own middle units are more generalized semigroups than the $(\mathfrak{M})$-inversible semigroups. Now, by a special middle unitary semigroup we mean a semigroup which is middle unitary and contains no idempotents except its own middle units. Then the purpose of the author in this paragraph is to expatiate on the general theory of the special middle unitary semigroups and to show that the problem determining the structure of such semigroups consequently reduced to that of the structure of the nonpotent semigroups. Hereafter S stands for a middle unitary semigroup and M_S for the set consisting of all middle units of S unless otherwise provided. Moreover, two symbols $\exists$, $\bar{\exists}$ stand for 'existence', 'non existence' respectively. Accordingly, for example, if we write '$\exists x$; —— ' we mean that there exists an element x satisfing the condition —— .

Any element of S is logically contained in one of the following S_1, S_2, S_3 and S_4.

$$S_1 = \{x \mid \exists x'';\, xx'' \in M_S, \exists x';\, x'x \in M_S\}$$

$$S_2 = \{x \mid \exists x'';\, xx'' \in M_S, \bar{\exists} x';\, x'x \in M_S\}$$

$$S_3 = \{x \mid \bar{\exists} x'';\, xx'' \in M_S, \exists x';\, x'x \in M_S\}$$

$$S_4 = \{x \mid \bar{\exists} x'';\, xx'' \in M_S, \bar{\exists} x';\, x'x \in M_S\}$$

Therefore,

$$S = S_1 \dot{+} S_2 \dot{+} S_3 \dot{+} S_4 \qquad (*)$$

(Remark)

The relation '$\exists x''$; $xx'' \in M_S$, $\exists x'$; $x'x \in M_S$' is equivalent to the relation '$\exists x^*$; $xx^* = x^*x \in M_S$'. Accordingly, as S_1 is (M_S)-inversible, it is $(\mathfrak{M})$-inversible.

[Lemma 2.1] If S is a special middle unitary semigroup, it follows that S_1 is a $(\mathfrak{M})$-inversible semigroup, $S_2 = \phi$, $S_3 = \phi$ and S_4 is an ideal of S.

Now the following theorem is concluded by (*) and [Lemma 2.1], because S_4 contains no idempotents. (We call such semigroup 'nonpotent semigroup')

[Theorem 2.1] If S is a special middle unitary semigroup, the relation

$$S = V \dot{+} T$$

is satisfied, where V is a $(\mathfrak{M})$-inversible subsemigroup and T is a nonpotent ideal of S.

(Remark)
In the above theorem, it is easy to see that V, T are uniquely determined by S.

Hereafter, V stand for a $(\mathfrak{M})$-inversible semigroup and T for a nonpotent semigroup.

A middle unitary semigroup S is said to be an extension V by T if S satisfies the following conditions.

(1) $S = V \dot{+} T$; both V and T are subsemigroups of S.
(2) S is a special middle unitary semigroup.
(3) $V = \{x \mid x \in S, \exists x^* \in S.\ xx^* = x^*x \in M_S\}$.

(Remark)
If S is an extension of V by T, T is an ideal of S.

In the next place we define the left and right translations of a semigroup.

Let K be any semigroup and $\lambda x (x\rho)$ is a mapping of K into itself, then $\lambda (\rho)$ is called a left (right) translation of K if it has the following property [5].

$$(\lambda x)y = \lambda(xy)\,[y(x\rho) = (yx)\rho]$$
$$\text{for any } x, y \in K$$

If λ_1 and λ_2 (ρ_1 and ρ_2) are left (right) translations of K, then so is their product. Moreover, the set consisting of all left (right) translations of K is a semigroup, which we shall call the left (right) translation semigroup of K [5].

Let λ be a left translation of K and ρ be a right translation of K, then we shall say that λ (ρ) is middle unitary if

$$y(\lambda x) = yx\,[(y\rho)x = yx] \text{ for any } x, y \in K$$

and say that λ and ρ are linked or commuted if they satisfy the following (1), (2) respectively.

(1) $x(\lambda y) = (x\rho)y$ for any $x, y \in K$.
(2) $(\lambda x)\rho = \lambda(x\rho)$ for any $x \in K$.

Henceforth, $\mathfrak{T}_L$ is used for the left translation semigroup of T and $\mathfrak{T}_R$ for the right translation semigroup of T, and A, B, C, etc. for elements of V and a, b, c, etc. for elements of T.

[Theorem 2.2] If S is an extension of V by T, there exist homomorphisms φ, ψ of V into $\mathfrak{T}_L$, respectively such as, if λ_A stands for $\varphi(A)$ and ρ_A for $\varphi(A)$,

(1) λ_A and ρ_A are linked for any $A \in V$.
(2) λ_A and ρ_B are commuted for any $A, B \in V$.
(3) Both λ_E and ρ_E are middle unitary for any $E \in M_V^2$.
(4) λ_E is a right unit of $\varphi(V)$ and ρ_E is a left unit of $\psi(V)$, for any $E \in M_V^2$.

are satisfied, and the product $\circ$ in S is represented by

$$\begin{cases} \alpha \circ \beta = \alpha \times \beta & \text{for any } \alpha, \beta \in V. \\ \quad\;\;\; = \alpha \Delta \beta & \text{for any } \alpha, \beta \in T. \\ \quad\;\;\; = \lambda_\alpha \beta & \text{for any } \alpha \in V, \beta \in T. \\ \quad\;\;\; = \alpha \rho_\beta & \text{for any } \alpha \in T, \beta \in V., \end{cases}$$

where M_V is the set consisting of all middle units of V and symbols $\times$, Δ stand for the product in V, T respectively.

Conversely,

[Theorem 2.3] If there are two homomorphisms φ, ψ of V into $\mathfrak{T}_L$, $\mathfrak{T}_R$ respectively such as, if λ_A stands for $\varphi(A)$ and ρ_A for $\psi(A)$,

(1) λ_A and ρ_A are linked for any $A \in V$.
(2) λ_A and ρ_B are commuted for any $A, B \in V$.
(3) Both λ_E and ρ_E are middle unitary for any $E \in M_V^2$
(4) λ_E is a right unit of $\varphi(V)$ and ρ_E is a left unit of $\psi(V)$, for any $E \in M_V^2$.

are satisfied, then the direct sum $V \dot{+} T$ becomes an extension of V by T if the product $\circ$ in $V \dot{+} T$ is defined as follows;

$$\begin{cases} \alpha \circ \beta = \alpha \times \beta & \text{for any } \alpha, \beta \in V. \\ \quad\;\;\; = \alpha \Delta \beta & \text{for any } \alpha, \beta \in T. \\ \quad\;\;\; = \lambda_\alpha \beta & \text{for any } \alpha \in V, \beta \in T. \\ \quad\;\;\; = \alpha \rho_\beta & \text{for any } \alpha \in T, \beta \in V., \end{cases}$$

where symbols $\times$, Δ stand for the products in V, T respectively.

(Remarks)
1. The set M_V^2 in the two theorems above-mentioned consists of all idempotent middle units of V.

2. By the [Theorem 2.3], it is easy to see that the following product $\circ$ in $V \dotplus T$ gives an extension of V by T.

$$\begin{cases} \alpha \circ \beta = \alpha \times \beta & \text{for any } \alpha, \beta \in V \\ \quad = \alpha \triangle \beta & \text{for any } \alpha, \beta \in T. \\ \quad = \alpha & \text{for any } \alpha \in T, \beta \in V. \\ \quad = \beta & \text{for any } \alpha \in V, \beta \in T \end{cases}$$

Accordingly, there exists at least one extension of V by T.

Any middle unitary semigroup S is uniquely decomposed to the direct sum of two sets V and T, where V is a $(\mathfrak{M})$-inversible subsemigroup of S and T is a nonpotent ideal of S. And from the [Theorem 2.2], it is obvious that there exist some homomorphisms φ, ψ of V into the left translation semigroup of T, the right translation semigroup of T respectively, such that the product $\circ$ of S is represented by the following relation.

$$\begin{cases} \alpha \circ \beta = \alpha \times \beta & \text{for any } \alpha, \beta \in V \\ \quad = \alpha \triangle \beta & \text{for any } \alpha, \beta \in T. \\ \quad = \lambda_\alpha \beta & \text{for any } \alpha \in V, \beta \in T. \\ \quad = \alpha \rho_\beta & \text{for any } \alpha \in T, \beta \in V, \end{cases}$$

where λ_α stands for $\varphi(\alpha)$, ρ_β for $\psi(\beta)$ and $\times$, $\triangle$ are the products in V, T respectively.
S is, therefore, determined by V, T, φ and ψ. We shall describe this sense as $S = ((V, T; \varphi, \psi))$

Now, in what cases are two special middle unitary semigroups isomorphic to each other? The following theorem is the answer of this question.

[Theorem 2.4] In order to two special middle unitary semigroups $S = ((V, T; \varphi, \psi))$, $S' = ((V', T'; \varphi', \psi'))$ are isomorphic to each other it is necessary and sufficient that there exist two isomorphisms ξ_V of V onto V' and ξ_T of T onto T', such that

$$\lambda_A = \xi_T^{-1} \lambda'_{\xi_V A} \xi_T$$

$$\rho_A = \xi_T \rho'_{\xi_V A} \xi_T^{-1}$$

are satisfied for any $A \in T$, where λ_A, ρ_A, $\lambda'_{\xi_V A}$, and $\rho'_{\xi_V A}$ stand for $\varphi(A)$, $\psi(A)$, $\varphi'(\xi_V A)$ and $\psi'(\xi_V A)$ respectively.

By the above observations, the structure of the special middle unitary semigroups have become clear on the whole. But, we have to determine the structure of the nonpotent semigroups in order to completely determine the structure of the special middle unitary semigroups. The author will touch upon this problem in another opportunity.

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By a right (left) group we mean a semigroup which is left regular and right simple (right regular and left simple). Such a semigroup is isomorphic to an outer product semigroup $G \times R$ $(G \times L)$, where G is a group and $R(L)$ is a right (left) singular semigroup.

[3] N.Kimura ; On some examples of semigroups. Kodai. Math. Semi. Rep. No.3(1954).

[4] By a completely non-commutative semigroup we mean a semigroup which has the following property.
$xy = yx$ implies $x = y$ for any two elements x, y
A semigroup K is a completely non-commutative when and only when it is isomorphic to some $L \times R$, where L is a left singular semigroup and R is a right singular semigroup. In other words, a completely non-commutative semigroup is a semigroup whose elements are all idempotent middle units of its own. This fact is proved in [6]. See [2] or [3] about the right (left) simple semigroups, the right (left) regular semigroups and the regular semigroups.

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